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Existence of least energy sign-changing solution for the nonlinear Schrödinger system with two types of nonlocal terms

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Abstract

In the paper, we are concerned with the system of Kirchhoff-Schrödinger-Poisson system under certain assumptions on V_1 , V_2 , K and f . We are interested in the existence of least energy sign-changing solutions to the system on \mathbb{R}^N . Because two kinds of nonlocal terms ϕ_u and $\int_{\mathbb{R}^N} |\nabla u|^2$ are involved in the system, the methods are different from the Kirchhoff or the Schrödinger-Poisson system. The two nonlocal terms $\int_{\mathbb{R}^N} |\nabla u|^2$ and ϕ_u make that the functional $J(u^+ + u^-) \neq J(u^+) + J(u^-)$. Moreover, the nonlocal term ϕ_u does not have the convergence property because of the assumption V_2 . In addition, the convergence of these two nonlocal terms are different. In the present paper, we unify the increasing property conditions on sign-changing solution in previous papers. We construct a new homotopy operator and then weaken the assumption that f is C^1 to that of f being only continuous. We prove that the system has a sign-changing solution via a constraint variational method combining with Brouwer's degree theory.

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Keywords: Schrödinger-Poisson system; Kirchhoff; sign-changing solution; Brouwer's degree; constraint variational

1 Introduction

In this paper, we consider the nonlinear Kirchhoff-Schrödinger-Poisson system

$$\begin{cases} -(1 + b \int_{\mathbb{R}^N} |\nabla u|^2) \Delta u + V_1(x)u + V_2(x)\phi u = K(x)f(u), & x \in \mathbb{R}^N, \\ -\Delta \phi = V_2(x)u^2, & x \in \mathbb{R}^N, \end{cases} \quad (1.1)$$

for $3 \leq N \leq 5$. System (1.1) with $b = 0$ has been introduced while looking for the existence of standing waves for the Schrödinger equation acting with an electrostatic field. We refer to [1] and the references therein for more details as regards the physics aspects. This problem is called the Schrödinger-Poisson system. If $V_2 = 0$, system (1.1) is the Kirchhoff equation, which is the stationary problem associated to the time-dependent problem, which models small vertical vibrations of an elastic string [2].

In recent years, Schrödinger equations with nonlocal terms have attracted much attention. There are several nonlocal Schrödinger equations such as the Kirchhoff and the Schrödinger-Poisson system, and fractional order differential equations. Because of the

nonlocal effect, the convergence and energy functional in a variational reduction are different from local ones. There have been papers studying fractional order differential equations; see [3–9]. When $b = 0$, (1.1) is the Schrödinger-Poisson system. While $V_2 = 0$, (1.1) is the famous Kirchhoff equation. There has been interest in studying problem (1.1) with $b = 0$ or $V_2 = 0$ under various assumptions on K and V_1 or V_2 . For $b = 0, K(x) = 1$, the existence of positive solutions, ground state solutions, and multiple solutions for (1.1) has been massively addressed in the mathematical literature [10–14]. When $V_2 = 0$, there have been many papers to study the existence of positive solution and infinitely many solutions; see [15–20]. Especially, [16] studied the existence of nonnegative solutions to critical Kirchhoff problem. In addition, the systems with two types of nonlocal terms has been studied; see [8, 21]. In [8], fractional p -Kirchhoff equations was considered. The sign-changing solutions to Kirchhoff problems are also considered. For example, in [22–24], the authors proved the existence of sign-changing solutions for $b = 0, V_2(x) = K(x) = 1$. When $V_2 = 0$, [25] studied the existence of the least energy sign-changing solution to (1.1) with $V_1(x) = K(x) = 1$. However, there have appeared few papers to study the case of $b \neq 0$ and $V_2 \neq 0$; see [21]. In [21], the authors considered infinitely many solutions to (1.1).

Motivated by the above work, the main aim of this paper is to study the existence of a sign-change of (1.1) when the potentials V_1 and K decay to zero as $|x| \rightarrow \infty$. Precisely, we suppose:

(V) $V_1 : \mathbb{R}^N \rightarrow \mathbb{R}$ is a smooth function and there exist $a, c > 0$ and $\tau \in (0, 2)$ such that

$$\frac{a}{1 + |x|^\tau} \leq V_1(x) \leq c, \quad \forall x \in \mathbb{R}^N,$$

and $V_2 \in L^\infty(\mathbb{R}^N) \cup L^{(6-N)/2N}(\mathbb{R}^N)$ is nonnegative;

