

# Cluster Derivation of the Parisi Scheme for Disordered Systems

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**Abstract.** We propose a general quantitative scheme in which systems are given the freedom to sacrifice energy equi-partitioning on the relevant time-scales of observation, and have phase transitions by separating autonomously into ergodic sub-systems (clusters) with different characteristic time-scales and temperatures. The details of the break-up follow uniquely from the requirement of zero entropy for the slower cluster. Complex systems, such as the Sherrington-Kirkpatrick model, are found to minimise their free energy by spontaneously decomposing into a hierarchy of ergodically equilibrating degrees of freedom at different (effective) temperatures. This leads exactly and uniquely to Parisi's replica symmetry breaking scheme. Our approach, which is somewhat akin to an earlier one by Sompolinsky, gives new insight into the physical interpretation of the Parisi scheme and its relations with other approaches, numerical experiments, and short range models. Furthermore, our approach shows that the Parisi scheme can be *derived* quantitatively and uniquely from plausible physical principles.

## I INTRODUCTION

The so-called Parisi scheme [1] for replica symmetry breaking (RSB) has been one of the most successful tools in the description of (the statics of) disordered systems in the non-ergodic or glassy phase. Originally proposed as the solution for the Sherrington-Kirkpatrick (SK)-model [2] for mean field spin glasses, it has been successfully applied to a wide range of disordered systems. The physical interpretation of the Parisi scheme has been a subject of many discussions, and has led to the introduction of concepts such as *disparate time-scales* [3], *effective temperatures* [4], *low entropy production* [5] and non-equilibrium *thermodynamics* [6].

In this paper, we will show how a general scheme in which systems are given the freedom to sacrifice energy equi-partitioning by separating autonomously into sub-systems with different characteristic time-scales and temperatures, not only yields the Parisi scheme, but also introduces all the above-mentioned concepts in a very natural way. Our assumptions are simple, and all the quantities that appear in the theory, have a clear physical meaning. In section II we briefly discuss systems with disparate time-scales, and in section III we apply this to the benchmark problem

of mean field disordered systems, viz. the SK-model [2]. In section IV we present numerical evidence for the existence of multiple disparate time-scales. Finally, in the discussion V we summarize the simple physical picture that naturally emerges from our scheme, and discuss the points that still need further investigation.

## II SYSTEMS WITH DISPARATE TIME SCALES

In this section we briefly describe how the formalism, as developed and applied for Ising spin systems with slowly evolving bonds [7], can be generalised to arbitrary stochastic systems with two or more disparate time-scales. In the case of systems with two infinitely disparate time-scales, the faster variables  $\vec{x}_f$  equilibrate before the slower ones  $\vec{x}_s$  can effectively change. Therefore, the  $\vec{x}_s$  evolve to an a Boltzmann-type equilibrium distribution with an effective energy which is the free energy of the  $\vec{x}_f$  given the  $\vec{x}_s$ , at an effective inverse temperature  $\beta_s$ , while the  $\vec{x}_f$  evolve to the normal Boltzmann equilibrium distribution with  $\beta_f = \beta$ :

$$\begin{aligned} \mathcal{Z} &= \text{Tr}_{\vec{x}_s} (\mathcal{Z}_f(\vec{x}_s))^{\tilde{m}} , \\ \mathcal{Z}_f(\vec{x}_s) &\equiv \text{Tr}_{\vec{x}_f} \exp(-\beta\mathcal{H}(\vec{x})) . \end{aligned} \quad (1)$$

