COMMUNICATIONS IN NUMERICAL METHODS IN ENGINEERING, Vol. 12, 461–470 (1996)

SYMBOLIC INTEGRATION OF POLYNOMIAL FUNCTIONS OVER A LINEAR POLYHEDRON IN EUCLIDEAN THREE-DIMENSIONAL SPACE

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SUMMARY

The paper concerns analytical integration of polynomial functions over linear polyhedra in three-dimensional space. To the authors' knowledge this is a first presentation of the analytical integration of monomials over a tetrahedral solid in 3D space. A linear polyhedron can be obtained by decomposing it into a set of solid tetrahedrons, but the division of a linear polyhedral solid in 3D space into tetrahedra sometimes presents difficulties of visualization and could easily lead to errors in nodal numbering, etc We have taken this into account and also the linearity property of integration to derive a symbolic integration formula for linear hexahedra in 3D space. We have also used yet another fact that a hexahedron could be built up in two, and only two, distinct ways from five tetrahedral shaped elements These symbolic integration formulas are then followed by an illustrative numerical example for a rectangular prism element, which clearly verifies the formulas derived for the tetrahedron and hexahedron elements.

KEY WORDS linear polyhedra; symbolic integration; polynomial functions; monomials; tetrahedron; hexahedron

1. INTRODUCTION

The computation of volumes, centre of mass, moment of inertia and other geometric properties of rigid homogeneous solids arises very frequently in a large number of engineering applications, in CAD/CAE/CAM applications in geometric modelling as well as in robotics. Quadrature formulas for multiple integrals have always been of great interest in computer applications. A good overview of various methods of evaluating volume (triple) integrals in this context is given by Lee and Requicha.[1] Timmer and Stern[2] have discussed a theoretical approach to the evaluation of volume integral by transforming the volume integral to a surface integral over the boundary of the integration domain. Lien and Kajiya[3] presented an outline of a closed formula of volume integration for tetrahedra and suggested that volume integration for a linear polyhedron can be obtained by decomposing it into a set of solid tetrahedrons. Cattani and Paoluzzi[4,5] gave a symbolic solution to both the surface and volume integration of polynomials by using a triangulation of solid based mainly on the concepts of Timmer and Stern.[2] In a recent paper, Bernardini[6] has presented the evaluation of integrals over n-dimensional polyhedra which are based on methods presented by Timmer and Stern[2] and Lien and Kajiya[3]. Closed integration formulas for polynomial functions are presented in this paper for a linear tetrahedral solid in 3D space. We have shown how this can be used effectively to compute the volume integrals over a linear polyhedral solid domain. The division of a linear polyhedral solid in 3D space

CCC 1069–8299/96/080461–10

Received 20 January 1995
Revised 3 January 1996

into tetrahedra sometimes presents difficulties of visualisation and could easily lead to errors in nodal numbering, etc. more convenient subdivision of space is into eight cornered brick elements.[7] Such elements could be assembled automatically from several tetrahedra and the process of creating these tetrahedra left to a simple logical program. The method of computing the volume integral is to map the arbitrary tetrahedron into a unit orthogonal tetrahedron. This explicit integration formula is followed by an example for which we have explained the detailed computational scheme.

2. PROBLEM STATEMENT

Most computational studies of volume integrals deal with problems in which the domain of integration is very simple, like cube or sphere, but the integrand is very complicated. However, in real applications, and even in the most powerful numerical technique – the finite element method of recent origin[7] – we confront the inverse problem; the integrating function, say, (x, y, z) is usually simple but the domain is very complicated. Hence, in this paper and even in other previous works,[3–6] we recognize the importance of obtaining practical explicit formulae for the exact evaluation of integrals,"

$$\iiint_p f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where p is a polyhedron in $\mathbb{R}^3$ $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is the differential volume. In general such simple function may be written as trivariate polynomial

$$f(x, y, z) = \sum_{\alpha=0}^{n} \sum_{\beta=0}^{m} \sum_{\gamma=0}^{p} a_{\alpha\beta\gamma} x^{\alpha} y^{\beta} z^{\gamma}$$

where α, β, γ are non-negative integers. However, the present paper is focused on the calculation of the following integral of monomials:

$$\mathrm{III}^{\alpha\beta\gamma} = \iiint_{\mathrm{T}_e} x^{\alpha} y^{\beta} z^{\gamma}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where T_e is an arbitrary tetrahedron shaped element with element number 'e' having its vertices at (x_i, y_i, z_i), (x_j, y_j, z_j), (x_k, y_k, z_k), (x_l, y_l, z_l). Now it is easy to see that an extension to the integral $\iiint_{\mathrm{T}_e} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ can be obtained by the linearity property of integrals. We have used the fact that an integration over the domain of an arbitrary tetrahedron can always be evaluated by transforming it into an integral over the domain of an unit orthogonal tetrahedron by means of a suitable mapping.

