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Multiple periodic solutions for a class of second-order neutral functional differential equations

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Abstract

In this paper we consider a class of second-order neutral functional differential equations. Under certain conditions, we establish the existence of multiple periodic solutions by means of Z_2 group index theory and variational methods. The main result is also illustrated with an example.

Keywords: neutral functional differential equations; Z_2 group index theory; periodic solution

1 Introduction

In this paper we consider a class of second-order neutral functional differential equations described by

$$\begin{cases} (p(t)u'(t-s\tau))' - q(t)u(t-s\tau) + \lambda f(t,u(t),u(t-\tau),\dots,u(t-2s\tau)) = 0, \\ u(0) - u(2k\tau) = u'(0) - u'(2k\tau) = 0, \end{cases} \tag{1.1}$$

where $p \in C^1([0,\tau],\mathbb{R}^+)$, $q \in C([0,\tau],\mathbb{R}^+)$ and they are τ-periodic. $f \in C(\mathbb{R}^{2(s+1)},\mathbb{R})$, $\lambda \in \mathbb{R}$, and $\tau > 0$. k and s are given positive integers with $k > s$. A function $u \in C^1(\mathbb{R},\mathbb{R})$ is a solution of system (1.1) if the function u satisfies (1.1).

The necessity to study delay differential equations is due to the fact that these equations are useful mathematical tools in modeling many real processes and phenomena studied in economics, biology, electronics, optimal control, mechanics, medicine, etc. [1, 2].

In recent years many researchers have focused on the existence of periodic solutions of delay differential equations; see, for example, [3–5]. Several available approaches to tackle the existence of periodic solutions for delay differential equations include the dual Lyapunov method, the Fourier analysis method, fixed point theory, and the coincidence degree theory [6–9]. Recently, some researchers have studied the existence of periodic solutions for delay differential equations via variational methods [10–12].

When $p(t) = q(t) = 1$ and $s = 1$, system (1.1) reduces to the following equation:

$$\begin{cases} u''(t-\tau) - u(t-\tau) + \lambda f(t,u(t),u(t-\tau),u(t-2\tau)) = 0, \\ u(0) - u(2k\tau) = u'(0) - u'(2k\tau) = 0. \end{cases}$$

In [12], Shu and Xu obtained the following result.

Theorem A *Assume that the following conditions are satisfied.*

(H1) $\frac{\partial f(t,u_1,u_2,u_3)}{\partial t} \neq 0$.

(H2) *There exists a function* $F(t,u_1,u_2) \in C^1(\mathbb{R}^3,\mathbb{R})$ *such that*

$$\frac{\partial F(t,u_1,u_2)}{\partial u_2} + \frac{\partial F(t,u_2,u_3)}{\partial u_2} = f(t,u_1,u_2,u_3).$$

(H3) $F(t,u_1,u_2)$ *is* τ*-periodic in* t.

(H4) F *satisfies* $F(t,-u_1,-u_2) = F(t,u_1,u_2)$ *and* $f(t,-u_1,-u_2,-u_3) = -f(t,u_1,u_2,u_3)$.

(H5) $F(t,u_1,u_2) = 0$ *if and only if* $(u_1,u_2) = 0, \forall t \in [0,\tau]$.

(H6) $\lim_{|u|\to 0} \frac{F(t,u_1,u_2)}{|u|^2} = 1$, *where* $|u| = (|u_1|^2 + |u_2|^2)^{\frac{1}{2}}, t \in [0,\tau]$.

(H7) *There exists a constant* $\alpha > 0$ *such that when* $|u_1|^2 + |u_2|^2 > \alpha^2$, $F(t,u_1,u_2) < 0, t \in [0,\tau]$.