One thus obtains a theory with a non-negative replica dimension  $\tilde{m} \equiv \beta_s/\beta$ . This procedure can be generalised to a system with  $L+1$  different levels of stochastic variables, time-scales and temperatures  $\{(\vec{x}_\ell, \tau_\ell, \beta_\ell) : \ell \in \{0, L\}\}$ . Assuming that each level is adiabatically slower than the next level ( $\tau_\ell/\tau_{\ell-1} = 0$ ), we obtain the following recursion relations

$$\begin{aligned} \mathcal{Z}_\ell &\equiv \text{Tr}_{\vec{x}_\ell} (\mathcal{Z}_{\ell+1})^{\tilde{m}_{\ell+1}} \quad (\ell < L), \\ \mathcal{Z}_L &\equiv \text{Tr}_{\vec{x}_L} \exp(-\beta_L \mathcal{H}(\{\vec{x}\})) , \end{aligned} \quad (2)$$

where  $\tilde{m}_\ell \equiv \beta_{\ell-1}/\beta_\ell$ ,  $\beta_L = \beta$ , and the total free energy of the system, defined on the longest time-scale, is given by

$$\mathcal{F} = -\frac{1}{\beta_0} \log(\mathcal{Z}_0) . \quad (3)$$

## III THE SK-MODEL

We will now apply this scheme to the SK-model [2], for which the Parisi scheme was originally developed. Therefore, we briefly recall the definitions:

$$\mathcal{H}(\vec{\sigma}) = -\sum_{i<j}^N J_{ij} \sigma_i \sigma_j , \quad (4)$$

with Gaussian couplings  $J_{ij}$ ,  $P(J_{ij}) = \mathcal{N}(J_0/N, J/\sqrt{N})$ , and Ising spins  $\sigma_j$ .

We assume that there are  $L+1$  levels of spins with corresponding disparate time-scales and temperatures  $\{\vec{\sigma}_\ell = \{\sigma_j \in I_\ell\}, |I_\ell| \equiv \epsilon_\ell, \tau_\ell, T_\ell\}, \ell = 0, \dots, L\}$  in the system, where  $\tau_\ell/\tau_{\ell-1} = 0$ , such that larger  $\ell$  correspond to faster spins. Although we expect the selection of time-scales for the spins to depend on the specific realisation of the couplings, at present we will make the simplest approximation: the system can only choose the relative sizes  $\epsilon_\ell$  of the levels. A more detailed study of the (annealed) selection of levels will be presented elsewhere. To deal with the quenched disorder average over the  $J_{ij}$  we use the replica trick

$$\overline{\mathcal{F}} = -\frac{1}{\beta_0} \overline{\log \mathcal{Z}_0} = -\lim_{\tilde{n} \rightarrow 0} \frac{1}{\tilde{n}\beta_0} \log \overline{\mathcal{Z}_0^{\tilde{n}}} . \quad (5)$$

Together with the recursion relations (2) this leads to a nested set of  $\tilde{n} \prod_{\ell=1}^L \tilde{m}_\ell$  replicas, in which spins at level  $\ell$  carry a set of replica indices  $\{a\}_\ell \equiv \{a_0, \dots, a_\ell\}$ . The index  $a_0 = 1, \dots, \tilde{n}$  comes from the disorder average, whereas  $a_\ell = 1, \dots, \tilde{m}_\ell = \beta_{\ell-1}/\beta_\ell$ . The asymptotic free energy per spin is then given by

$$f = \lim_{\tilde{n} \rightarrow 0} \frac{-1}{\tilde{n}\beta_0} \left[ \frac{-J^2\beta^2}{4} \sum_{\{a\}_L, \{b\}_L} q_{\{b\}_L}^{\{a\}_L}{}^2 + \sum_{\ell=0}^L \epsilon_\ell \log(\mathcal{K}_\ell) \right] , \quad (6)$$

$$\mathcal{K}_\ell \equiv \text{Tr}_{\sigma^{\{c\}_\ell}} \exp \left[ \frac{J^2\beta^2}{2} \sum_{\{a\}_L, \{b\}_L} q_{\{b\}_L}^{\{a\}_L} \sigma^{\{a\}_\ell} \sigma^{\{b\}_\ell} \right] , \quad q_{\{b\}_L}^{\{a\}_L} \equiv \frac{1}{N} \sum_{\ell=0}^L \sum_{j \in I_\ell} \sigma_j^{\{a\}_\ell} \sigma_j^{\{b\}_\ell} . \quad (7)$$

