Matter Fields in Curved Space-Time[1]

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Abstract. We study the geometry of a two-sheeted space-time within the framework of non-commutative geometry. As a prelude to the Standard Model in curved space-time, we present a model of a left- and a right- chiral field living on the two sheeted-space time and construct the action functionals that describe their interactions.

I. Introductory Remarks

It has been said by many, many times and in many words that our current picture of space-time is unsatisfactory and inadequate for the description of elementary particle interactions at all scales. The reasons are many-fold. The most noted one is the fact that, in spite of great deal of effort, the twin pillars of the twentieth century physics, namely, the general theory of gravity (governing the dynamics of classical space-time coupled to the dynamics of matter fields in it) and quantum field theories (with the rules quantization to be applied in principle to all degrees of freedom including gravity) are found to be mutually incompatible. Attempts over the past several decades at quantizing general relativity have yet to meet with success. Superstring theory, as the candidate for a consistent quantum theory of gravity along with a unified description of all elementary particle interactions, has remained so far only a promise. In spite of its recent new insights through duality, of an underlying unity among a diversity of string theories, described by a so called M-theory, is as yet far from a convincing physical theory with predictable and experimentally verifiable consequences.

The mathematical framework underlying both, the general theory of relativity and quantum field theories, is based on a continuum picture of space-time. A pseudo-Riemannian manifold endowed with a metric structure based on a continuum picture underlies the general theory of relativity. Likewise, quantum fields and their interactions are local operators that are functions of continuous commuting space-time coordinates. There are several reasons to believe why such a continuum picture of space-time is untenable at all distance scales. The problem

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of singularities in the curvature tensor in general relativity and the ultra-violet divergences in quantum field theories are too well known to merit discussion. To these we might add two other problems that have received considerable attention in recent years. One is the problem of black hole entropy and the enumeration of the black hole degrees of freedom and the other, the problem of localization of an event consistent with quantum mechanics. To elaborate briefly on the latter, we note that when we perform precise measurements of the space-time localization of an event, up to uncertainties, $\Delta X^0, \Delta X^1, \Delta X^2, \Delta X^3$, we must transfer energy of the order $E \approx 1/\Delta X$. This energy creates a gravitational field, and assuming spherical symmetry for simplicity, the associated Schwarzschild radius $R \approx E \approx 1/\Delta X$. Consequently signals originating inside R cannot reach an outside observer. Such arguments point to the existence of fundamental space-time uncertainty relations (STURS) [1],

$$\Delta X^0 \Delta X \approx \ell_p^2 \quad ; \qquad\qquad \Delta X^1 \Delta X^2 \approx \ell_p^2$$

where ℓ_p is the Planck length.

Clearly such uncertainty relations are incompatible with classical commuting coordinates of a Lorentzian or a Riemannian manifold. They seem to imply [2] that the quantum theories of space, time and matter are all interwoven. Hence, the ultimate goal of a fundamental theory should be a generalized quantum theory, a theory that does not, at the outset, begin with a continuum space-time as an input, but a theory that gives rise to the classical continuum of space-time in an appropriate limit. However, at present, we have no real candidate for such a unified quantum theory of space, time and matter, although there are promising guideposts and indications of progress from several different points of view.

One such promising approach has been provided by Alain Connes based on, what is called, non-commutative geometry (NCG) [3,4]. Connes' ideas hinge upon the well-known theorem due to Gelfand [3,5], which states that the classical topological space based on a continuum can be completely recovered by the abelian algebra of smooth functions defined on that space. As a natural generalization, Connes considers non-commutative, but associative and involutive algebras as the starting point for the description of more general spaces with non-commuting coordinates, or spaces with both continuous and discrete degrees of freedom. He reconstructs the standard objects of the conventional differential geometry in a purely algebraic way, setting up the basis for his non-commutative geometry. It has given rise to the description of the Standard Model with a geometrical interpretation of the Higgs field on the same footing as the gauge fields with spontaneous symmetry breaking following as a natural consequence.

In what follows, we present a brief account of work in progress, which uses the basic ideas of Connes to construct Lagrangians and action functionals of interacting matter fields in curved space-time. This is an extension of our previous work [6] that dealt with the geometry of a two-sheeted space-time within the framework of NCG and the consequent models for gravitational interactions. In this report, we include matter fields. For simplicity, we consider two chiral fields interacting with abelian gauge fields. Their setting in curved space-time brings into play the gravitational interactions and together they provide a rich and complex model for further study.