(K) $K : \mathbb{R}^N \rightarrow \mathbb{R}$ is a smooth function and there exist $\xi > \tau, d > 0$ such that

$$0 < K(x) \leq \frac{d}{1 + |x|^\xi}, \quad \forall x \in \mathbb{R}^N.$$

As regards the function f , we assume $f \in C(\mathbb{R}, \mathbb{R})$ and we have the following hypotheses:

- (f₁) $f(t) = o(|t|)$ as $t \rightarrow 0$;
- (f₂) $\lim_{t \rightarrow \infty} \frac{f(t)}{t^3} = +\infty$, where $F(t) = \int_0^t f(s) ds$;
- (f₃) there exists a $\theta \in [0, 1)$ such that $\frac{K(x)f(t) - \theta V(x)t}{|t|^3}$ is nondecreasing on $(-\infty, 0)$ and $(0, \infty)$, respectively;
- (f₄) $|f(t)| \leq C(|t| + |t|^p), p > 3$.

The conditions (f₁)-(f₄) with $\theta = 0$ are usual for the Kirchhoff equation or the Schrödinger-Poisson system; see [22, 25]. The condition (f₃) is equivalent to the one in [26, 27]

$$(V') \quad K(x) \left[\frac{f(\tau)}{\tau^3} - \frac{f(t\tau)}{(t\tau)^3} \right] \text{sign}(1 - t) + \theta V(x) \frac{|1-t^2|}{(t\tau)^2} \geq 0, \quad t > 0, \tau \neq 0.$$

In fact, we have the next remark.

Remark 1.1 Condition (f₃) is equivalent to (V').

Proof We only prove the case of $t > 0$. For $t \in (0, 1]$, $t\tau \leq \tau$, then

$$\begin{aligned} & K(x) \left[\frac{f(\tau)}{\tau^3} - \frac{f(t\tau)}{(t\tau)^3} \right] \text{sign}(1-t) + \theta V(x) \frac{|1-t^2|}{(t\tau)^2} \\ &= K(x) \left[\frac{f(\tau)}{\tau^3} - \frac{f(t\tau)}{(t\tau)^3} \right] + \theta V(x) \frac{1-t^2}{(t\tau)^2} \\ &= \frac{K(x)f(\tau) - \theta V(x)\tau}{\tau^3} - \frac{K(x)f(t\tau) - \theta V(x)t\tau}{(t\tau)^3}. \end{aligned}$$

If $t > 1$, then by $t\tau \geq \tau$, we have

$$\begin{aligned} & K(x) \left[\frac{f(\tau)}{\tau^3} + \frac{f(t\tau)}{(t\tau)^3} \right] \text{sign}(1-t) + \theta V(x) \frac{|1-t^2|}{(t\tau)^2} \\ &= -K(x) \left[\frac{f(\tau)}{\tau^3} + \frac{f(t\tau)}{(t\tau)^3} \right] - \theta V(x) \frac{1-t^2}{(t\tau)^2} \\ &= -\frac{K(x)f(\tau) - \theta V(x)\tau}{\tau^3} + \frac{K(x)f(t\tau) - \theta V(x)t\tau}{(t\tau)^3}. \end{aligned}$$

Therefore, the conclusion holds. □

In addition, we must notice that the condition on V_2 cannot ensure the nonlocal term $\phi_{u_n} \rightarrow \phi_u$ if $u_n \rightharpoonup u$ in $L^{2N/(N-2)}$. The least energy sign-changing solution to the Kirchhoff-Schrödinger-Poisson system has not been studied under the conditions (V) and (K). We unify the conditions $\theta = 0$ and $\theta > 0$ and generalize to the problem (1.1). Throughout this paper, we consider the weighted space L^q_K of measurable $u : \mathbb{R}^N \rightarrow \mathbb{R}$ such that

$$|u|_{q,K} := \left[\int_{\mathbb{R}^N} K(x) |u(x)|^q dx \right]^{\frac{1}{q}} < \infty.$$

If K is a constant, L^p_K is the usual space L^p and the norm is denoted by $|\cdot|_p$. The weighted Sobolev spaces E defined by setting

$$E = \left\{ u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_1(x)u^2 dx < +\infty \right\},$$

make E a Hilbert space with the inner product and the norm

$$(u, v) = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V_1(x)uv) dx, \quad \|u\| = (u, u)^{\frac{1}{2}}.$$

Then we have the following embedding theorem, which is often used later.