Moreover, if there exists an integer $m > 0$ *such that* λ *satisfies*

$$\lambda > \frac{m^2(\pi^2 + k^2\tau^2)}{4k\tau^2}, \tag{1.2}$$

then the system

$$\begin{cases} u''(t-\tau) - u(t-\tau) + \lambda f(t,u(t),u(t-\tau),u(t-2\tau)) = 0, \\ u(0) - u(2k\tau) = u'(0) - u'(2k\tau) = 0, \end{cases} \tag{1.3}$$

possesses at least $2m$ *non-zero solutions with the period* $2k\tau$.

Remark 1.1 Based on our analysis, (1.2) should be replaced by

$$\lambda > \frac{m^2(\pi^2 + k^2\tau^2)}{4k^2\tau^2}.$$

Compared to system (1.3), the neutral functional differential system (1.1) admits four control parameters p, q, λ, s. We aim to derive conditions in terms of these four control parameters for the existence and multiplicity of periodic solutions of a class of second-order neutral functional differential equation (1.1).

Our approach is based on the Z_2 group index theory and some techniques of mathematical analysis. We remark that the Z_2 group index theory and the variational method have also been employed to prove the existence of multiple periodic solutions of mixed type differential equations in [12]. When (1.1) reduces to the special case, see system (1.3), we obtain a more accurate result (see Remark 1.1). Moreover, our result generalizes the existence result obtained in [12] as the equation considered in [12] is a special case of our system (1.1) with $p(t) = q(t) = s = 1$.

To prove our main result, we first make the following assumptions.

(H1) $\frac{\partial f(t,u_1,u_2,\dots,u_{2s+1})}{\partial t} \neq 0$.

(H2) There exists a function $F(t,u_1,u_2,\ldots,u_{s+1})\in C^1(\mathbb{R}^{s+2},\mathbb{R})$ such that

$$\begin{aligned}&\frac{\partial F(t,u_1,u_2,\ldots,u_{s+1})}{\partial u_{s+1}}+\frac{\partial F(t,u_2,\ldots,u_{s+1},u_{s+2})}{\partial u_{s+1}}\\&\quad+\cdots+\frac{\partial F(t,u_{s+1},u_{s+2},\ldots,u_{2s+1})}{\partial u_{s+1}}=f(t,u_1,u_2,\ldots,u_{2s+1}).\end{aligned}$$

(H3) $F(t,u_1,u_2,\ldots,u_{s+1})$ is τ-periodic in t.

(H4) F satisfies

$$F(t,-u_1,-u_2,\ldots,-u_{s+1})=F(t,u_1,u_2,\ldots,u_{s+1})$$

and

$$f(t,-u_1,-u_2,\ldots,-u_{2s+1})=-f(t,u_1,u_2,\ldots,u_{2s+1}).$$

(H5) $F(t,u_1,u_2,\ldots,u_{s+1})=0$ if and only if $(u_1,u_2,\ldots,u_{s+1})=0,\forall t\in[0,\tau]$.

(H6) $\lim_{|u|\to 0}\frac{F(t,u_1,u_2,\ldots,u_{s+1})}{|u|^2}=1$, where $|u|=(|u_1|^2+\cdots+|u_{s+1}|^2)^{\frac{1}{2}}, t\in[0,\tau]$.

(H7) There exists a constant $\alpha>0$ such that $F(t,u_1,u_2,\ldots,u_{s+1})<0$ for $t\in[0,\tau]$ when $|u_1|^2+\cdots+|u_{s+1}|^2>\alpha^2$.

Note that system (1.1) is equivalent to the following system:

$$\begin{aligned}&\bigl(p(t)u'(t-s\tau)\bigr)'-q(t)u(t-s\tau)+\lambda\bigl(F'_{s+1}\bigl(t,u(t),u(t-\tau),\ldots,u(t-s\tau)\bigr)\\&\quad+\cdots+F'_1\bigl(t,u(t-s\tau),u\bigl(t-(s+1)\tau\bigr),\ldots,u(t-2s\tau)\bigr)\bigr)=0.\end{aligned}\tag{1.4}$$

The rest of this paper is organized as follows. In Section 2, we present some preliminaries, which will be used to prove our main results. In Section 3 we state and prove our main results. Finally, we provide one example to illustrate the applicability of our results.

