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Wijsman $\lambda-$statistical convergence of interval numbers

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abstract: In this paper we introduce and study the concepts of Wijsman $\lambda-$statistical convergence and Wijsman strong $\lambda-$statistical convergence of sequences for interval numbers and prove some inclusion relations.

Key Words: $\lambda-$sequence, interval numbers, statistical convergence.

Contents

1. Introduction

The notion of statistical convergence was introduced by Fast [10] and Schoenberg [19] independently. A lot of developments have been made in this areas after the works of Šalát [18] and Fridy [11]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. In the recent years, generalization of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-Čech compactification of the natural numbers.

A real or complex number sequence $x=(x_k)$ is said to be *statistically convergent* to L if for every $\varepsilon>0$

$$\lim_n \frac{1}{n}\left|\{k\leq n:|x_k-L|\geq\varepsilon\}\right|=0.$$

In this case, we write $S-\lim x=L$ or $x_k\to L(S)$ and S denotes the set of all statistically convergent sequences.
The generalized de la Vallée-Poussin mean is defined by

$$t_n(x)=\frac{1}{\lambda_n}\sum_{k\in I_n}x_k$$

where $I_n=[n-\lambda_n+1,n]$. A sequence $x=(x_k)$ is said to be $(V,\lambda)-$summable to number L [13] if $t_n(x)\to L$ as $n\to\infty$. If $\lambda_n=n$, then $(V,\lambda)-$summability

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reduces to (C,1)-summability.

Mursaleen [14] defined $\lambda-$statistically convergent sequence as follows: A sequence $x = (x_k)$ is said to be $\lambda-$statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\{k \in I_n : \ |x_k - L| \geq \varepsilon\}\right| = 0.$$

Let S_λ denotes the set of all $\lambda-$statistically convergent sequences. If $\lambda_n = n$, then S_λ is the same as S.

Let (X,ρ) be a metric space. For any point $x \in X$ and any non-empty subset $A \subset X$, the distance from x to A is defined by

$$d(x, A) = \inf_{y\in A} \rho\,(x, y)\,.$$

For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman convergent to A if $\lim_k d(x, A_k) = d(x, A)$ for each $x \in X$. In this case we write $W - \lim A_k = A$. The concepts of Wijsman statistical convergence and boundedless for the sequence (A_k) were given by Nuray and Rhoades [17] as follows: Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman statistical convergent to A if the sequence $(d(x, A_k))$ is statistically convergent to $d(x, A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$

$$\lim_n \frac{1}{n} \left|\{k \leq n : \ |d(x, A_k) - d(x, A)| \geq \varepsilon\}\right| = 0.$$

In this case, we write $st - \lim_k A_k = A$ or $A_k \to A\,(WS)$. The sequence (A_k) is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$. The set of all bounded sequences of sets denoted by L_∞.

2. Preliminaries

A set consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analytical properties. We denote the set of all real valued closed intervals by $\mathbb{IR}$. Any elements of $\mathbb{IR}$ is a closed interval and denoted by $\overline{A}$. That is $\overline{A} = \{x \in \mathbb{R} : a \leq x \leq b\}$. An interval number $\overline{A}$ is a closed subset of real numbers [19]. Let x_l and x_r be first and last points of $\overline{A}$ interval number, respectively. For $\overline{A}, \overline{B} \in \mathbb{IR}$, we have $\overline{A} = \overline{B} \Leftrightarrow x_{1_l} = x_{2_l}$, $x_{1_r} = x_{2_r}$. $\overline{A} + \overline{B} = \{x \in \mathbb{R} : x_{1_l} + x_{2_l} \leq x \leq x_{1_r} + x_{2_r}\}$, and if $\alpha \geq 0$, then $\alpha\overline{A} = \{x \in \mathbb{R} : \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\}$ and if $\alpha < 0$, then $\alpha\overline{A} = \{x \in \mathbb{R} : \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\}$,

$$\begin{aligned}\overline{A} \cdot \overline{B} = \{x \in \mathbb{R} : \min\{x_{1_l} \cdot x_{2_l}, x_{1_l} \cdot x_{2_r}, x_{1_r} \cdot x_{2_l}, x_{1_r} \cdot x_{2_r}\} \leq x \\ \leq \max\{x_{1_l} \cdot x_{2_l}, x_{1_l} \cdot x_{2_r}, x_{1_r} \cdot x_{2_l}, x_{1_r} \cdot x_{2_r}\}\}\,.\end{aligned}$$

The set of all interval numbers $\mathbb{IR}$ is a complete metric space defined by

$$d\left(\overline{A},\overline{B}\right)=\max\left\{\left|x_{1_l}-x_{2_l}\right|,\left|x_{1_r}-x_{2_r}\right|\right\}\quad[7].$$

In the special case $\overline{A}=[a,a]$ and $\overline{B}=[b,b]$, we obtain usual metric of $\mathbb{R}$.