Proposition 1.2 ([28]) *Let $N \geq 3$ and suppose that (V), (K) hold with $\tau \in (0, 2]$ and $\xi > 0$, respectively. Then, for $p \in [1, \frac{N+2}{N-2}]$, there is $C > 0$ such that*

$$|u|_{p+1,K} \leq C \|u\|, \quad u \in E.$$

Furthermore, the embedding $E \hookrightarrow L^{p+1}_K$ is compact for $p \in (1, \frac{N+2}{N-2})$.

Let $g(x, t) = K(x)f(t) - \theta V(x)t$ and $G(x, t) = \int_0^t g(x, s) ds$, then we must notice the following fact.

Remark 1.3 If $u_n \rightharpoonup u$ in E , then $\int_{\mathbb{R}^3} KF(u_n) \rightarrow \int_{\mathbb{R}^3} KF(u)$ and $\int_{\mathbb{R}^3} Kf(u_n)u_n \rightarrow \int_{\mathbb{R}^3} Kf(u)u$, for the proof see Lemma 2.1. However, $G(x, t)$ and $g(x, t)$ do not satisfy these properties, due to the term $\int_{\mathbb{R}^3} V(x)u_n^2$. Therefore, we cannot replace $Kf(u) - \theta V(x)u$ with $g(x, u)$.

For every $u \in E$, the Lax-Milgram theorem implies that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^N)$, satisfying $-\Delta\phi_u = V_2u^2$. Thus, we can define the corresponding functional to (1.1) as follows:

$$\begin{aligned} J(u) &:= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_1(x)u^2) + \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^2 + \frac{1}{4} \int_{\mathbb{R}^N} V_2(x)\phi_u u^2 - \int_{\mathbb{R}^N} KF(u) \\ &= \frac{1}{2} \|u\|^2 + \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^2 + \frac{1}{4} \int_{\mathbb{R}^N} V_2(x)\phi_u u^2 - \int_{\mathbb{R}^N} KF(u). \end{aligned}$$

Here $G(x, u) = \int_0^u g(x, t) dt$. It is obvious that $J \in C^1(E, \mathbb{R})$ and

$$\begin{aligned} \langle J'(u), \varphi \rangle &= \int_{\mathbb{R}^N} (\nabla u \nabla \varphi + V_1(x)u\varphi) + b \int_{\mathbb{R}^N} |\nabla u|^2 \int_{\mathbb{R}^N} \nabla u \cdot \nabla \varphi \\ &\quad + \int_{\mathbb{R}^N} V_2(x)\phi_u u\varphi - \int_{\mathbb{R}^N} Kf(u)\varphi \\ &= (u, \varphi) + b \int_{\mathbb{R}^N} |\nabla u|^2 \int_{\mathbb{R}^N} \nabla u \cdot \nabla \varphi + \int_{\mathbb{R}^N} V_2(x)\phi_u u\varphi - \int_{\mathbb{R}^N} K(x)f(u)\varphi. \end{aligned}$$

Clearly, critical points of J are the weak solutions to nonlocal problem (1.1). Furthermore, if $u \in E$ is a solution of (1.1) and $u^\pm \neq 0$, then u is sign-changing, where

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}.$$

In order to get a least energy sign-changing solution for the system (1.1), we first try to seek a minimizer of the energy functional J with the following constraint:

$$M := \{u \in E : u^\pm \neq 0, \langle J'(u), u^+ \rangle = \langle J'(u), u^- \rangle = 0\} \tag{1.2}$$

and then we show that the minimizer is a sign-changing solution of (1.1). But in our problem (1.1), since f is not C^1 , which is assumed in [22, 25], it is rather difficult to show that $M \neq \emptyset$. Thus, we must introduce some new ideas to get a sign-changing solution for the problem (1.1). In the present paper, we use a new homotopy operator which is C^1 .

Now, we state our main theorem.

Theorem 1.4 *If we have the assumptions (V), (K), and (f₁)-(f₄), then the problem (1.1) possesses at least one sign-changing solution which is the least energy sign-changing solution.*

The paper is organized as follows. In Section 2, we prove several lemmas, which are crucial to prove our main results. In Section 3, we show that the minimizer of the constrained problem is a sign-changing solution by a new homotopy operator.

2 Preliminaries

Firstly, we give the convergence of the nonlinear term.