2 Some preliminaries

Let

$$H^1_{2k\tau}=\bigl\{u:\mathbb{R}\to\mathbb{R}|u,u'\in L^2\bigl(\bigl([0,2k\tau]\bigr),\mathbb{R}\bigr),u(0)=u(2k\tau),u'(0)=u'(2k\tau)\bigr\}.$$

Then $H^1_{2k\tau}$ is a separable and reflexive Banach space and the inner product

$$(u,v)=\int_0^{2k\tau}\bigl(p(t)u'(t)v'(t)+q(t)u(t)v(t)\bigr)\,dt,$$

induces the norm

$$\|u\|_{H^1_{2k\tau}}=\biggl(\int_0^{2k\tau}\bigl(p(t)\bigl|u'(t)\bigr|^2+q(t)\bigl|u(t)\bigr|^2\bigr)\,dt\biggr)^{\frac{1}{2}}.$$

We introduce the following notations. Denote

$$F'_{s+1}(t,u_1,u_2,\ldots,u_{s+1})=\frac{\partial F(t,u_1,u_2,\ldots,u_{s+1})}{\partial u_{s+1}},$$

$$F_s'(t,u_2,\dots,u_{s+1},u_{s+2}) = \frac{\partial F(t,u_2,\dots,u_{s+1},u_{s+2})}{\partial u_{s+1}},$$

$$\dots$$

$$F_1'(t,u_{s+1},u_{s+2},\dots,u_{2s+1}) = \frac{\partial F(t,u_{s+1},u_{s+2},\dots,u_{2s+1})}{\partial u_{s+1}}.$$

Define a functional φ as

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_0^{2k\tau}\bigl(p(t)\bigl|u'(t)\bigr|^2 + q(t)\bigl|u(t)\bigr|^2\bigr)\,dt \\ &\quad - \lambda\int_0^{2k\tau} F\bigl(t,u(t),\dots,u(t-s\tau)\bigr)\,dt, \quad u \in H^1_{2k\tau}.\end{aligned} \tag{2.1}$$

Then φ is Fréchet differentiable at any $u \in H^1_{2k\tau}$. For any $v \in H^1_{2k\tau}$, by a simple calculation, we have

$$\begin{aligned}\varphi'(u)(v) = {}& \int_0^{2k\tau}\bigl(p(t)u'(t)v'(t) + q(t)u(t)v(t)\bigr)\,dt - \lambda\int_0^{2k\tau}\bigl(F_1'\bigl(t,u(t),\dots,u(t-s\tau)\bigr)v(t) \\ & + F_2'\bigl(t,u(t),u(t-\tau),\dots,u(t-s\tau)\bigr)v(t-\tau) \\ & + \cdots \\ & + F_{s+1}'\bigl(t,u(t),u(t-\tau),\dots,u(t-s\tau)\bigr)v(t-s\tau)\bigr)\,dt.\end{aligned}$$

From (H3), $p(t) \in C^1([0,\tau],\mathbb{R}^+)$, $q(t) \in C([0,\tau],\mathbb{R}^+)$, and their periodicity, we have