3. VOLUME INTEGRATION OVER AN ARBITRARY TETRAHEDRON

In this Section, we first obtain the volume integral of a scalar function $f(x, y, z) = x^{\alpha} y^{\beta} z^{\gamma}$, where α, β, γ are positive integers, over an arbitrary tetrahedron by transforming it into an orthogonal unit tetrahedron. That is, here we are concerned about the evaluation of integral

$$\mathrm{III}_{\mathrm{T}_e}^{\alpha\beta\gamma} = \iiint_{\mathrm{T}_e} x^{\alpha} y^{\beta} z^{\gamma}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{1}$$

where T_e is an arbitrary tetrahedron with element number 'e' in the x, y, z Cartesian co-ordinate system.

3.1. Theorem 1. A structure product $\mathrm{III}_{\mathrm{T}_e}{}^{\alpha,\beta,\gamma}$ over the volume of an arbitrary tetrahedron T_e is a polynomial combination of its vertices (x_i, y_i, z_i), $(x_j, {}_j, z_j)$, (x_k, y_k, z_k), and (x_l, y_l, z_l) and is given by the formula (see Figure 1)

$$\mathrm{III}_{\mathrm{T}_e}^{\alpha\beta\gamma} = |\det J^e| \sum_{n=0}^{\alpha+\beta+\gamma} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \mathrm{III}_{\bar{\mathrm{T}}}^{r,s,t} G^e(r,s,t) \tag{2}$$

where

$$\mathrm{III}_{\bar{\mathrm{T}}}^{r,s,t} = \int_0^1 \int_0^{1-\xi} \int_0^{1-\xi-\eta} \xi^r \eta^s \zeta^t \, d\xi \, d\eta \, d\zeta = \frac{\lfloor r \lfloor s \lfloor t}{\lfloor r+s+t+3}$$

$$G^e(r, s, t) = \left(\left\{ \frac{\partial^n f(\xi, \eta, \zeta)}{\partial \xi^r \, \partial \eta^s \, \partial \zeta^t} \right\} \Big/ (\lfloor r \lfloor s \lfloor t) \right)_{(0,0,0)}$$

$$r + s + t = n \tag{3}$$

$\bar{\mathrm{T}}$ is the unit orthogonal tetrahedron in ξ, η, ζ space (Figure 2) panned by vertices i, j, k, l a $\langle (1,0,0), (0,1,0), (0,0,1), (0,0,0) \rangle$

$$\det J^e = \begin{vmatrix} x_{il} & x_{jl} & x_{kl} \\ y_{il} & y_{jl} & y_{kl} \\ z_{il} & z_{jl} & z_{kl} \end{vmatrix} \tag{4}$$

$$x_{pq} = x_p - x_q, \; y_{pq} = y_p - y_q, \; z_{pq} = z_p - z_q, \; p, q = i, j, k, l \tag{5}$$

$$f(\xi, \eta, \zeta) = x^\alpha(\xi, \eta, \zeta) y^\beta(\xi, \eta, \zeta) z^\gamma(\xi, \eta, \zeta) \tag{6}$$

$$\begin{aligned} x(\xi, \eta, \zeta) &= x_l + x_{il}\xi + x_{jl}\eta + x_{kl}\zeta \\ y(\xi, \eta, \zeta) &= y_l + y_{il}\xi + y_{jl}\eta + y_{kl}\zeta \\ z(\xi, \eta, \zeta) &= z_l + z_{il}\xi + z_{jl}\eta + z_{kl}\zeta \end{aligned} \tag{7}$$