Lemma 2.1 *Assume that $u_n \rightharpoonup u$ in E , then $\int_{\mathbb{R}^N} K(x)f(u_n)u_n \rightarrow \int_{\mathbb{R}^N} K(x)f(u)u$ and $\int_{\mathbb{R}^3} K(x)F(u_n) \rightarrow \int_{\mathbb{R}^3} K(x)F(u)$.*

Proof By Proposition 1.2, $u_n \rightarrow u$ strongly in L_K^q for $q \in [2, 6)$. Thus after passing to a subsequence, $u_n(x) \rightarrow u(x)$ for a.e. $x \in \mathbb{R}^N$. From the convergence of $u_n \rightarrow u$ in L_K^2 and L_K^{p+1} , there exist two functions $a \in L^2$ and $b \in L^{p+1}$ such that $K(x)u_n^2 \leq a^2(x)$ and $K(x)|u_n|^{p+1} \leq [b(x)]^{p+1}$ for a.e. $x \in \mathbb{R}^N$. Therefore, (f₄) implies that

$$\begin{aligned} \left| \int_{\mathbb{R}^N} K(x)f(u_n)u_n \right| &\leq \int_{\mathbb{R}^N} C|K(x)|u_n|^2 + CK(x)|u_n|^{p+1} \\ &\leq \int_{\mathbb{R}^N} [C(a(x))^2 + C(b(x))^{p+1}] dx. \end{aligned}$$

According to the dominated convergence theorem, $\int_{\mathbb{R}^3} K(x)f(u_n)u_n \rightarrow \int_{\mathbb{R}^3} K(x)f(u)u$. A similar argument implies that $\int_{\mathbb{R}^3} K(x)F(u_n) \rightarrow \int_{\mathbb{R}^3} K(x)F(u)$. The proof is completed. \square

The next lemma shows that $M \neq \emptyset$.

Lemma 2.2 *Assume that (f₁)-(f₄) hold. If $u \in E$ with $u^\pm \neq 0$, then there is a unique pair (t, s) of positive numbers such that $tu^+ + su^- \in M$.*

Proof Fix $u \in E$ with $u^\pm \neq 0$, $\int_{\mathbb{R}^N} \nabla \phi_{u^+} \cdot \nabla \phi_{u^-} = \int_{\mathbb{R}^N} V_2(x)\phi_{u^+}(u^-)^2$; since $-\Delta \phi_{u^-} = V_2(x)(u^-)^2$, we can obtain

$$\int_{\mathbb{R}^N} V_2(x)\phi_{u^+}|u^-|^2 dx = \int_{\mathbb{R}^N} V_2(x)\phi_{u^-}|u^+|^2 dx = \int_{\mathbb{R}^N} \nabla \phi_{u^+} \cdot \nabla \phi_{u^-}.$$

Denote $A := b \int_{\mathbb{R}^N} |\nabla u^+|^2 dx \int_{\mathbb{R}^N} |\nabla u^-|^2 dx + \int_{\mathbb{R}^N} V_2(x)\phi_{u^-}|u^+|^2 dx$ for convenience. We introduce the following notations:

$$a_1 = \|u^+\|^2, \quad a_2 = b \left(\int_{\mathbb{R}^N} |\nabla u^+|^2 \right)^2; \tag{2.1}$$

$$b_1 = \|u^-\|^2, \quad b_2 = b \left(\int_{\mathbb{R}^N} |\nabla u^-|^2 \right)^2. \tag{2.2}$$

Since $-\Delta \phi_u = V_2(x)u^2 = V_2(x)(u^+)^2 + V_2(x)(u^-)^2 = -\Delta(\phi_{u^+} + \phi_{u^-})$, $\phi_u = \phi_{u^+} + \phi_{u^-}$. Then, $tu^+ + su^-$ is contained in M if and only if

$$\begin{cases} g_1(t, s) := a_1t^2 + a_2t^4 + At^2s^2 - \int_{\mathbb{R}^N} Kf(tu^+)(tu^+) = 0, \\ g_2(t, s) := b_1s^2 + b_2s^4 + At^2s^2 - \int_{\mathbb{R}^N} Kf(su^-)(su^-) = 0. \end{cases} \tag{2.3}$$

We have from (f₁), (f₂), and (2.3) $g_1(t, t) > 0$ and $g_2(t, t) > 0$ for $t > 0$ small and $g_1(s, s) < 0$, $g_2(s, s) < 0$ for $s > 0$ large. Thus there exist $0 < \delta < R$ such that

$$g_1(\delta, \delta) > 0, \quad g_1(R, R) < 0; \quad g_2(\delta, \delta) < 0, \quad g_2(R, R) < 0. \tag{2.4}$$

From (2.3), it follows that $g_1(\delta, s), g_1(R, s)$ both are increasing for $s \geq 0$ and $g_2(t, R), g_2(t, \delta)$ both are increasing for $t \geq 0$. Therefore, by (2.4), we have

$$g_1(\delta, s) > 0, \quad g_1(R, s) < 0, \quad s \in [\delta, R], \tag{2.5}$$

and

$$g_2(t, \delta) > 0, \quad g_2(t, R) < 0, \quad t \in [\delta, R]. \tag{2.6}$$

By (2.5), (2.6), and Miranda’s theorem [29], there exists a point (t_u, s_u) with $t_u, s_u \in (\delta, R)$ such that $g_1(t_u, s_u) = g_2(t_u, s_u) = 0$.