$$\begin{aligned}\varphi'(u)(v) = {}& \int_0^{2k\tau}\bigl(-\bigl(p(t)u'(t)\bigr)' + q(t)u(t)\bigr)v(t)\,dt - \lambda\int_0^{2k\tau}\bigl(F_1'\bigl(t,u(t),\dots,u(t-s\tau)\bigr)v(t) \\ & + F_2'\bigl(t,u(t+\tau),u(t),\dots,u\bigl(t-(s-1)\tau\bigr)\bigr)v(t) \\ & + \cdots \\ & + F_{s+1}'\bigl(t,u(t+s\tau),u\bigl(t+(s-1)\tau\bigr),\dots,u(t)\bigr)v(t)\bigr)\,dt \\ = {}& \int_0^{2k\tau}\bigl(-\bigl(p(t)u'(t)\bigr)' + q(t)u(t) - \lambda\bigl(F_1'\bigl(t,u(t),\dots,u(t-s\tau)\bigr) \\ & + F_2'\bigl(t,u(t+\tau),u(t),\dots,u\bigl(t-(s-1)\tau\bigr)\bigr) \\ & + \cdots \\ & + F_{s+1}'\bigl(t,u(t+s\tau),u\bigl(t+(s-1)\tau\bigr),\dots,u(t)\bigr)\bigr)\bigr)v(t)\,dt.\end{aligned}$$

Therefore, the corresponding Euler equation of functional φ is

$$\begin{aligned}&\bigl(p(t)u'(t)\bigr)' - q(t)u(t) + \lambda\bigl(F_1'\bigl(t,u(t),\dots,u(t-s\tau)\bigr) \\ &\quad + F_2'\bigl(t,u(t+\tau),u(t),\dots,u\bigl(t-(s-1)\tau\bigr)\bigr) \\ &\quad + \cdots \\ &\quad + F_{s+1}'\bigl(t,u(t+s\tau),u\bigl(t+(s-1)\tau\bigr),\dots,u(t)\bigr)\bigr) = 0.\end{aligned} \tag{2.2}$$

Note that system (1.4) is equivalent of system (2.2). Hence, critical points of the functional φ are classical $2k\tau$-periodic solutions of system (1.1).

Definition 2.1 [13]. Let E be a real reflexive Banach space, and

$$\Sigma = \{A \mid A \subset E \setminus \{0\} \text{ is closed, symmetric set}\}.$$

Define $\gamma : \Sigma \to \mathbb{Z}^+ \cup \{+\infty\}$ as follows:

$$\gamma(A) = \begin{cases} \min\{n \in \mathbb{Z}^+ : \text{there exists an odd continuous map} \\ \quad \varphi : A \to \mathbb{R}^n \setminus \{0\}\}; \\ 0, \quad \text{if } A = \emptyset; \\ +\infty, \quad \text{if there is no odd continuous map } \varphi : A \to \mathbb{R}^n \setminus \{0\} \\ \quad \text{for any } n \in \mathbb{Z}^+. \end{cases} \tag{2.3}$$

Then we say γ is the genus of Σ.

Denote $i_1(\varphi) = \lim_{a \to -0} \gamma(\varphi_a)$ and $i_2(\varphi) = \lim_{a \to -\infty} \gamma(\varphi_a)$ where $\varphi_a = \{u \in E \mid \varphi(u) \le a\}$.

Lemma 2.2 ([14]) *Let E be a real Banach space, $\varphi \in C^1(E, \mathbb{R})$ with φ even functional and satisfying the Palais-Smale* (*PS*) *condition. Suppose $\varphi(0) = 0$ and*

(i) *if there exist an m-dimensional subspace X of E and a constant $r > 0$ such that*

$$\sup_{u \in X \cap B_r} \varphi(u) < 0, \tag{2.4}$$

where B_r is an open ball of radius r in E centered at 0, then we have $i_1(\varphi) \ge m$;

(ii) *if there exists a j-dimensional subspace V of E such that*

$$\inf_{u \in V^\perp} \varphi(u) > -\infty, \tag{2.5}$$

then we have $i_2(\varphi) \le j$.

Moreover, if $m \ge j$, then φ possesses at least $2(m - j)$ distinct critical points.

3 Main result

In this section, we state and prove our main result. Set $P = \max_{t \in [0,\tau]} p(t)$, $Q = \max_{t \in [0,\tau]} q(t)$.