In the next section, we discuss briefly the basic elements in constructing the action funtionals in the usual Riemannian geometry (R-geometry) to set the stage for their adoption in the framework of non-commutative geometry. In Section III, we consider a two-sheeted, non-commutative space-time with matter fields and discuss the construction of action functionals involving fermionic, gauge and gravitational interactions. The final section is devoted to some concluding remarks.

II. Action functionals in R-Geometry; Basic ideas of NCG

The starting point in the conventional R-geometry is the construction of tangent and cotangent vector spaces $T_x(M)$ and $T_x^*(M)$ at an arbitrary local point, x. They are spanned respectively by the coordinate derivative operators $\{\partial/\partial x^\mu\}$, and the coordinate differentials $\{dx^\mu\}, \mu = 3D0, 1, 2, 3, = 85.$, n-1, in an n-dimensional manifold, so that an arbitrary tangent vector X and an arbitrary cotangent vector or one-form ω are linear combinations with real or complex coefficients,

$$X = 3DX^\mu \partial_\mu \ , \qquad\qquad \omega = 3Ddx^\mu \omega_\mu \ . \tag{2.1}$$

These vector spaces are dual via the scalar product

$$dx^\mu(\partial_\mu) \equiv \langle dx^\mu \ , \partial_\nu \rangle = 3D\delta^\mu_\nu,$$

and hence

$$\omega(X) \equiv \langle \omega, X \rangle = 3D\langle dx^\mu \omega_\mu, X^\nu \partial_\mu \rangle = 3D\omega^*_\mu X^\nu. \tag{2.2}$$

Higher rank tensors T of type (p, q) and a differential n-form, ω, as totally anti-symmetric tensor of type (n, o) are defined in the standard way,

$$T = 3DT^{\mu_1 \dots \mu_q}_{\nu_1 \dots \nu_p} \ dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_p} \otimes \partial_{\mu_1} \cdots \otimes \partial_{\mu_q}, \tag{2.3}$$

$$\omega = 3Ddx^{\mu_1} \wedge dx^{\mu_2} \cdots \wedge dx^{\mu_n} \omega_{\mu_1 \mu_2 \cdots \mu_n} \ . \tag{2.4}$$

With tensor and wedge products, we can define products between tensors of different ranks and different n-forms, and hence construct algebras over tensors and differential forms, which we will denote as $\Lambda_x(M)$ and $\Omega_x(M)$ respectively:

$$\Lambda_x(M) = 3D \oplus_{p,q} \Lambda^{(p,q)}_x(M) \ ; \quad \Omega_x(M) = 3D \oplus_n \Omega^n_x(M). \tag{2.5}$$

The scalar product defined in (2.2) implies an isomorphism between the tangent and cotangent vector spaces. This isomorphism lends itself to the introduction of a metric leading to the definition of the scalar products between two one- forms (or equivalently between two tangent vectors),

$$g(dx^\mu, dx^\nu) = 3Dg^{\mu\nu}(x) \ , \qquad g^{\mu\nu}(x) \epsilon C^\infty(M) \tag{2.6}$$

$$g(\omega_1, \omega_2) = 3Dg(dx^\mu \omega_{1\mu}, dx^\nu \omega_{2\nu}) = 3D(\omega_{1\mu})^* g^{\mu\nu} \omega_{2\mu}. \tag{2.7}$$

While the local coordinate basis (2.1) is the customary basis to describe curved space-time, it is advantageous for our purposes to utilize locally flat, orthonormal tetrad system $\{e_a, \theta^a\}$ defined by the transformations, e^μ_a, called the vierbines,

$$e_a = 3De^\mu_a \partial_\mu \ , \qquad \theta^a = 3Ddx^\mu e^a_\mu \ ,$$

with

$$e^\mu_a e^b_\mu = 3D\delta^b_a \ , \qquad e^\mu_a e^a_\nu = 3D\delta^\mu_\nu. \tag{2.8}$$

It then follows that the metric in tetrad system is given by

$$g(\theta^a, \theta^b) = 3D(e^a_\mu)^* g^{\mu\nu} e^b_\nu \equiv \eta^{ab}, \tag{2.9}$$

with the Lorentzian signature chosen to be η =3D diag(-1, 1, 1,=85,1).