3.1.2. Proof. The detailed proof may be referred to in the authors' recent paper.[8]

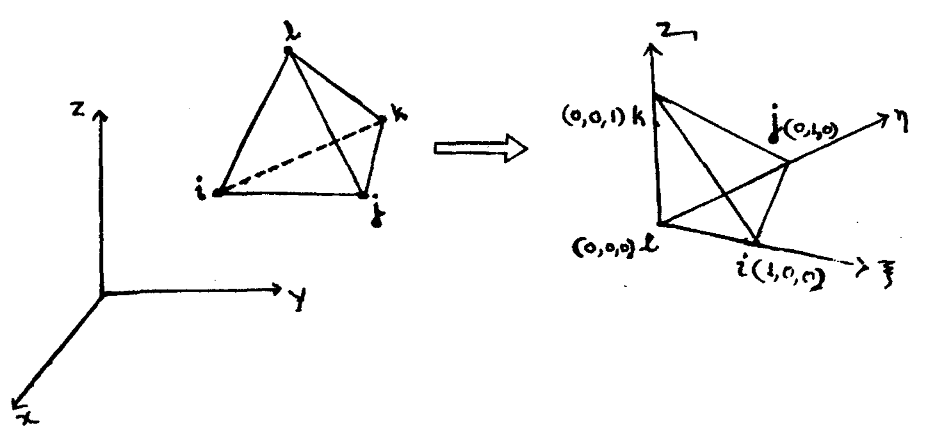

Figure 1. Three-dimensional mapping of an arbitrary tetrahedron in xyz-space into a unit orthogonal tetrahedron in $\xi\eta\zeta$ space

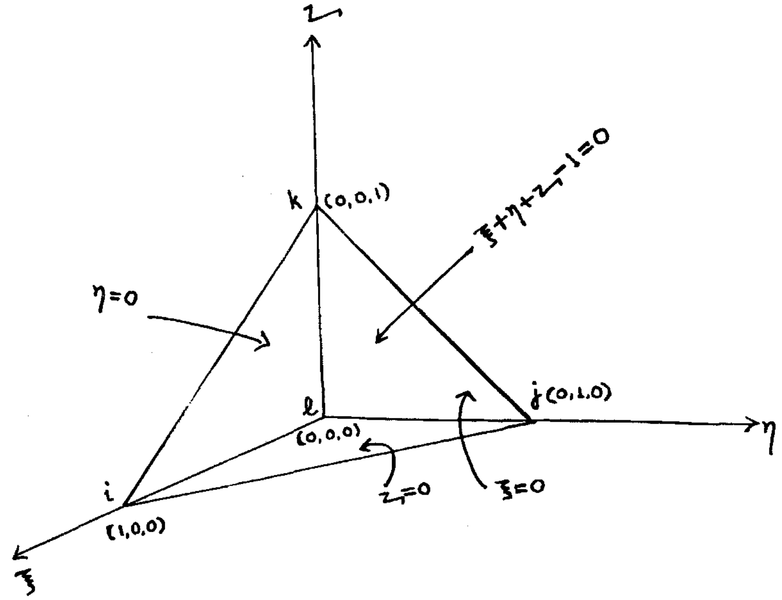

Figure 2. Unit orthogonal tetrahedron in $\xi\eta\zeta$ space

3.2. *Determination of $G^e(r,s,t)$*

In our recent research work,[8] it is shown that

$$\left(\frac{\partial^n f(\xi,\eta,\zeta)}{\partial\xi^r\,\partial\eta^s\,\partial\zeta^t}\right)_{0,0,0} = \sum_{r_1=0}^{r}\sum_{r_2=0}^{r-r_1}\frac{\lfloor r}{\lfloor r_1\,\lfloor r_2\,\lfloor r_3}\sum_{s_1=0}^{s}\sum_{s_2=0}^{s-s_1}\frac{\lfloor s}{\lfloor s_1\,\lfloor s_2\,\lfloor s_3}\sum_{t_1=0}^{t}\sum_{t_2=0}^{t-t_1}\frac{\lfloor t}{\lfloor t_1\,\lfloor t_2\,\lfloor t_3}$$
$$\cdot\lambda^e(r_1,s_1,t_1)\mu^e(r_2,s_2,t_2)\delta^e(r_3,s_3,t_3) \tag{8}$$

where

$$\begin{aligned} t&=n-r-s\geqslant 0\\ r_3&=r-r_1-r_2\geqslant 0\\ s_3&=s-s_1-s_2\geqslant 0\\ t_3&=t-t_1-t_2\geqslant 0\\ r_3+s_3+t_3&=n-r_1-s_1-t_1-r_2-s_2-t_2 \end{aligned} \tag{9}$$