Theorem 3.1 *Assume that* (H1)-(H7) *are satisfied. If there exists an integer $m > 0$ such that λ satisfies*

$$\lambda > \frac{m^2(P\pi^2 + Qk^2\tau^2)}{2(s+1)k^2\tau^2}, \tag{3.1}$$

then system (1.1) *admits at least $2m$ non-zero solutions with the period $2k\tau$.*

Proof We apply Lemma 2.2 to finish the proof. Under the assumptions (H4), it is easy to see that if function u is a solution of the system (1.1), then the function $-u$ is also a solution of the system (1.1). Therefore, the solutions of the system (1.1) are a set which is symmetric with respect to the origin in $H^1_{2k\tau}$. It follows directly from (2.1) and (H5) that φ is even in u and $\varphi(0) = 0$. The rest of the proof is divided into three steps.

Step 1: We show that the functional φ satisfies the assumption (ii) of Lemma 2.2.

It follows from (H7) that there exists a constant $M > 0$ such that

$$\max_{t\in\mathbb{R}} F\bigl(t,u(t),\dots,u(t-s\tau)\bigr) \le \max_{(t,u_1,u_2,\dots,u_{s+1})\in\Omega} F(t,u_1,\dots,u_{s+1}) \le M, \tag{3.2}$$

where $\Omega = [0,\tau] \times [-\alpha,\alpha] \times [-\alpha,\alpha] \times \cdots \times [-\alpha,\alpha]$. Combining (2.1) and (3.2), we get

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\|u\|^2_{H^1_{2k\tau}} - \lambda \int_0^{2k\tau} F\bigl(t,u(t),\dots,u(t-s\tau)\bigr)\,dt \\ &\ge \frac{1}{2}\|u\|^2_{H^1_{2k\tau}} - 2\lambda Mk\tau > -\infty, \end{aligned} \tag{3.3}$$

which implies that φ is bounded from below. By the condition (ii) of Lemma 2.2, we have $i_2(\varphi) = 0$.

Step 2: We show that the functional φ satisfies PS condition.

For any given sequence $\{u_n\} \in H^1_{2k\tau}$ such that $\{\varphi(u_n)\}$ is bounded and $\lim_{n\to\infty} \varphi'(u_n) = 0$, there exists a constant C_1 such that

$$\bigl|\varphi(u_n)\bigr| \le C_1, \qquad \bigl\|\varphi'(u_n)\bigr\|_{(H^1_{2k\tau})^*} \le C_1, \quad \forall n \in \mathbb{N},$$

where $(H^1_{2k\tau})^*$ is the dual space of $H^1_{2k\tau}$.

Therefore, we have

$$\frac{1}{2}\|u\|^2_{H^1_{2k\tau}} \le C_1 + 2\lambda Mk\tau.$$

It follows that $\|u_n\|_{H^1_{2k\tau}}$ is bounded.

Since $H^1_{2k\tau}$ is a reflexive Banach space, we can pick $\{u_n\}$ be a weakly convergent sequence to u in $H^1_{2k\tau}$ and $\{u_n\}$ converges uniformly to u in $C[0,2k\tau]$. So, we have

$$\begin{aligned} &\int_0^{2k\tau} \bigl(F_1'\bigl(t,u_n(t),\dots,u_n(t-s\tau)\bigr) - F_1'\bigl(t,u(t),\dots,u(t-s\tau)\bigr)\bigr)\bigl(u_n(t)-u(t)\bigr)\,dt \to 0, \\ &\int_0^{2k\tau} \bigl(F_2'\bigl(t,u_n(t),\dots,u_n(t-s\tau)\bigr) - F_2'\bigl(t,u(t),\dots,u(t-s\tau)\bigr)\bigr) \\ &\quad \times \bigl(u_n(t-\tau)-u(t-\tau)\bigr)\,dt \to 0, \\ &\dots \\ &\int_0^{2k\tau} \bigl(F_{s+1}'\bigl(t,u_n(t),\dots,u_n(t-s\tau)\bigr) - F_{s+1}'\bigl(t,u(t),u(t-\tau),\dots,u(t-s\tau)\bigr)\bigr) \\ &\quad \times \bigl(u_n(t-s\tau)-u(t-s\tau)\bigr)\,dt \to 0, \\ &u_n(t)-u(t) \to 0 \quad \text{as } n\to\infty, t\in[0,2k\tau]. \end{aligned} \tag{3.4}$$

Therefore, by (3.4), we have $\|u_n - u\|_{H^1_{2k\tau}} \to 0$. Hence the functional φ satisfies the PS condition.