With the definitions  $m_\ell \equiv \prod_{f=\ell}^L \tilde{m}_f = \beta_{\ell-1}/\beta$  we have  $\beta_0 \tilde{n} = \beta n$ , and the connection with the original Parisi scheme becomes clear. Note that  $0 \leq \tilde{m}_\ell \leq 1$ , as slower clusters cannot have a lower temperature than faster ones, because otherwise the latter would act as a heat bath. We now assume full ergodicity at each level in the hierarchy of time-scales:

$$q_{\{b\}_L}^{\{a\}_L} = q_\ell , \quad \ell \equiv \sum_{r=0}^L r \delta_{\{a\}_{r-1}, \{b\}_{r-1}} (1 - \delta_{a_r, b_r}) , \quad (8)$$

(i.e.  $\ell$  is the slowest level at which  $\{a\}_L$  and  $\{b\}_L$  differ), to obtain

$$f = \frac{J^2\beta}{2} \sum_{\ell=0}^L \left[ \frac{m_{\ell+1}}{2} (q_{\ell+1}^2 - q_\ell^2) - \epsilon_\ell [q]_\ell \right] - \frac{1}{m_1\beta} \sum_{\ell=0}^L \epsilon_\ell \int D z_0 \log(\mathcal{N}_\ell^1) , \quad (9)$$

where

$$\mathcal{N}_\ell^r \equiv \begin{cases} \int D z_r (\mathcal{N}_\ell^{r+1})^{\frac{m_r}{m_{r+1}}} , & r \leq \ell \\ 2 \cosh(J\beta m_{\ell+1} \sum_{s=0}^{\ell} z_s \sqrt{q_s - q_{s-1}}) , & r = \ell + 1 \end{cases} , \quad (10)$$

$$[q]_\ell \equiv \sum_{r=\ell}^L m_{r+1} (q_{r+1} - q_r) , \quad (11)$$

The physical meaning of the  $q_\ell$  is given by

$$q_\ell = \lim_{N \rightarrow \infty} \sum_j \overline{\langle \dots \langle \langle \langle \dots \langle \sigma_j \rangle_L \dots \rangle_{\ell+1} \rangle_\ell^2 \rangle_{\ell-1} \dots \rangle_0}, \quad (12)$$

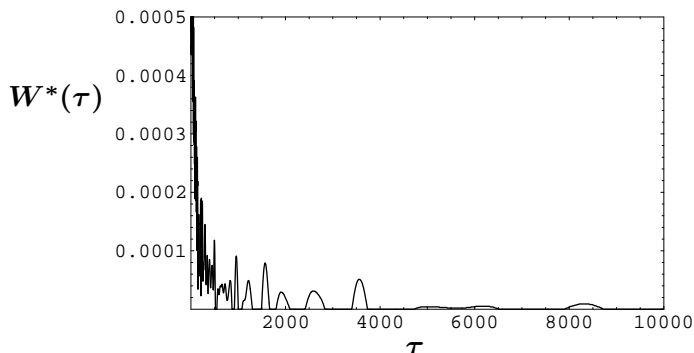
where  $\langle \cdot \rangle_r$  denotes the average over the level  $r$  process, and  $\overline{\quad}$  denotes the disorder average. The minimum of the free energy with respect to the  $\epsilon_\ell$  (with  $\sum_{\ell=0}^L \epsilon_\ell = 1$ , for  $\tilde{n}, \tilde{m}_\ell$  positive integers), is at  $\{\epsilon_L^* = 1, \epsilon_\ell^* = 0 \ \forall \ell < L\}$ , and we exactly recover the  $L$ -th order Parisi solution. Furthermore, the extremization with respect to the  $\tilde{m}_\ell$  is now recognized to express the fact that, for self-consistency, the entropy of the spins slower than level  $\ell$  is zero (they are effectively fixed on the time-scale  $\tau_\ell$ ).