$$\lambda^e(r_1,s_1,t_1)=\left(\frac{\partial^{r_1+s_1+t_1}X(\xi,\eta,\zeta)}{\partial\xi^{r_1}\,\partial\eta^{s_1}\,\partial\zeta^{t_1}}\right)_{0,0,0}=\frac{\lfloor\alpha}{\lfloor\alpha-(r_1+s_1+t_1)}\,x_l^{\alpha-(r_1+s_1+t_1)}\cdot x_{il}^{r_1}x_{jl}^{s_1}x_{kl}^{t_1} \tag{10}$$

$$\mu^e(r_2,s_2,t_2)=\left(\frac{\partial^{r_2+s_2+t_2}Y(\xi,\eta,\zeta)}{\partial\xi^{r_2}\,\partial\eta^{s_2}\,\partial\zeta^{t_2}}\right)_{0,0,0}=\frac{\lfloor\beta}{\lfloor\beta-(r_2+s_2+t_2)}\,x_l^{\beta-(r_2+s_2+t_2)}\cdot x_{il}^{r_2}x_{jl}^{s_2}x_{kl}^{t_2} \tag{11}$$

$$\delta^e(r_3,s_3,t_3)=\left(\frac{\partial^{r_3+s_3+t_3}Z(\xi,\eta,\zeta)}{\partial\xi^{r_3}\,\partial\eta^{s_3}\,\partial\zeta^{t_3}}\right)_{0,0,0}=\frac{\lfloor\gamma}{\lfloor\gamma-(r_3+s_3+t_3)}\,x_l^{\gamma-(r_3+s_3+t_3)}\cdot x_{il}^{r_3}x_{jl}^{s_3}x_{kl}^{t_3} \tag{12}$$

where $X(\xi,\eta,\zeta)=x^\alpha(\xi,\eta,\zeta)$, $Y(\xi,\eta,\zeta)=y^\beta(\xi,\eta,\zeta)$ and $Z(\xi,\eta,\zeta)=z^\gamma(\xi,\eta,\zeta)$.

Now it is clear that equation (8) can be explicitly determined in terms of x_l, y_l, z_l, x_{il}, y_{il}, z_{il}, $x_{jl}, y_{jl}, z_{jl}, x_{kl}, y_{kl}, z_{kl}$.

4. HEXAHEDRON AS AN ASSEMBLAGE OF TETRAHEDRA

The division of a space volume into individual tetrahedra sometimes presents difficulties of visualisation and could easily lead to errors in nodal numbering, etc. A more convenient subdivision of space is into eight cornered brick elements. Such elements could be assembled automatically from several tetrahedra and the process of creating these tetrahedra left to a simple logical program. It will be readily appreciated from the 'exploded' view that an element of this shape (hexahedron) could be built up in two, and only two, distinct ways from five tetrahedral shape elements. This has been clearly illustrated in the book by Zienkiewicz.[7] Both possible divisions into tetrahedral elements are illustrated in Figures 3 and 4 which could be conveniently used in computer aided design and stress analysis, etc.

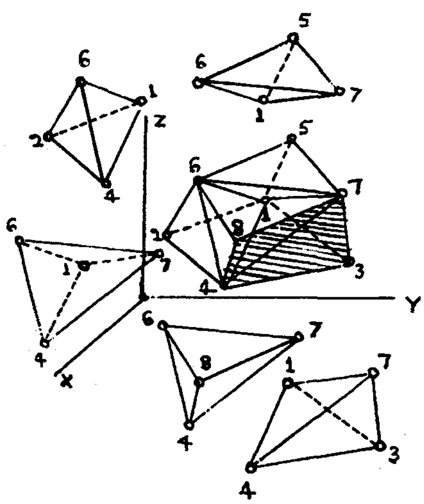

Figure 3. A systematic way of splitting an eight cornered hexahedron shaped brick into five tetrahedra

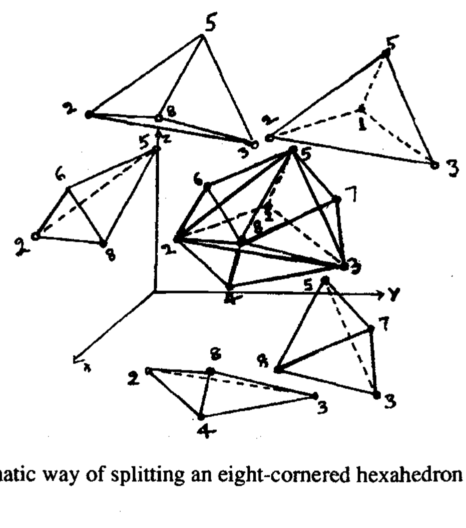

Figure 4. An alternative systematic way of splitting an eight-cornered hexahedron shaped brick into five tetrahedra

4.1. Volume integration over an arbitrary hexahedron and a linear polyhedron

4.1.1. Theorem 2. A structure product $III_V^{\alpha\beta\gamma}$ over linear three polyhedron V equals the sum of integrals over all tetrahedra formed by disjoint decomposition of the polyhedron.