Let us define transformation $f:\mathbb{N}\to\mathbb{IR}$ by $k\to f(k)=\overline{A}$, $\overline{A}=\left(\overline{A}_k\right)$. Then $\overline{A}=\left(\overline{A}_k\right)$ is called sequence of interval numbers. The $\overline{A}_k$ is called k^{th} term of sequence $\overline{A}=\left(\overline{A}_k\right)$. w^i denotes the set of all interval numbers with real terms and the algebraic properties of w^i can be found in [2].

Recently, Şengönül and Eryılmaz [20], Esi [3,4,5] Esi and Braha [6], Esi and Yasemin [7] and Esi and Hazarika [8] introduced and studied some properties of interval numbers. After then, Esi and Hazarika [9], Hazarika and Esi [12] and Hazarika et.al [13] studied different properties of Wijsman convergent sequences. Chiao [1] introduced sequence of interval numbers and defined usual convergence of sequences of interval numbers.

Now we give the definition of convergence of interval numbers:

Definition 2.1. *[15] A sequence $\overline{A}=\left(\overline{A}_k\right)$ of interval numbers is said to be convergent to the interval number $\overline{A}_o$ if for each $\varepsilon>0$ there exists a positive integer k_o such that $d\left(\overline{A}_k,\overline{A}_o\right)<\varepsilon$ for all $k\geq k_o$ and we denote it by $\lim_k\overline{A}_k=\overline{A}_o$.*

Thus, $\lim_k\overline{A}_k=\overline{A}_o\Leftrightarrow\lim_k x_{k_l}=x_{o_l}$ and $\lim_k x_{k_r}=x_{o_r}$.

3. Main results

In this paper, we introduce and study the concepts of Wijsman strongly $\lambda-$convergence and Wijsman $\lambda-$statistically convergence for interval numbers.

Definition 3.1. *Let $\lambda=(\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1}\leq\lambda_n+1$, $\lambda_1=1$, $\lambda_n\to\infty$ as $n\to\infty$ and $I_n=[n-\lambda_n+1,n]$. The sequence $\overline{A}=\left(\overline{A}_k\right)$ of interval numbers is said to be Wijsman strongly $\lambda-$summable if there is an interval number $\overline{A}_o$ such that*

$$\lim_n\frac{1}{\lambda_n}\sum_{k\in I_n}\left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|=0.$$

In which case we say that the sequence $\overline{A}=\left(\overline{A}_k\right)$ of interval numbers is said to be Wijsman strongly $\lambda-$summable to interval number $\overline{A}_o$. If $\lambda_n=n$, then strongly $\lambda-$summable reduces to Wijsman strongly Cesaro summable defined as follows:

$$\lim_n\frac{1}{n}\sum_{k=1}^{n}\left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|=0.$$

Definition 3.2. *A sequence $\overline{A}=\left(\overline{A}_k\right)$ of interval numbers is said to be Wijsman $\lambda-$statistically convergent to interval number $\overline{A}_o$ if for every $\varepsilon>0$*

$$\lim_n\frac{1}{\lambda_n}\left|\left\{k\in I_n:\ \left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|\geq\varepsilon\right\}\right|=0.$$

In this case we write $s_\lambda - \lim \overline{A}_k = \overline{A}_o$. If $\lambda_n = n$, then Wijsman $\lambda-$statistically convergence reduces to Wijsman statistically convergence as follows:

$$\lim_n \frac{1}{n} \left|\left\{k \leq n : \ \left|d\left(\overline{A}_k, x\right) - d\left(\overline{A}_o, x\right)\right| \geq \varepsilon\right\}\right| = 0.$$

In this case we write $s - \lim \overline{A}_k = \overline{A}_o$.