## IV NUMERICAL EVIDENCE

In numerical simulations of the SK-model we have measured the distribution of the number of flips  $f$  at time  $t$  per spin:  $\rho_{\text{sim}}(f, t)$ . Assuming a characteristic time-scale  $\tau_j$  for each spin  $\sigma_j$  and a distribution  $W(\tau)$  of these time-scales, we obtain a theoretical prediction of the distribution of the number of flips per spin at time  $t$ :

$$\rho_{\text{th}}(W, f, t) \simeq \int_0^\infty d\tau W(\tau) \binom{t}{f} \left(\frac{1}{\tau}\right)^f \left(1 - \frac{1}{\tau}\right)^{t-f}. \quad (13)$$

Minimization of  $\sum_{f=0}^t [\rho_{\text{sim}}(f, t) - \rho_{\text{th}}(W(\tau), f, t)]^2$  with respect to  $W(\tau)$  then yields the most probable distribution of time-scales  $W^*(\tau)$ , see fig. 1.

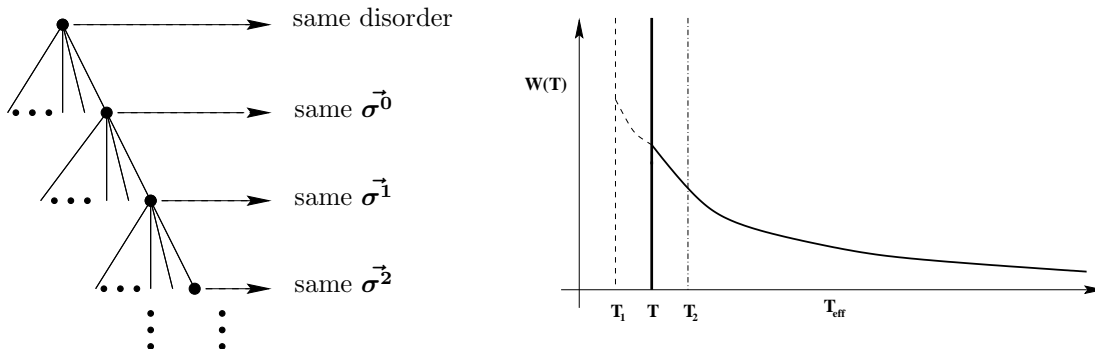


**FIGURE 1.** The most probable distribution of time-scales for a simulation of the SK-model with  $N = 6000$ , after  $t = 5 \cdot 10^5$  Monte-Carlo updates per spin, for  $T = 0.25$ .

We have found that both the number of peaks (in agreement with full RSB), and the separation between peaks (in agreement with infinitely disparate time-scales) seem to grow with increasing system size and/or time. The total fraction of the slow spins, however, seems to remain finite, which implies that a more precise analytical treatment for the choice of clusters or simulations with larger system sizes and/or times are needed.

## V DISCUSSION

We have shown that the Parisi solution can be derived from simple physical principles, and can be interpreted as describing a system with an infinite hierarchy of time-scales where a vanishingly small fraction of slow spins act as effective disorder for the faster ones. The block-sizes  $m_\ell$  at level  $\ell$  of the Parisi matrix are found to be the ratio of the effective temperature  $T_\ell$  of that level and the ambient temperature  $T$ . Extremization with respect to  $m_\ell$  expresses the fact that for self-consistency the entropy of the spins slower than level  $\ell$  is zero (they are frozen on the time-scale  $\tau_\ell$ ). It follows from physical considerations (i.e. the absence of heat flow in equilibrium) that  $m_\ell \leq 1$ ,  $\forall \ell$ . The fact that the fraction of slow spins vanishes, indicates that the cumulative entropy of the slow spins is less than extensive, and hence that the so-called *complexity* is zero (at least in this full-RSB model).