4.1.2. Proof. This follows from the regularity of the integration domain and the continuity of the integrating function.

Using Theorem 2, we can write

$$III_V^{\alpha,\beta,\gamma} = \iiint_V x^\alpha y^\beta z^\gamma \, dx \, dy \, dz = \sum_{e=1}^{E} III_{T_e}^{\alpha,\beta,\gamma} \tag{13}$$

where E is the number of tetrahedra obtained by decomposition of V.

Letting H denote an arbitrary hexahedron, we can also write (referring to Figures 3 and 4)

$$III_H^{\alpha,\beta,\gamma} = \sum_{e=1}^{5} III_{T_e}^{\alpha,\beta,\gamma} \tag{14}$$

4.2. Hexahedral element and its subdivision

A hexahedral (eight cornered brick) element can be systematically divided *into five* tetrahedra. This is illustrated in Figures 3 and 4, and the nodal connectivity is shown in Tables I and II.

Table I. Nodal connectivity for hexahedron of Figure 3 as a subdivision into five tetrahedra

	Nodes			
Element no.	*i*	*j*	*k*	*l*
1	1	4	2	6
2	1	4	3	7
3	6	7	5	1
4	6	7	8	4
5	1	4	6	7

Table II. Nodal connectivity for hexahedron of Figure 4 as a subdivision into five tetrahedra

	Nodes			
Element no.	*i*	*j*	*k*	*l*
1	1	2	3	5
2	2	6	5	8
3	2	3	4	8
4	3	5	7	8
5	2	3	5	8

5. APPLICATION EXAMPLE

Let us consider the structure products

$$\mathrm{III}_R^{\alpha\beta\gamma} = \iiint_R x^\alpha y^\beta z^\gamma \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{15}$$

where R is the rectangular prism element of dimension h_x, h_y, h_z as shown in Figure 5. Evaluation of integral (15) is simple and we can write

$$\begin{aligned}\mathrm{III}_R^{\alpha,\beta,\gamma} &= \int_0^{h_x}\int_0^{h_y}\int_0^{h_z} x^\alpha y^\beta z^\gamma \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\ &= h_x^{\alpha+1} h_y^{\beta+1} h_z^{\gamma+1}/\{(\alpha+1)(\beta+1)(\gamma+1)\}.\end{aligned} \tag{16}$$

We shall assume $h_x = 2,\ h_y = 3,\ h_z = 4,\ \alpha = 2,\ \beta = 1,\ \gamma = 0$ in equation (16), so that we obtain

$$\mathrm{III}_R^{2,1,0} = 48 \tag{17}$$

We shall verify the result of equation (17) with reference to equation (14), so that we may now write

$$\mathrm{III}_R^{2,1,0} = \sum_{e=1}^{5} \mathrm{III}_{\mathrm{T}_e}^{2,1,0}. \tag{18}$$

We now visualise the rectangular prism element of Figure 5 as an assemblage of five tetrahedra of Figures 3, 4 and the nodal connectivity as shown in Tables I and II.
From Theorem I of Section 3.1 and the determination of $G^e\ (r, s, t)$ as explained in section 3.2, we can write

$$\mathrm{III}_{\mathrm{T}_e}{}^{2,1,0} = |\det j^e|\,\{G_e\}^{\mathrm{T}}\{\mathrm{III}_{\bar{\mathrm{T}}}\} \tag{19}$$