Theorem 3.3. *Let $\overline{A} = \left(\overline{A}_k\right)$ and $\overline{B} = \left(\overline{B}_k\right)$ be sequences of interval numbers.*

(i) *If $s_\lambda - \lim \overline{A}_k = \overline{A}_o$ and $\alpha \in \mathbb{R}$, then $s_\lambda - \lim \alpha \overline{A}_k = \alpha \overline{A}_o$.*

(ii) *If $s_\lambda - \lim \overline{A}_k = \overline{A}_o$ and $s_\lambda - \lim \overline{B}_k = \overline{B}_o$, then $s_\lambda - \lim \left(\overline{A}_k + \overline{B}_k\right) = \overline{A}_o + \overline{B}_o$.*

Proof: **(i)** Let $\alpha \in \mathbb{R}$. We have $d\left(\alpha\overline{A}_k, \alpha\overline{A}_o\right) = |\alpha|\, d\left(\overline{A}_k, \overline{A}_o\right)$. For a given $\varepsilon > 0$

$$\frac{1}{\lambda_n} \left|\left\{k \in I_n : \ \left|d\left(\alpha\overline{A}_k, x\right) - d\left(\alpha\overline{A}_o, x\right)\right| \geq \varepsilon\right\}\right| \leq \frac{1}{\lambda_n} \left|\left\{k \in I_n : d\left(\overline{A}_k, \overline{A}_o\right) \geq \frac{\varepsilon}{|\alpha|}\right\}\right|.$$

Hence $s_\lambda - \lim \alpha\overline{A}_k = \alpha\overline{A}_o$.

(ii) Suppose that $s_\lambda - \lim \overline{A}_k = \overline{A}_o$ and $s_\lambda - \lim \overline{B}_k = \overline{B}_o$. We have

$$d\left(\overline{A}_k + \overline{B}_k, \overline{A}_o + \overline{B}_o\right) \leq d\left(\overline{A}_k, \overline{A}_o\right) + d\left(\overline{B}_k, \overline{B}_o\right).$$

Therefore given $\varepsilon > 0$, we have

$$\begin{aligned} &\frac{1}{\lambda_n} \left|\left\{k \in I_n : d\left(\overline{A}_k + \overline{B}_k, \overline{A}_o + \overline{B}_o\right) \geq \varepsilon\right\}\right| \\ \leq &\frac{1}{\lambda_n} \left|\left\{k \in I_n : d\left(\overline{A}_k, \overline{A}_o\right) + d\left(\overline{B}_k, \overline{B}_o\right) \geq \varepsilon\right\}\right| \\ \leq &\frac{1}{\lambda_n} \left|\left\{k \in I_n : \ d\left(\overline{A}_k, \overline{A}_o\right) \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{\lambda_n} \left|\left\{k \in I_n : \ d\left(\overline{B}_k, \overline{B}_o\right) \geq \frac{\varepsilon}{2}\right\}\right|. \end{aligned}$$

Thus, $s_\lambda - \lim \left(\overline{A}_k + \overline{B}_k\right) = \overline{A}_o + \overline{B}_o$. □

Theorem 3.4. *If an interval sequence $\overline{A} = \left(\overline{A}_k\right)$ is Wijsman strongly $\lambda-$summable to interval number $\overline{A}_o$, then it is Wijsman strongly $\lambda-$ Cauchy summable.*

Proof: Suppose that $\overline{A} = \left(\overline{A}_k\right)$ is Wijsman strongly $\lambda-$summable to interval number $\overline{A}_o$. Then it follows that

$$\frac{1}{\lambda_n} \sum_{k \in I_n} |d(\overline{A_k}, x) - d(\overline{A_0}, x)| < \frac{\epsilon}{2},$$

and if $N \in \mathbb{N}$ is chosen such that

$$\frac{1}{\lambda_n} \sum_{k \in I_n} |d(\overline{A_N}, x) - d(\overline{A_0}, x)| < \frac{\epsilon}{2}.$$

Then we get that:

$$\frac{1}{\lambda_n}\sum_{k\in I_n}|d(\overline{A_k},x)-d(\overline{A_N},x)|<\epsilon,$$

for all $n\geq n_0$. □

Theorem 3.5. *If an interval sequence $\overline{A}=\left(\overline{A}_k\right)$ is Wijsman $\lambda-$statistically convergent to interval number $\overline{A}_o$, and $d(\overline{A_k},x)=d(\overline{B_k},x)$ starting from some $k=k_0$, then it follows that $\overline{B}=\left(\overline{B}_k\right)$ is Wijsman strongly $\lambda-$statistically convergent to interval number $\overline{A}_o$.*

Proof: Let us consider that $d(\overline{A_k},x)=d(\overline{B_k},x)$ starting from some $k=k_0$. And $s_\lambda-\lim_k\overline{B}_k=\overline{A}_0$.