**FIGURE 2.** Left: the ultra-metric tree, which here is a direct consequence of the hierarchy of time-scales. Right: qualitative picture of the distribution  $W(T)$  of (effective) temperatures (time-scales increase with  $T_{\text{eff}}$ ) at external temperature  $T$  ( $T_1, T_2$ , resp.) in the spin glass phase.

Although a more careful treatment of the selection of clusters is obviously required, the main consequences of our interpretation do not crucially depend on it. Firstly, ultra-metricity (see fig. 2a) is a direct consequence of the existence of a hierarchy of time-scales. At each level  $\ell$ , the different descendants of a node represent different configurations of the  $\vec{\sigma}_{\ell+1}$ , which, however, all share the same realisation of disorder and slower spins. Furthermore, a large fraction of the spins (see fig. 2b) evolves at the fastest (microscopic) time-scale at the ambient temperature  $T$ , while a small fraction of the spins evolves at (infinitely) slower time-scales at higher effective temperatures. Therefore, cooling to a temperature  $T_1 < T$  and heating back to  $T$  will leave the spins with  $T_{\text{eff}} > T$  unchanged, explaining memory effects. On the other hand, after heating to  $T_2 > T$  and cooling back to  $T$ , the original configuration of the spins with  $T \leq T_{\text{eff}} \leq T_2$  will be erased, which may explain thermo-cycling experiments [8]. Note that since the spins are discrete variables, finding a small number of flips for a given spin implies long periods of stationarity (persistence) with only short periods of activity (avalanches).

We expect the qualitative features of our picture to survive in short range systems, where the time-scales need not be infinitely disparate due to activated processes. It is as yet unclear whether each level would represent a single cluster or a family of clusters. The origin of the slow time-scales of these clusters can only be understood if they are coupled much stronger internally, than (effectively) to the rest of the system (i.e. a softened version of the completely disconnected clusters which give rise to so-called Griffiths singularities in diluted systems [9]). In short range systems, the clusters would have to be spatially localised, which is in line with the *droplet* picture for short range spin glasses as proposed by Fisher and Huse [10]. The fact that the time-scale of a clusters increases with  $T_{\text{eff}} - T$ , explains why the effective age of a system at a certain  $T$  is found to be either older or younger when it spends some time at a  $T_1 (< T)$ ,  $T_2 (> T)$  respectively.

A more careful treatment of the selection of clusters is clearly needed (and currently been carried out), both for full- and 1-RSB models. This may allow us to calculate the complexity in such systems. Furthermore, it needs to be investigated whether slow clusters survive above the thermodynamical (spin) glass temperature  $T_{\text{sg}}$ . Our results suggest further numerical experiments for mean field and short range models, concentrating on quantities such as spin flip frequencies, avalanches, spatial correlations, and cluster persistency.

To conclude, we have shown how a scheme based on an autonomous selection of infinitely disparate time-scales can be used succesfully to describe the statics of disordered systems with quenched disorder and discrete and/or continuous stochastic variables. It allows us to *derive* the Parisi scheme from simple physical principles, and interpret its ingredients in such a way that it becomes compatible with the droplet picture for short range models.

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## REFERENCES

1. G. Parisi, Phys. Lett. **73A**, 203, (1979).
2. D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792, (1975).
3. H. Sompolinsky, Phys. Rev. Lett. **47**, 935, (1981).
4. L.L. Bonilla, F.G. Padilla and F. Ritort, Physica A **250**, 315, (1998).
5. L.F. Cugliandolo and J. Kurchan, Phys. Rev. Lett. **71**, 173, (1993); Prog. Theor. Phys. Supp. **126**, 407, (1997).
6. T.M. Nieuwenhuizen, Phys. Rev. E **61**, 267, (2000).
7. R.W. Penney, A.C.C. Coolen and D. Sherrington, J. Phys. A **26**, 3681, (1993).
8. for a recent review see e.g. M. Picco, F. Ricci-Tersenghi and F. Ritort cond-mat/0005541, and references therein.
9. R. B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969).
10. D.S. Fisher and D.A. Huse, Phys. Rev. Lett. **56**, 1601, (1986).