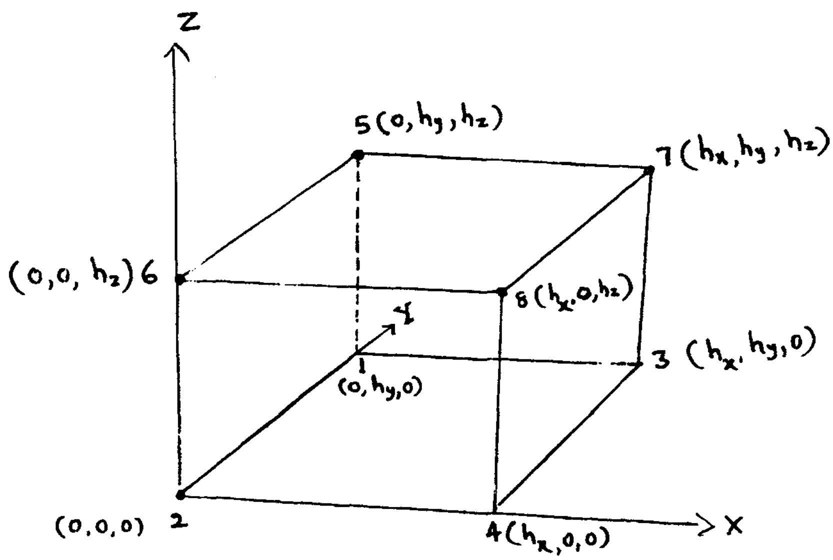

Figure 5. Rectangular prism with dimension h_x, h_y, h_z

where

$$\det J^e = \begin{vmatrix} x_{il} & x_{jl} & x_{kl} \\ y_{il} & y_{il} & y_{kl} \\ z_{il} & z_{jl} & z_{kl} \end{vmatrix} \tag{20}$$

$$\begin{aligned}\{G_e\}^{\mathrm{T}} = (&G_e(0,0,0),\ G_e(1,0,0),\ G_e(0,1,0),\ G_e(0,0,1),\\ &G_e(1,1,0),\ G_e(1,0,1),\ G_e(0,1,1),\ G_e(2,0,0),\\ &G_e(0,2,0),\ G_e(0,0,2),\ G_e(1,1,1),\ G_e(2,1,0),\\ &G_e(0,2,1),\ G_e(1,0,2),\ G_e(2,0,1),\ G_e(0,1,2),\\ &G_e(1,2,0),\ G_e(3,0,0),\ G_e(0,3,0),\ G_e(0,0,3))\end{aligned} \tag{21}$$

$$\{\mathrm{III}_{\dagger}\}^{\mathrm{T}} = (\tfrac{1}{6}, \tfrac{1}{24}, \tfrac{1}{24}, \tfrac{1}{24}, \tfrac{1}{120}, \tfrac{1}{120}, \tfrac{1}{120}, \tfrac{1}{60}, \tfrac{1}{60}, \tfrac{1}{60}, \tfrac{1}{720}, \tfrac{1}{360}, \tfrac{1}{360}, \tfrac{1}{360}, \tfrac{1}{360}, \tfrac{1}{360}, \tfrac{1}{360}, \tfrac{1}{120}, \tfrac{1}{120}, \tfrac{1}{120}) \tag{22}$$