Then for each $\epsilon>0$ and for every n we have:

$$\left\{k\in I_n:|d(\overline{A}_k,x)-d(\overline{A}_0,x)|\geq\epsilon\right\}\subset\left\{k\in I_n:d(\overline{A}_k,x)\neq d(\overline{B}_k,x)\right\}$$

$$\cup\left\{k\in I_n:|d(\overline{B}_k,x)-d(\overline{A}_0,x)|\geq\epsilon\right\}.$$

From fact that $s_\lambda-\lim\overline{B}_k=\overline{A}_0$, it follows that set $\left\{k\in I_n:|d(\overline{B}_k,x)-d(\overline{A}_0,x)|\leq\epsilon\right\}$ has finite numbers which are not depended from n, hence

$$\frac{\left|\left\{k\in I_n:|d(\overline{B}_k,x)-d(\overline{A}_0,x)|\leq\epsilon\right\}\right|}{\lambda_n}\to 0,n\to\infty.$$

On the other hand, from $d(\overline{A_k},x)=d(\overline{B_k},x)$ for almost all k, we get:

$$\frac{\left|\left\{k\in I_n:d(\overline{A_k},x)\neq d(\overline{B_k},x)\right\}\right|}{\lambda_n}\to 0,n\to\infty.$$

From last two relations follows that:

$$\frac{\left|\left\{k\in I_n:|d(\overline{A}_k,x)-d(\overline{A}_0,x)|\geq\epsilon\right\}\right|}{\lambda_n}\to 0,n\to\infty.$$

□

Theorem 3.6. *Under conditions that $\frac{n}{\lambda_n}\to 0$ as $n\to\infty$, then space of all Wijsman strongly $\lambda-$ convergent sequences is a normal space.*

Proof: Let us suppose that are given two sequence of intervals: (B_n) and (A_n), such that $|B_n|\leq|A_n|$ and (A_n) is Wijsman strongly $\lambda-$ convergent sequence. Then it follows that:

$$\frac{1}{\lambda_n}\sum_{k=1}^{n}|d(A_k,x)-d(A_0,x)|\to 0,n\to\infty.$$

On the other hand:

$$\frac{1}{\lambda_n}\sum_{k=1}^{n}|d(B_k,x)-d(B_0,x)| \le \frac{1}{\lambda_n}\sum_{k=1}^{n}|d(B_k,B_0)| \le \frac{1}{\lambda_n}\sum_{k=1}^{n}|d(B_k,0)+d(B_0,0)|$$

$$= \frac{1}{\lambda_n}\sum_{k=1}^{n}||B_k|+|B_0|| \le \frac{1}{\lambda_n}\sum_{k=1}^{n}||A_k|+|A_0|| = \frac{1}{\lambda_n}\sum_{k=1}^{n}|d(A_k,0)+d(A_0,0)| \le$$

$$\frac{1}{\lambda_n}\sum_{k=1}^{n}|d(A_k,x)-d(A_0,x)| + \frac{2n(|x|+d(A,x))}{\lambda_n} \to 0, n \to \infty.$$

□

Theorem 3.7. *The space of all interval sequences $\overline{A}=\left(\overline{A}_k\right)$ which are Wijsman strongly $\lambda-$summable to interval number $\overline{A}_o$, is not sequence algebra.*

Proof: Let us consider the following example: Let $(\overline{A}_n)=\left[-\frac{1}{n},\frac{1}{n}\right]$ and $(\overline{B}_n)=\left[-\frac{1}{n},\frac{1}{n}\right]$. After some calculations it follows that $(\overline{A}_n),(\overline{B}_n)$ are Wijsman strongly $\lambda-$ summable sequences. But their product $(\overline{C}_n)=(\overline{A}_n)\cdot(\overline{B}_n)$, is not. Really,

$$\frac{1}{\lambda_n}\sum_{k=1}^{n}|d(C_k,0)-d(C_0,0)| = \frac{1}{\lambda_n}\sum_{k=1}^{n}||C_k|-|C_0|| = \frac{1}{\lambda_n}\sum_{k=1}^{n}||C_k|| =$$

$$\frac{1}{\lambda_n}\sum_{k=1}^{n}\frac{1}{k^2} = \frac{n(n+1)(2n+1)}{6\lambda_n} \to \infty, n \to \infty.$$

□

In the following theorems, we exhibit some connections between Wijsman strongly $\lambda-$summable and Wijsman $\lambda-$statistically convergence of sequences of interval numbers.