By sesquilinearity the metric structure can be extended for the vector spaces of higher forms. Of particular interest for us to construct the action functionals is the two form $\Omega^2(M)$, in which case, $\{\theta^a \Lambda \theta^b\}$ provide the basis elements for the space of two-forms, and we define the scalar product

$$\langle \theta^a \Lambda \theta^b, \ \theta^c \Lambda \theta^d \rangle = 3D\eta^{ad}\eta^{bc} - \eta^{ac}\eta^{bd}. \tag{2.10}$$

The next important concept in R-Geometry is the covariant derivative operator (or affine connection), =D1, that assigns to each one-form $\omega \epsilon \Lambda^1(M)$, a (0, 2) tensor field, $\nabla \omega$,

$$\nabla : \ \Lambda'(M) \to \Lambda'(M) \otimes \Lambda'(M) \ , \tag{2.11}$$

where the image of $\nabla \omega$ on vector fields X, Y, has the usual linearity properties. Skipping details that are too well known in the literature, we note

$$\nabla \theta^a = 3D\theta^b \otimes \omega^a_b = 3D\theta^b \otimes \theta^c \omega^a_{bc} \ , \tag{2.12}$$

where ω^a_b are the connection one-forms and ω^a_{bc} the connection coefficients (Christoffel symbols in coordinate frame). With this, the Cartan Structure Equations defining torsion T, and curvature R are as follows:

$$\begin{aligned} T^a &= 3Dd\theta^a - \theta^b \omega^a_b \ , \\ R^a_b &= 3Dd\omega^a_b + \omega^a_c \wedge \omega^c_b \ . \end{aligned} \tag{2.13}$$

As well known, if torsion is assumed to vanish, the connection coefficients can be expressed in terms of the vierbeins and their derivatives(that is, in terms of metric components and their derivatives). Starting with appropriate one-forms, we can then calculate the required curvature tensors using the second of the Cartan structure equations. For the abelian gauge sector, for instance, the one-form

$$A = 3D dx^{\mu} A_{\mu} \tag{2.14}$$

leads to the curvature

$$F = 3D dA + A \wedge A \ , \tag{2.15}$$

and the Lagrangian and the action

$$\mathcal{L}_A = 3D \langle F, F \rangle \ ; \ S_A = 3D \int d^4x \sqrt{-g} \langle F, F \rangle . \tag{2.16}$$

Likewise, in the gravitational sector, the connection one-forms ω^a_b are the Riemannian connections in (2.13), leading to the Riemannian curvature two-forms

$$R^a_b = 3D \frac{1}{2} R^a_{bcd} \theta^e \wedge \theta^d \ , \tag{2.17}$$

and the Einstein-Hilbert Lagrangian and the action

$$\mathcal{L}_{EH} = 3D \langle R_{ab}, \theta \overset{a1}{\wedge} \theta^b \rangle \ , \ S_{EH} = 3D \int d^4x \frac{\sqrt{-g}}{16\pi G} R \tag{2.18}$$

where

$$R = 3D \frac{1}{2} (\eta^{bc} R^d_{bcd} - \eta^{bd} R^c_{bcd}) \ .$$

Next to obtain an action functional for a fermionic field in curved space-time, consider a Dirac spinor, $\Psi(x)$, at an arbitrary point $x \epsilon M$, and a homomorphism γ on the Clifford algebra $C(1,3)$ with the multiplication defined on the basis one-forms

$$\theta^a \theta^b + \theta^b \theta^a = 3D 2 \eta^{ab} \ .$$

This provides the usual γ-matrices in flat space-time with

$$\gamma(\theta^a) = 3D \gamma^a \ , \qquad \{\gamma^a, \gamma^b\} = 3D 2 \eta^{ab} 1 \ . \tag{2.19}$$

The γ-matrices in curved space-time are then given by

$$\gamma^{\mu}(x) = 3D e^{\mu}_a(x) \gamma^a \ , \qquad \{\gamma^{\mu}, \gamma^{\nu}\} = 3D 2 g^{\mu\nu} 1. \tag{2.20}$$

Taking into account the correct transformation properties, with the derivative operator, ∂_a, in flat space-time replaced by, ∇_a, where

$$\nabla_a = 3D e^{\mu}_a (\partial_{\mu} + ieA_{\mu} + \frac{1}{2i} e^a_{\mu} \omega_{abc} \sigma^{bc}), \tag{2.21}$$

we have from the Dirac Lagrangian in flat space-time,

$$\mathcal{L}_D = 3Di\bar{\Psi}\gamma^a\partial_a\Psi \ ,$$

transformed into the action in curved space-time that includes gauge and gravitational interactions,

$$\mathcal{L}_D \mapsto \mathcal{L}_\Psi = 3Di\bar{\Psi}e_a^\mu(\partial_\mu + ieA_\mu + \frac{1}{2i}e_\mu^a\omega_{abc}\sigma^{bc}) \ \Psi. \tag{2.22}$$