Step 3: We show that the functional φ satisfies the assumption (i) of Lemma 2.2.

Let $\beta_j(t) = \frac{k\tau}{j\pi}\sin\frac{j\pi}{\kappa\tau}t, j = 1, 2, \dots, m$. By calculation, we obtain

$$\int_0^{2k\tau} \left|\beta_j(t)\right|^2 dt = \left(\frac{k\tau}{j\pi}\right)^2 k\tau$$

and

$$\int_0^{2k\tau} \left|\beta_j'(t)\right|^2 dt = k\tau.$$

Define the m-dimensional linear subspace as follows:

$$E_m = \operatorname{span}\left\{\beta_1(t), \beta_2(t), \dots, \beta_m(t)\right\}.$$

It is clear to see that E_m is a symmetric set. Take $r > 0$, when $u(t) \in E_m \cap S_r$, where S_r denotes the boundary of B_r, $u(t)$ has an expansion $u(t) = \sum_{j=1}^m b_j\beta_j(t)$, and

$$\begin{aligned} r^2 = \left\|u(t)\right\|^2_{H^1_{2k\tau}} &= \int_0^{2k\tau} \left(p(t)\left|u'(t)\right|^2 + q(t)\left|u(t)\right|^2\right) dt \\ &\le k\tau \sum_{j=1}^m b_j^2 \left(P + \frac{Qk^2\tau^2}{j^2\pi^2}\right). \end{aligned} \tag{3.5}$$

By (H6), for given ε with $0 < \varepsilon < \frac{\lambda m^2}{2(s+1)k^2\tau^2}(\frac{2(s+1)k^2\tau^2}{m^2} - \frac{P\pi^2+Qk^2\tau^2}{\lambda})$, there exists $\delta > 0$ such that when $(|u(t)|^2 + \cdots + |u(t-s\tau)|^2)^{\frac{1}{2}} < \delta$, we have

$$\lambda F\left(t, u(t), \dots, u(t-s\tau)\right) > (\lambda - \varepsilon)\left(\left|u(t)\right|^2 + \cdots + \left|u(t-s\tau)\right|^2\right). \tag{3.6}$$

Combining (2.1), (3.5), and (3.6), when $u(t) \in E_m \cap S_r$, we have

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\int_0^{2k\tau} \left(p(t)\left|u'(t)\right|^2 + q(t)\left|u(t)\right|^2\right) dt - \lambda \int_0^{2k\tau} F\left(t, u(t), \dots, u(t-s\tau)\right) dt \\ &\le \frac{1}{2}\|u\|^2_{H^1_{2k\tau}} - (\lambda - \varepsilon)\int_0^{2k\tau} \left(\left|u(t)\right|^2 + \cdots + \left|u(t-s\tau)\right|^2\right) dt \\ &\le \frac{k\tau}{2}\sum_{j=1}^m b_j^2\left(P + \frac{Qk^2\tau^2}{j^2\pi^2}\right) - \frac{(\lambda-\varepsilon)(s+1)k^2\tau^2}{m^2\pi^2} k\tau \sum_{j=1}^m b_j^2 \\ &\le \frac{k\tau}{2\pi^2}\sum_{j=1}^m b_j^2\left(P\pi^2 + Qk^2\tau^2 - \frac{2(\lambda-\varepsilon)(s+1)k^2\tau^2}{m^2}\right) \\ &= \frac{\lambda k\tau}{2\pi^2}\sum_{j=1}^m b_j^2\left(\frac{P\pi^2 + Qk^2\tau^2}{\lambda} - \frac{2(s+1)k^2\tau^2}{m^2} + \varepsilon\frac{2(s+1)k^2\tau^2}{\lambda m^2}\right) \\ &< 0. \end{aligned}$$

Therefore $i_1(\varphi) \ge m$. Consequently, system (1.1) admits at least $2m$ non-zero $2k\tau$-periodic solutions. □

Next we provide an example to illustrate the applicability of our result.