Theorem 3.8. *If an interval sequence $\overline{A}=\left(\overline{A}_k\right)$ is Wijsman strongly $\lambda-$summable to interval number $\overline{A}_o$, then it is Wijsman $\lambda-$statistically convergent to interval number $\overline{A}_o$. Conversely is not true.*

Proof: Let $\varepsilon>0$. Since

$$\sum_{k\in I_n}\left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right| \ge \sum_{\substack{k\in I_n \\ \left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|\ge\varepsilon}}\left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|$$

$$\ge \left|\left\{k\in I_n: \ \left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|\ge\varepsilon\right\}\right|\varepsilon$$

if $\overline{A}=\left(\overline{A}_k\right)$ is Wijsman strongly $\lambda-$ summable to $\overline{A}_o$, then it is Wijsman $\lambda-$statistically convergent to $\overline{A}_o$. To prove conversely we will show this by following example.

Example 3.9. *Let* $\overline{A}=\left(\overline{A}_k\right)$ *defined as follows:*

$$\overline{A}_k=\begin{cases}\overline{1}, & \text{if } n\text{-}\left[\sqrt{\lambda_n}\right]\leq k\leq n,\ k=1,2,3,\ldots\\ \overline{0}, & \text{otherwise}\end{cases}.$$

Then $\frac{1}{\lambda_n}\left|\left\{k\in I_n:\ \left|d\left(\overline{A}_k,x\right)-d\left(\overline{0},x\right)\right|\geq\varepsilon\right\}\right|\ \leq\ \frac{\left[\sqrt{\lambda_n}\right]}{\lambda_n}\ \rightarrow\ 0$ *as* $n\rightarrow\infty$ *i.e.,* $\left(\overline{A}_k\right)$ *is Wijsman* $\lambda-$*statistically convergent to interval number* $\overline{0}$. *But* $\frac{1}{\lambda_n}\sum_{k\in I_n}\left|d\left(\overline{A}_k,x\right)-d\left(\overline{0},x\right)\right|\rightarrow\frac{1}{2}\neq 0$, *hence* $\left(\overline{A}_k\right)$ *is not Wijsman strongly* $\lambda-$*summable to* $\overline{0}$.

□

Theorem 3.10. *If* $\overline{A}=\left(\overline{A}_k\right)\in L_\infty$ *and* $\overline{A}=\left(\overline{A}_k\right)$ *is Wijsman* $\lambda-$*statistically convergent to interval number* $\overline{A}_o$*, then it is Wijsman strongly* $\lambda-$*summable to* $\overline{A}_o$ *and hence* $\overline{A}=\left(\overline{A}_k\right)$ *is Wijsman strongly Cesaro summable to* $\overline{A}_o$*, where* $L_\infty=\left\{\overline{A}=\left(\overline{A}_k\right):\ \sup_k d\left(\overline{A}_k,\overline{A}_o\right)<\infty\right\}$.

Proof: Suppose that $\overline{A}=\left(\overline{A}_k\right)\in L_\infty$ and Wijsman $\lambda-$statistically convergent to interval number $\overline{A}_o$. Since $\overline{A}=\left(\overline{A}_k\right)\in L_\infty$, we write $d\left(\overline{A}_k,\overline{A}_o\right)\leq A$ for all $k\in\mathbb{N}$. Given $\varepsilon>0$, we have

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n}\left|d\left(\alpha\overline{A}_k,x\right)-d\left(\alpha\overline{A}_o,x\right)\right| &= \frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ \left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|\geq\varepsilon}}\left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|\\ &\quad+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)<\varepsilon}}\left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|\\ &\leq\frac{A}{\lambda_n}\left|\left\{k\in I_n:\ \left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|\geq\varepsilon\right\}\right|+\varepsilon\end{aligned}$$

which implies that $\overline{A}=\left(\overline{A}_k\right)$ is $\lambda-$ strongly $\lambda-$summable to $\overline{A}_o$. Further we have