To summarize, the basic ingredients to construct matter fields and their interactions including gravity in R-geometry are, the algebra of differential forms, $\Lambda^*(M)$, the affine connection, ∇, curvature(or field strengths) and an inner product on tangent space, whose image is a Lorentz-invariant scalar. These geometrical constructs depend strictly upon the local properties defined in the neighborhood of a point in the manifold. To generalize these notions to non-commutative spaces, one needs to depart from this localization. In Connes version of NCG, this is accomplished by first reformulating the above concepts in algebraic terms and then an operator-theoretic representation of the algebra on a Hilbert space. Thus, it involves three basic elements $(\mathcal{A}, \mathcal{H}, \mathcal{D})$,which are collectively called a spectral triple. In what follows, we shall describe the main ideas to be applied to a two-sheeted space-time in the following section. For a more extensive treatment of these topics, see [2,6].

A). The Manifold $M \to \mathcal{A}$, an associative, involutive, commutative or noncommutative algebra with a unit element, whose elements are to be represented as operators on a Hilbert space $\mathcal{H}$. A universal differential algebra, $\Omega^*\mathcal{A}$, is generated by all $a\epsilon\mathcal{A}$, and a symbol δ, such that

$$\delta(1) = 3D0 \ ; \delta(ab) = 3D(\delta a)b + a(\delta b), \forall a, b\epsilon\mathcal{A} \ . \tag{2.23}$$

By definition, the algebra of zero forms is $\Omega^\circ\mathcal{A} \equiv \mathcal{A}$; δa belongs to the space of universal one-forms $\Omega'\mathcal{A}$, whose general element is a linear combination

$$\omega = 3D\Sigma_i\delta a_i b_i \ , \qquad a_i, b_i\epsilon\mathcal{A} \ . \tag{2.24}$$

In NCG, the natural generalization of a vector space of one-form is an $\mathcal{A}$-module, where the linear combinations with real or complex coefficients are replaced by linear combinations with coefficients belonging to the algebra. Likewise scalar multiplication is defined with elements of the algebra from the right (left) for right-(left-) $\mathcal{A}$-modules. Using the Leibnitzian property of δ, and repeated multiplication of one-forms, we can construct higher p-forms on the algebra, and can multiply two such arbitrary forms, generating a graded universal algebra, $\Omega^*\mathcal{A} \equiv \oplus\Omega^p\mathcal{A}$, corresponding to $\Omega_x(M)$ in (2.5). Further, by defining δ as a linear operator,

$$\Omega^p \to \Omega^{p+1} \ ,$$

$$\delta(\delta a_1 \cdots \delta a_p b) = 3D\delta a_1 \cdots \delta a_p \delta b \ ,$$

we can transform the graded algebra of forms, $\Omega^*\mathcal{A}$, into a differential algebra.

Next, we note that in the standard treatment, the physical matter fields are described as sections of vector and covector bundles. The space of such sections on a continuum manifold is always a finite, projective module over the commutative algebra of smooth functions. We assume that in their generalization in NCG, they are replaced by finite, projective $\mathcal{A}$-modules.

B). A self-adjoint operator, called the Dirac operator, $\mathcal{D}$, on the Hilbert space $\mathcal{H}$, such that the commutator $[\mathcal{D}, a]$ is a bounded operator, $\forall a \epsilon \mathcal{A}$. The objects of the universal differential algebra are then represented as operators on by the following graded homomorphism,

$$\Pi : \Omega^* \mathcal{A} \to \mathcal{L}(\mathcal{H}) \ ,$$

$$\Pi_p(\delta a_1 \cdots \delta a_p b) = 3D \prod_{i=3D1}^{p} [\mathcal{D}, \Pi_0(a_i)]\Pi_0(b) \tag{2.25}$$

where $\mathcal{L}(\mathcal{H})$ denotes the space of bounded operators on $\mathcal{H}$ and Π_0, the representation of $\mathcal{A}$ on $\mathcal{H}$.