In the following, we prove the uniqueness of (t, s) . If there are two pairs (t_1, s_1) and (t_2, s_2) such that $t_1 u^+ + s u^- \in M$ and $t_2 u^+ + s u^- \in M$, then $(t_1/t_2)t_2 u^+ + (s_1/s_2)s_2 u^- \in M$. Therefore, we may assume that $u \in M$ and $tu^+ + su^- \in M$. We need only to prove that $t = s = 1$. We assume that $0 < t \leq s$. Then, by (2.3) and $\theta < 1$, we have $b_1 - \theta \int_{\mathbb{R}^N} V(x)(u^-)^2 > 0$ and

$$\left[b_1 - \theta \int_{\mathbb{R}^N} V(x)(u^-)^2 \right] \frac{1}{s^2} + b_2 + A \geq \int_{\mathbb{R}^N} \frac{[Kf(su^-) - \theta V(x)su^-]}{(su^-)^3} (u^-)^4.$$

Since $u \in M$,

$$\begin{aligned} & \left[b_1 - \theta \int_{\mathbb{R}^N} V(x)(u^-)^2 \right] \left[\frac{1}{s^2} - 1 \right] \\ & \geq \int_{\mathbb{R}^N} \left\{ \frac{[Kf(su^-) - \theta V(x)su^-]}{(su^-)^3} - \frac{[Kf(u^-) - \theta V(x)u^-]}{(u^-)^3} \right\} (u^-)^4. \end{aligned}$$

According to condition (f_3) , $s \leq 1$. On the other hand, since $t \leq s$ and $u \in M$, similarly, we can obtain

$$\begin{aligned} & \left[a_1 - \theta \int_{\mathbb{R}^N} V(x)(u^+)^2 \right] \left[\frac{1}{t^2} - 1 \right] \\ & \leq \int_{\mathbb{R}^N} \left\{ \frac{[Kf(tu^+) - \theta V(x)tu^+]}{(tu^+)^3} - \frac{[Kf(u^+) - \theta V(x)u^+]}{(u^+)^3} \right\} (u^+)^4. \end{aligned}$$

Therefore, $t \geq 1$. Consequently, $t = s = 1$. □

Lemma 2.3 *For $u \in E$ with $u^\pm \neq 0$, the pair (t_u, s_u) obtained in Lemma 2.2 is the maximum point of the function $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ defined as $\psi(t, s) = J(tu^+ + su^-)$, where $\mathbb{R}_+ = [0, \infty)$.*

Proof From the proof of Lemma 2.2, (t_u, s_u) is a critical point of ψ in $\mathbb{R}_+ \times \mathbb{R}_+$. By assumptions (f_1) , (f_2) , and the proof of Lemma 2.2, we deduce that $J(tu^+ + su^-) > 0$ for $t, s > 0$ small and $\psi(t, s) \rightarrow -\infty$ as $|(t, s)| \rightarrow \infty$. Therefore, (t_u, s_u) is the maximum point. In the following, we prove that $t_u, s_u > 0$. It is sufficient to check that a maximum point cannot be achieved on the boundary of $\mathbb{R}_+ \times \mathbb{R}_+$. Without loss of generality, we may assume that $(0, s_0)$ is a maximum point of ψ . Then, for $t > 0$ small, by condition (f_1) ,

$$\begin{aligned} \psi(t, s_0) &= J(tu^+ + s_0u^-) \\ &= \frac{t^2}{2} \|u^+\|^2 + \frac{t^4}{4} b |\nabla u^+|_2^4 + \frac{t^4}{4} \int_{\mathbb{R}^N} V_2(x) \phi_{u^+}(u^+)^2 - \int_{\mathbb{R}^N} K(x) F(tu^+) dx \end{aligned}$$