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n}\left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|\\ &=\frac{1}{n}\sum_{k=1}^{n-\lambda_n}\left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|+\frac{1}{n}\sum_{k\in I_n}\left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|\\ &\leq\frac{1}{\lambda_n}\sum_{k=1}^{n-\lambda_n}\left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|+\frac{1}{\lambda_n}\sum_{k\in I_n}\left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|\\ &\leq\frac{2}{\lambda_n}\sum_{k\in I_n}\left|d\left(\overline{A}_k,x\right)-d\left(\overline{A}_o,x\right)\right|.\end{aligned}$$

Hence $\overline{A}=\left(\overline{A}_k\right)$ is Wijsman strongly Cesaro summable to $\overline{A}_o$. □

Theorem 3.11. *If a interval sequence $\overline{A} = \left(\overline{A}_k\right)$ is Wijsman statistically convergent to interval number $\overline{A}_o$ and $\liminf_n \frac{\lambda_n}{n} > 0$ then it is Wijsman $\lambda-$statistically convergent to $\overline{A}_o$.*

Proof: For given $\varepsilon > 0$, we have

$$\left\{k \le n: \left|d\left(\overline{A}_k, x\right) - d\left(\overline{A}_o, x\right)\right| \ge \varepsilon\right\} \supset \left\{k \in I_n: \left|d\left(\overline{A}_k, x\right) - d\left(\overline{A}_o, x\right)\right| \ge \varepsilon\right\}.$$

Therefore

$$\frac{1}{n}\left|\left\{k \le n: \left|d\left(\overline{A}_k, x\right) - d\left(\overline{A}_o, x\right)\right| \ge \varepsilon\right\}\right| \ge \frac{1}{n}\left|\left\{k \in I_n: \left|d\left(\overline{A}_k, x\right) - d\left(\overline{A}_o, x\right)\right| \ge \varepsilon\right\}\right|$$
$$\ge \frac{\lambda_n}{n}.\frac{1}{\lambda_n}\left|\left\{k \in I_n: \left|d\left(\overline{A}_k, x\right) - d\left(\overline{A}_o, x\right)\right| \ge \varepsilon\right\}\right|.$$

Taking limit as $n \to \infty$ and using $\liminf_n \frac{\lambda_n}{n} > 0$, we get that $\overline{A} = \left(\overline{A}_k\right)$ is Wijsman $\lambda-$statistically convergent to $\overline{A}_o$. □

Finally we conclude this paper by stating a definition which generalizes Definition 3.1 of this section and two theorems related to this definition.

Definition 3.12. *Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \le \lambda_n + 1$, $\lambda_1 = 1$, $\lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$ and $p \in (0, \infty)$. The sequence $\overline{A} = \left(\overline{A}_k\right)$ of interval numbers is said to be Wijsman strongly $\lambda p-$summable if there is an interval number $\overline{A}_o$ such that*

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[\left|d\left(\overline{A}_k, x\right) - d\left(\overline{A}_o, x\right)\right|\right]^p = 0.$$

In this case we say that the sequence $\overline{A} = \left(\overline{A}_k\right)$ of interval numbers is said to be Wijsman strongly $\lambda p-$summable to interval number $\overline{A}_o$. If $\lambda_n = n$, then Wijsman strongly $\lambda p-$summable reduces to Wijsman strongly $p-$Cesaro summable defined as follows:

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \left[\left|d\left(\overline{A}_k, x\right) - d\left(\overline{A}_o, x\right)\right|\right]^p = 0.$$

The following theorems is similar to that of Theorem 3.8 and Theorem 3.10, so we state the theorems without proof.

Theorem 3.13. *If an interval sequence $\overline{A} = \left(\overline{A}_k\right)$ is Wijsman strongly $\lambda p-$summable to interval number $\overline{A}_o$, then it is Wijsman $\lambda-$statistically convergent to interval number $\overline{A}_o$.*

Theorem 3.14. *If $\overline{A} = \left(\overline{A}_k\right) \in L_\infty$ and $\overline{A} = \left(\overline{A}_k\right)$ is Wijsman $\lambda-$statistically convergent to interval number $\overline{A}_o$, then it is Wijsman strongly $\lambda p-$summable to $\overline{A}_o$ and hence $\overline{A} = \left(\overline{A}_k\right)$ is Wijsman strongly $p-$Cesaro summable to $\overline{A}_o$..*

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