In the case of a 4-dimensional, pseudo-Riemannian, spin manifold, M, using the canonicle spectral triple, where $\mathcal{A} = 3DC^\infty(M, R), \mathcal{H} = 3DL^2(M, S)$ the space of square integrable sections of the spinor bundle, and $D = 3D\gamma^\mu \nabla_\mu$, we can reproduce the action functionals (2.16, 2.18, and 2.22), derived using standard techniques [7].

III. Two-sheeted Space-time

A two sheeted space-time may be interpreted as a Kaluza-Klein theory with an internal space of two discrete points in the fifth dimension. As stated earlier, apart from providing a non-trivial extension of R-geometry and an example of a noncommutative space, it is also physically motivated from the fact that, due to parity violation, there is an intrinsic difference between left- and right-chiral fields. We might imagine that they live on two separate copies of space-time. For simplicity, we consider here two chiral spin-1/2 fields, each coupled to two an abelian gauge field in the presence of gravity. Extension to the full Standard Model will be dealt with elsewhere. In what follows, we shall use M, N, .., and A, B,=85, to mark the indices in curvilinear frame and the locally flat tetrad system respectively, with $M = 3D\mu, 5$(μ =3D 0,1,2,3) and $A = 3Da,\dot{5}$($a = 3D$0,1,2,3).
As discussed in the previous section, the basic ingredient in NCG is the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. The spectral triple for our model is as follows:

$$\text{Algebra :} \qquad \mathcal{A} = 3DC^\infty(M) \otimes (R \oplus R) = 3DC^\infty(M, R) \oplus C^\infty(M, R)$$

with the elements of the algebra given by

$$F(x) = 3Df_+(x)\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right) + f_-(x)\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) = 3D\left(\begin{array}{cc} f_1(x) & 0 \\ 0 & f_2(x) \end{array}\right) \ . \tag{3.1}$$

$$\text{Hilbert Space :} \qquad \mathcal{H} = \dot{3}D\ L^2(M,S) \oplus L^2(M,S)$$

consisting of left and right, square-integrable sections of a spinor bundle. Elements of the algebra, $\mathcal{A}$, act as operators,

$$F(x)\Psi(x) = 3D\left(\begin{array}{cc} f_1(x) & 0 \\ 0 & f_2(x) \end{array}\right)\left(\begin{array}{c} \psi_L(x) \\ \psi_R(x) \end{array}\right) \ . \tag{3.2}$$

$$\text{Dirac Operator :} \qquad \mathcal{D} = 3D\Gamma^M D_M = 3D\Gamma^\mu D_\mu + \Gamma^5 D_5 \ ,$$

where the derivative operators D_μ, D_5 are given by

$$D_\mu = 3D\left(\begin{array}{cc} \nabla_\mu & 0 \\ 0 & \nabla_\mu \end{array}\right) \ , \ D_5 = 3D\left(\begin{array}{cc} m & 0 \\ 0 & m \end{array}\right) \tag{3.3}$$

The parameter m in (3.3) has the dimensions of mass to conform to the dimensions in D_μ.
In complete analogy with the spinor formulation of R-geometry, we introduce two sets of gamma matrices, Γ^A (locally flat space-time), and Γ^M (curved space-time),

$$\Gamma^a = 3D\left(\begin{array}{cc} \gamma^a & 0 \\ 0 & \gamma^a \end{array}\right) \ , \ \Gamma^{\dot{5}} = 3D\left(\begin{array}{cc} 0 & \gamma^5 \\ \gamma^5 & 0 \end{array}\right)$$

and

$$\Gamma^M = 3D\Gamma^A E_A^M \ , \tag{3.4}$$

where E_A^M are the generalized vierbeins that define the metric. Without any loss of generality, we can choose them to be

$$E_\mu^a(x) = \dot{3}D\left(\begin{array}{cc} e_{1\mu}^a(x) & 0 \\ 0 & e_{2\mu}^a(x) \end{array}\right) \ , \ E_5^a(x) = \dot{3}D0$$

$$E_5^{\dot{5}}(x) = \dot{3}D\Phi(x) = 3D\left(\begin{array}{cc} \phi_1(x) & 0 \\ 0 & \phi_2(x) \end{array}\right) \ , \quad \begin{array}{ll} E_\mu^{\dot{5}}(x) & = 3D\left(\begin{array}{cc} a_{1\mu}(x) & 0 \\ 0 & a_{2\mu}(x) \end{array}\right)\Phi(x), \\ & = \dot{3}DA_\mu\Phi \ . \end{array} \tag{3.5}$$