$$\begin{aligned}
G_e(0,0,0) &= x_l^2 y_l\\
G_e(1,0,0) &= \lambda^e(1,0,0)\mu^e(0,0,0) + \lambda^e(0,0,0)\mu^e(1,0,0)\\
G_e(0,1,0) &= \lambda^e(0,1,0)\mu^e(0,0,0) + \lambda^e(0,0,0)\mu^e(0,1,0)\\
G_e(0,0,1) &= \lambda^e(0,0,1)\mu^e(0,0,0) + \lambda^e(0,0,0)\mu^e(0,0,1)\\
G_e(1,1,0) &= \lambda^e(1,1,0)\mu^e(0,0,0) + \lambda^e(1,0,0)\mu^e(0,1,0) + \lambda^e(0,1,0)\mu^e(1,0,0)\\
G_e(1,0,1) &= \lambda^e(1,0,1)\mu^e(0,0,0) + \lambda^e(1,0,0)\mu^e(0,0,1) + \lambda^e(0,0,1)\mu^e(1,0,0)\\
G_e(0,1,1) &= \lambda^e(0,1,1)\mu^e(0,0,0) + \lambda^e(0,1,0)\mu^e(0,0,1) + \lambda^e(0,0,1)\mu^e(0,1,0)\\
G_e(2,0,0) &= \lambda^e(1,0,0)\mu^e(1,0,0) + \tfrac{1}{2}\lambda^e(2,0,0)\mu^e(0,0,0)\\
G_e(0,2,0) &= \lambda^e(0,1,0)\mu^e(0,1,0) + \tfrac{1}{2}\lambda^e(0,2,0)\mu^e(0,0,0)\\
G_e(0,0,2) &= \lambda^e(0,0,1)\mu^e(0,0,1) + \tfrac{1}{2}\lambda^e(0,0,2)\mu^e(0,0,0)\\
G_e(1,1,1) &= \lambda^e(1,1,0)\mu^e(0,0,1) + \lambda^e(1,0,1)\mu^e(0,1,0) + \lambda^e(0,1,1)\mu^3(1,0,0)\\
G_e(2,1,0) &= \tfrac{1}{2}\lambda^e(2,0,0)\mu^e(0,1,0) + \lambda^e(1,1,0)\mu^e(1,0,0)\\
G_e(0,2,1) &= \tfrac{1}{2}\lambda^e(0,2,0)\mu^e(0,0,1) + \lambda^e(0,1,1)\mu^e(0,1,0)\\
G_e(1,0,2) &= \lambda^e(1,0,1)\mu^e(0,0,1) + \tfrac{1}{2}\lambda^e(0,0,2)\mu^e(1,0,0)\\
G_e(2,0,1) &= \tfrac{1}{2}\lambda^e(2,0,0)\mu^e(0,0,1) + \lambda^e(1,0,1)\mu^e(1,0,0)\\
G_e(0,1,2) &= \lambda^e(0,1,1)\mu^e(0,0,1) + \tfrac{1}{2}\lambda^e(0,0,2)\mu^e(0,1,0)\\
G_e(1,2,0) &= \lambda^e(1,1,0)\mu^e(0,1,0) + \tfrac{1}{2}\lambda^e(0,2,0)\mu^e(1,0,0)\\
G_e(3,0,0) &= \tfrac{1}{2}\lambda^e(2,0,0)\mu^e(1,0,0)\\
G_e(0,3,0) &= \tfrac{1}{2}\lambda^e(0,2,0)\mu^e(0,1,0)\\
G_e(0,0,3) &= \tfrac{1}{2}\lambda^e(0,0,2)\mu^e(0,0,1)
\end{aligned} \tag{23}$$

$$\begin{aligned}
\lambda^e(0,0,0) &= x_l^2\\
\lambda(1,0,0) &= 2x_l x_{il}\\
\lambda^e(0,1,0) &= 2x_l x_{jl}\\
\lambda^e(0,0,1) &= 2x_l x_{kl}\\
\lambda^e(1,1,0) &= 2x_{il} x_{jl}\\
\lambda^e(1,0,1) &= 2x_{il} x_{kl}\\
\lambda^e(0,1,1) &= 2x_{jl} x_{kl}
\end{aligned}$$

$$\lambda^e(0,2,0) = 2x_{jl}^2$$
$$\lambda^e(0,0,2) = 2x_{kl}^2$$
$$\mu^e(0,0,0) = y_l$$
$$\mu^e(1,0,0) = y_{il}$$
$$\mu^e(0,1,0) = y_{jl}$$
$$\mu^e(0,0,1) = y_{kl} \qquad (24)$$

Table III. Numerical computation of integrals $\mathrm{III}_{T_e}^{2,1,0} = \iiint_{T_e} x^2 y \, dx \, dy \, dz$ with reference to Figure 3 and nodal connectivity of Table I

e (element no.)	$\{G^e\}, \lvert\det J^e\rvert$	$\mathrm{III}_{T_e}{}^{2,10} = \lvert\det J^3\rvert \{G^e\}^T\{\mathrm{III}_{\bar{T}}\}$
1	(0,0,0, 0,0,0, 0,0, 0,0,0, 0,0,0, 0,0, 12,0,0,0), 24	$\frac{4}{5}$
2	(0,0,0, 0,0,0,0,0, 12,0,0,0,0,0,0,0, −12,0,0,0), 24	$20\frac{4}{5}$
3	(0,0,0,0,0,0,0,0, 12,0,0,0,0,0,0,0 −12,0,0,0), 24	4
4	(0,0, 12,0, −24,0,0,0, 0,0,0, 12,0,0,0,0, 0,0,0,0), 24	8
5	(12,−24,−12, −36, 24, 48, 24, 12,0,36, −24 −12, 0,−24,−12, −12,0, 0, 0,−12), 48	$14\frac{2}{5}$
	$\sum_{e=1}^{5} \mathrm{III}_{T_e}^{2,1,0} = \frac{4}{5} + 20\frac{4}{5} + 4 + 8 + 14\frac{2}{5} = 48$	