Example 3.2 Consider (1.1) with $s = 2$, $p(t) = q(t) = (2 + \sin\frac{2\pi t}{\tau})$,

$$\begin{aligned} &f\bigl(t, u(t), \ldots, u(t-4\tau)\bigr) \\ &\quad = 6u(t-2\tau) - 4\left(2 + \cos\frac{2\pi t}{\tau}\right)u(t-2\tau) \\ &\qquad \times \bigl(u^2(t) + 2u^2(t-\tau) + 3u^2(t-2\tau) + 2u^2(t-3\tau) + u^2(t-4\tau)\bigr) \end{aligned}$$

and

$$F(t, u_1, u_2, u_3) = u_1^2 + u_2^2 + u_3^2 - \left(2 + \cos\frac{2\pi t}{\tau}\right)\bigl(u_1^2 + u_2^2 + u_3^2\bigr)^2.$$

It is easy to verify that f, F satisfy the assumptions of Theorem 3.1. Therefore system (1.1) admits at least $2m$ non-zero solutions with the period $2k\tau$. Note $u = 0$ is also a solution of system (1.1).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors made an equal contribution.

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References

1. Hale, JK: Theory of Functional Differential Equations. Springer, New York (1977)
2. Kuang, Y: Delay Differential Equations with Applications in Population Dynamics. Academic Press, New York (1993)
3. Olach, R: Positive periodic solutions of delay differential equations. Appl. Math. Lett. **26**, 1141-1145 (2013)
4. Yu, JS, Xiao, HF: Multiple periodic solutions with minimal period 4 of the delay differential equation $\dot{x} = -f(t, x(t-1))$. J. Differ. Equ. **254**, 2158-2172 (2013)
5. Sella, E: Periodic solutions for some nonlinear differential equations of neutral type. Nonlinear Anal. **17**, 139-151 (1991)
6. Yang, BX, Ma, RY, Gao, CH: Positive periodic solutions of delayed differential equations. Appl. Math. Comput. **218**, 4538-4545 (2011)
7. Li, XM, Yuan, XP: Quasi-periodic solutions for perturbed autonomous delay differential equations. J. Differ. Equ. **252**, 3752-3796 (2012)
8. Wu, K, Wu, X: Multiplicity results of periodic solutions for a class of first order delay differential equations. J. Math. Anal. Appl. **390**, 427-438 (2012)
9. Wu, J, Wang, ZC: Two periodic solutions of second-order neutral functional differential equations. J. Math. Anal. Appl. **329**, 677-689 (2007)
10. Xiao, HF: Multiplicity and minimality of periodic solutions to delay differential system. Electron. J. Differ. Equ. **39**, 1 (2014)
11. Shu, XB, Xu, YT, Huang, LH: Infinite periodic solutions to a class of second-order Sturm-Liouville neutral differential equations. Nonlinear Anal. **68**, 905-911 (2008)
12. Shu, XB, Xu, YT: Multiple periodic solutions to a class of second-order functional differential equations of mixed type. Acta Math. Appl. Sin. **5**, 821-831 (2006)
13. Mawhin, J, Willem, M: Critical Point Theory and Hamiltonian Systems. Springer, Berlin (1989)
14. Chang, KC: Critical Point Theory and Its Applications. Shanghai scientific and Technical Publishers, Shanghai (1986)