The orthogonality relations

$$E_M^A E_A^N = 3D\delta_M^N \ , \ \ E_M^A E_B^M = 3D\delta_B^A$$

then determine

$$\begin{array}{ll} E^{\mu}_{a}(x)=3D(E^{a}_{\mu}(x))^{-1} & ,E^{5}_{a}(X)=3D-E^{\mu}_{A}A_{\mu} \\ E^{\mu}_{5}(x)=3D0 & ,E^{5}_{5}(x)=3D\Phi^{-1}(x). \end{array} \tag{3.6}$$

The Dirac operator takes the form

$$D=3D\Gamma^{M}D_{M}=3D\begin{pmatrix} e^{\mu}_{1a}\gamma^{a}\nabla^{\mu} & \gamma^{5}m \\ \gamma^{5}m & e^{\mu}_{2a}\gamma^{a}\nabla_{\mu} \end{pmatrix} . \tag{3.7}$$

The generalized metric has the form

$$\begin{array}{lll} G^{MN} & =3D & E^{M}_{A}\eta^{AB}E^{N}_{B}=3D\begin{pmatrix} G^{\mu\nu} & G^{\mu 5} \\ G^{5\mu} & G^{55} \end{pmatrix}, \\ G_{MN} & =3D & E^{A}_{M}\eta_{AB}E^{B}_{N}=3D\begin{pmatrix} G_{\mu\nu} & G_{\mu 5} \\ G_{5\mu} & G_{55} \end{pmatrix} , \end{array}$$

where

$$G^{\mu\nu}(x)=3D\begin{pmatrix} g^{\mu\nu}_{1} & 0 \\ 0 & g^{\mu\nu}_{2}(x) \end{pmatrix} , \quad G^{\mu 5}=3DG^{5\mu}=3D-A^{\mu},G^{55}=3DA^{2}+\Phi^{-2}$$

and

$$G_{\mu\nu}(x)=3D\begin{pmatrix} g_{1\mu\nu}(x) & 0 \\ 0 & g_{2\mu\nu}(x) \end{pmatrix}+A_{\mu}(x)A_{\nu}(x)\Phi^{2}(x),G_{\mu 5}=3DG_{5\mu}=3DA_{\mu}(x)\Phi^{2}(x),$$

$$G_{55}(x)=3D\Phi^{2}(x). \tag{3.8}$$

Thus we find that, to describe the two-sheeted space-time, we need in general, two vierbeins (i.e., two tensor fields), two vector fields and two scalar fields. We shall call them collectively as metric fields.
The generalized one-forms are give by

$$U=3D\Gamma^{A}U_{A}=3D\Gamma^{M}E^{A}_{M}U_{A}=3D\Gamma^{M}U_{M} \tag{3.9}$$

with U_M being the elements of the algebra. Further, as a direct generalization of the covariant derivative operator in the standard R-geometry, we define

$$\nabla\Gamma^{A}=3D\Gamma^{B}\otimes\Omega^{A}_{B} ,$$

where Ω^{A}_{B} are the generalized connection one-forms corresponding to ω^{a}_{b} in (2.12).
The Cartan structure equations generalize easily to take the form

$$\begin{array}{lll} T^{A} & =3D & D\Gamma^{A}+\Gamma^{B}\wedge\Omega^{A}_{B} , \\ R^{A}_{B} & =3D & D\Omega^{A}_{B}+\Omega^{A}_{C}\wedge\Omega^{C}_{B} \end{array} . \tag{3.10}$$

The connection one-forms are arbitrary functions to be specified or to be determined by additional conditions. As noted previously, in R-geometry, metric compatibility and the condition that torsion vanish lead to the determination of these forms and the associated scalars in terms of the metric. We have shown in our previous work [6], a direct generalization of these conditions in the present case leads to constraints on the metric fields in the form

$$e^a_{1\mu}(x) = 3D\beta(x)e^a_{2\mu}(x)\ ,\ a_{1\mu}(x) = 3Da_{2\mu}(x)\alpha(x)\ ,\ \phi_1(x) = 3D\frac{\phi_2(x)}{\alpha(x)} \qquad (3.11)$$

where $\beta(x)$ and $\alpha(x)$ are arbitrary functions. With $\alpha = 3D\beta = 3D1$, we were able to reproduce the exact zero mode of the Kaluza-Klein theory. Seeking a more general formulation that contains all the metric component fields independently and a Lagrangian and an action that incorporates the full set of tensor, vector and scalar fields, Ai Viet has found a set of minimal constraints [8],