Table IV. Numerical computation of integrals $\mathrm{III}_{T_e}^{2,1,0} = \iiint_{T_e} x^2 y \, dx \, dy \, dz$ with reference to Figure 4 and nodal connectivity of Table II

e (element no.)	$\{G^e\}, \lvert\det J^e\rvert$	$\mathrm{III}_{T_e}{}^{2,10} = \lvert\det J^3\rvert \{G^e\}^T\{\mathrm{III}_{\bar{T}}\}$
1	(0,0,0, 0,0,0, 0,0, 0,12,0,0,0,0,0, −12, 0,0,0,0), 24	4
2	(0,0,0, 12, 0, −24, −24, 0,0, −24,24,0, 12, 24, 12, 24, 0, 0,0, 12), 24	$\frac{4}{5}$
3	(0,0, 12, 0, −24, 0,0,0, 0,0,0, 12, 0,0,0,0, 0,0,0,0,), 24	8
4	(0, 12, 12, 12, −24, 0, −24, 0, −24, 0,0,0, 12, 0,0,0, 12, 0, 12, 0), 24	$20\frac{4}{5}$
5	(0,0, 12, 12, −24, −24, −24, 0,0, −24, 24, 12, 0,24,12,12,0,0,0,12), 48	$14\frac{2}{5}$
	$\sum_{e=1}^{5} \mathrm{III}_{T_e}^{2,1,0} = 4 + \frac{4}{5} + 8 + 20\frac{4}{5} + 14\frac{2}{5} = 48$	

Using equations (19)–(24) and the geometry and nodal connectivity of Tables I and II, we can compute the integrals $III_{T_e}^{2,1,0}$ $(e = 1, 2, 3, 4, 5)$ for the Hexahedron as a subdivision into five tetrahedra of Figures 3 and 4. These numerical computations re shown in Tables III and IV.

It is clear from these Tables that

$$\sum_{e=1}^{5} III_{T_e}^{2,1,0} = 48. \tag{25}$$

Thus equations (17, and 25) clearly verify the result of equation (18).

6. CONCLUSIONS

The theorems we have presented in this paper are interesting for various reasons. Our formulas are more compact than those of previous researchers and require less computer arithmetic, as is evident by comparing the summations required in earlier studies and the present one.

Explicit formulas are first derived to compute integrals of monomials over a linear arbitrary tetrahedron in three-dimensional Euclidean space in which we have used a direct mapping to transform the integral over a linear arbitrary tetrahedron in 3D space into an integral over a unit orthogonal tetrahedron in new 3D space.

Volume integration over a linear polyhedron can be obtained by decomposing it into a set of solid tetrahedrons, but the division of a linear polyhedral solid in 3D space into tetrahedra sometimes presents difficulties of visualisation. We have suggested a means of overcoming this difficulty in the construction of a hexahedron which can be built in two and only two distinct ways from five tetrahedral shaped elements. Symbolic integration formulas thus derived are verified for an illustrative numerical example for a rectangular prism. Theorem 2 is proved in this paper and the subdivision of a hexahedron into five tetrahedral elements is confirmed by Zienkiewicz;[7] the numerical scheme proposed here is equally valid for an arbitrary hexahedron.

REFERENCES

1. Y. T. Lee and A. A. G. Requicha, 'Algorithms for computing the volume and other integral properties of solids I: Known methods and open issues', *Commun. ACM*, **25**, (9), 635–641 (1982).
2. H. G. Timmer and J. M. Stern, 'Computation of Global geometric properties of solid objects', *Comput.-Aided Des.*, *12*, (6), 301–304 (1980).
3. S. Lien and J. T. Kajiy, 'A symbolic method for calculating the integral properties of arbitrary nonconvex polyhedra', *IEEE Comput. Graph. Applic.*, *4*, (9), 35–41 ().
4. C. Cattani and A. Paoluzzi, 'Boundary integration over linear polyhedra', *Comput.—Aided Des.*, **22**, (2), 130–135 (1990).
5. C. Cattani and A. Paoluzzi, 'Symbolic analysis of linear polyhedra', *Eng. Comput.*, **6**, 17–29 (1990).
6. F. Bernardini, 'Integration of polynomials: n-dimensional polyhedra', *Comput.—Aided Des.*, **23**, (1), 51–58 (1991).
7. O. C. Zienkiewicz, *The Finite Element Method*, McGraw-Hill, England, 1977, p. 97.
8. H. T. Rathod and H. S. Govinda Rao 'Integration of polynomials over an arbitrary tetrahedron in Euclidean three dimensional space', *Comput. Struct.*, **59**, 55–65 (1996).