a) Metric compatibility

$$\nabla G = 3D0 \quad \Rightarrow \quad \Omega^+_{AB} = 3D - \Omega_{BA}$$

b) Torsion components involving the 4-dimensional space-time components vanish

$$T^a = 3D0$$

as the natural generalization of the R-geometry

c) The condition

$$\Omega_{AB\dot{5}} = 3D0$$

With the above conditions, he has shown that the rest of the connection one-forms and the non-vanishing components of torsion are determined in terms of metric fields (analogous to the determination of the connection coefficients in terms of the metric in R-geometry). This defines our gravity sector with six independent metric fields:

$$E^\mu_{\pm a}(x) = 3D\frac{e^\mu_{1a}(x) \pm e^\mu_{2a}(x)}{2}\ ,\ A_{\pm\mu}(x) = 3D\frac{a_{1\mu}(x) \pm a_{2\mu}(x)}{2}\ ,$$

$$\Phi_\pm(x) = 3D\frac{\phi_1(x) \pm \phi_2(x)}{2} \qquad (3.12)$$

While the full exploration of the ensuing general theory of gravity is in progress, we would like to point out one general feature. It is that one set of the component tensor, vector and scalar fields is massless where as the other is massive. There are essentially two parameters, the Newton's constant G and the parameter m of the dimensions of mass. Next,taking this as the background curved space-time, we proceed to construct the Lagrangians for the gauge and fermionic sectors. The resulting expressions are too long and involved to give their full forms here. We present here only their qualitative features, leaving the full details in papers to be published [7,8].

Gauge Sector

With the one-form

$$B = 3D\Gamma^M B_M = 3D\Gamma^A B_A \ , \ B_A = 3DE_A^M B_M \ ,$$

where

$$B_\mu(x) = 3D \begin{pmatrix} b_{1\mu}(x) & 0 \\ 0 & b_{2\mu}(x) \end{pmatrix} \ , \ B_5(x) = 3D \begin{pmatrix} h_1(x) & 0 \\ 0 & h_2(x) \end{pmatrix} = 3DH(x) \quad (3.13)$$

we have two abelian gauge fields and two Higgs fields. The curvature given by

$$G(x) = 3DDB(x) + B(x) \wedge B(x) = 3D\theta^A \wedge \theta^B G_{AB} \quad (3.14)$$

and hence the Lagrangian

$$< G, G > = 3DG_{AB}G_{CD}\left(\eta^{AD}\eta^{BC} - \eta^{AC}\eta^{BD}\right) \ . \quad (3.15)$$

Written in terms of the linear combinations $B_{\pm\mu} = 3D\frac{b_{1\mu} \pm b_{2\mu}}{2}$, one finds that $b_{+\mu}$ is massless, whereas $b_{-\mu}$ is massive. Proper kinetic terms for the gauge and Higgs fields and Lorentz covariant interactions ensure that we have a physically meaningful Lagrangian. In addition to the expected gauge and Higgs fields interactions, it has the new feature involving the interactions with vector and scalar components, $a_\mu, \ \phi$, that are parts of the metric.

We have a Higgs potential in the form

$$V(x) = 3D\left(\Phi\tilde{\Phi}\right)^{-2}\left(\tilde{H}H + m(H + \tilde{H})\right)^2 \quad (3.16)$$

which with $h_1 = 3D\overline{h}_2$ reduces to the required standard form for spontaneous symmetry breaking, namely,

$$\frac{1}{2} V(x) = 3D\left(\phi_1\phi_2\right)^{-2}\left(\eta(x)\overline{\eta}(x) - m^2\right)^2 \ ,$$

where

$$\eta = 3D\eta_1 + i\eta_2 \ , \ \eta_1 = 3D\frac{h_1 + h_2}{2} - m \ , \ \eta_1 = 3D\frac{h_1 + h_2}{2} - m \ , \ \eta_2 = 3D\frac{h_1 - h_2}{2i} \ . \quad (3.17)$$

Fermionic Sector

Beginning with the generalized Lagrangian,

$$\mathcal{L}_F = 3Di\overline{\Psi}\Gamma^A\left(E_A^M(D_M + B_M) - \frac{1}{4}\,\Gamma^B\Gamma^C\Omega_{BCA}\right)\Psi\ , \tag{3.18}$$

we find in it the expected part,

$$\begin{aligned} i\overline{\Psi}\Big\{&\gamma^a\left(e^\mu_{+a}\left(\partial_\mu + b_{+\mu}\right) + e^\mu_{-a}b_{-\mu}\right) - \frac{1}{8}\,\gamma^b\gamma^c\left(\omega_{1bca} + \omega_{2bca}\right) \\ &+\gamma^5\gamma^a\left(e^\mu_{-a}\left(\partial_\mu + b_{+\mu}\right) + e^\mu_{+}b_{-\mu}\right) - \frac{1}{8}\,\gamma^b\gamma^c\left(\omega_{1bca} - \omega_{2bca}\right)\Big\}\Psi\ , \end{aligned}$$

which is the Dirac Lagrangian in the curved space-time of our model. It contains host of other terms arising from the discrete part involving the Higgs fields, $h_1,\ h_2$, and the vector and scalar components $a_1,\ a_2$ and $\phi_1,\ \phi_2$ respectively. Assuming e^μ_{+a} defines the physical metric, it is interesting to note that the massless $b_{+\mu}$ has only the vector interaction, whereas the massive $b_{-\mu}$ has both vector and axial vector interactions. Thus, in our simplified model of two chiral fields, $b_{+\mu}$ represents the parity conserving photon and $b_{-\mu}$ represents the massive parity violating Z-vector boson. We also note the following terms of special interest

$$\begin{aligned} &i\overline{\Psi}\left[\frac{1+\gamma_5}{2}\,(m+h_2)\phi_2^{-1} - \frac{1-\gamma_5}{2}\,(m+h_1)\phi_1^{-1}\right]\Psi \\ &-i\overline{\Psi}\left[\frac{1+\gamma_5}{2}\,(m+h_1)\gamma^a a_{1a} + \frac{1-\gamma_5}{2}\,(m+h_2)\gamma^a a_{2a}\right]\Psi \end{aligned}$$

that involve interactions with the vector and scalar components of the gravity sector. They violate parity, and consequently raise the intriguing possibility of gravitation as the origin or part of the origin of parity violation.

IV. Concluding remarks

In this brief account of work in progress, we have presented an extension of the usual Riemannian geometry to a two-sheeted space-time, that is, a continuum space-time with a discrete two-point internal space. As a simple but nontrivial example of a noncommutative space, it has provided an extremely valuable model to study non-commutative geometric approach of Connes as well as the interplay of gravitational and other elementary particle interactions. Additional strong physical motivation for our model arises from the fact that parity violation implies different physical quantum numbers for left and right chiral fields in the Standard Model, so that we can imagine the two chiral fields live on two copies of space-time.

We have noted in our previous work, that this discretized Kaluza-Klein theory with a finite field content generates a finite mass spectrum. The appearance of new vector and scalar interaction terms with correlated strengths and nonlinear in nature lends itself to the speculation of possible softening of divergences in the gravitational sector and their effect on renormalization. As an extended theory of gravity, it has obviously cosmological implications. When matter fields are introduced, we find that the NCG framework allows spontaneous symmetry breaking and leads to a rich and complex variety of interactions that certainly merit a great deal of further study. Of particular importance is the appearance of parity violating interactions due to gravitational vector and scalar fields.

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References

1. S. Doplicher, K. Fredenhagen, and J. E. Roberts, Physics Letters B 331 (1994) 39-44.

2. J. Frohlich, in PASCOS 94-Proc.IV Int. Symp. On Particles, Strings and Cosmology (Syracuse, 1994),ed. K.C. Wali (World Scientific, Singapore, 1995), pp. 443

3. A. Connes, Non-Commutative Geometry (Academic Press, 1994)

4. A. Connes, " Essay on Physics and Non-Commutative Geometry," The Interface of Mathematics and Particle Physics, D.G. Quillen, G.B. Segal, S.T. Tsou editors (Clarendon Press, Oxford, 1990)

5. G.F. Simmons, Introduction to Topology and Modern Analysis (Krieger, 1983)

6. N.A. Viet and K.C.Wali, Intl.J.Modern Physics. A, 11 (1996), pp. 533-551 and ibid. A, 11 (1996), pp. 2403-2418

7. Construction of Action Functionals in Non-Commutative Geometry, J. A. Javor, N.A. Viet, and K.C. Wali (SU 4240-692)

8. A New Minimal Set of Constraints on the Connections and Torsion in Non-Commutative Geometry, N. A. Viet (SU-4240-566)