$$\begin{aligned}
 & + \frac{t^2 s_0^2}{2} b |\nabla u^+|_2^2 |\nabla u^-|_2^2 + \frac{t^2 s_0^2}{2} \int_{\mathbb{R}^N} V_2(x) \phi_{u^+}(u^-)^2 dx \\
 & + \frac{s_0^2}{2} \|u^-\|^2 + \frac{s_0^4}{4} b |\nabla u^-|_2^4 + \frac{s_0^4}{4} \int_{\mathbb{R}^N} V_2(x) \phi_{u^-}(u^-)^2 - \int_{\mathbb{R}^N} K(x) F(s_0 u^-) dx \\
 & > \frac{s_0^2}{2} \|u^-\|^2 + \frac{s_0^4}{4} b |\nabla u^-|_2^4 + \frac{s_0^4}{4} \int_{\mathbb{R}^N} V_2(x) \phi_{u^-}(u^-)^2 - \int_{\mathbb{R}^N} K(x) F(s_0 u^-) dx \\
 & = \psi(0, s_0).
 \end{aligned}$$

The pair $(0, s_0)$ is not a maximum point of ψ in $\mathbb{R}_+ \times \mathbb{R}_+$. □

Lemma 2.4

- (i) If $\langle J'(u), u^\pm \rangle < 0$, then $(t_u, s_u) \in (0, 1)^2$.
- (ii) If $u^\pm \neq 0$ is such that $\langle J'(u), u^\pm \rangle \leq 0$, then $(t_u, s_u) \in (0, 1]^2$.
- (iii) If $\langle J'(u), u^\pm \rangle > 0$, then $(t_u, s_u) \in (1, \infty)^2$.

Proof Under the assumptions of (i), (ii), and (iii), we have $u^\pm \neq 0$. Let $c_1 = a_1 - \theta \times \int_{\mathbb{R}^N} V(x)(u^+)^2$ and $d_1 = b_1 - \theta \int_{\mathbb{R}^N} V(x)(u^-)^2$, then $c_1, d_1 > 0$. Since $t_u u^+ + s_u u^- \in M$, we have from (2.3)

$$\begin{aligned}
 \langle J'(u), u^+ \rangle & = c_1 \left[1 - \frac{1}{t_u^2} \right] + A \left[1 - \frac{s_u^2}{t_u^2} \right] \\
 & \quad - \int_{\mathbb{R}^N} \left[\frac{K(x)f(u^+) - \theta V(x)(u^+)}{(u^+)^3} - \frac{K(x)f(t_u u^+) - \theta V(x)(t_u u^+)}{(t_u u^+)^3} \right] (u^+)^4 \quad (2.7)
 \end{aligned}$$

and

$$\begin{aligned}
 \langle J'(u), u^- \rangle & = d_1 \left[1 - \frac{1}{s_u^2} \right] + A \left[1 - \frac{t_u^2}{s_u^2} \right] \\
 & \quad - \int_{\mathbb{R}^N} \left[\frac{K(x)f(u^-) - \theta V(x)(u^-)}{(u^-)^3} - \frac{K(x)f(s_u u^-) - \theta V(x)(s_u u^-)}{(s_u u^-)^3} \right] (u^-)^4. \quad (2.8)
 \end{aligned}$$

For convenience, we introduce some notations as follows:

$$\begin{aligned}
 D_{11} & = c_1 \left[1 - \frac{1}{t_u^2} \right] + A \left[1 - \frac{s_u^2}{t_u^2} \right], \\
 D_{12} & = \int_{\mathbb{R}^N} \left[\frac{K(x)f(u^+) - \theta V(x)(u^+)}{(u^+)^3} - \frac{K(x)f(t_u u^+) - \theta V(x)(t_u u^+)}{(t_u u^+)^3} \right] (u^+)^4, \\
 D_{21} & = d_1 \left[1 - \frac{1}{s_u^2} \right] + A \left[1 - \frac{t_u^2}{s_u^2} \right], \\
 D_{22} & = \int_{\mathbb{R}^N} \left[\frac{K(x)f(u^-) - \theta V(x)(u^-)}{(u^-)^3} - \frac{K(x)f(s_u u^-) - \theta V(x)(s_u u^-)}{(s_u u^-)^3} \right] (u^-)^4.
 \end{aligned}$$

Without loss of generality, we may assume that $t_u \geq s_u$.

(i) If $t_u \geq 1$, then $D_{11} \geq 0$ and by the condition (f_3) , we have $D_{12} \leq 0$. It follows from (2.7) that $\langle J'(u), u^+ \rangle \geq 0$. This is a contradiction. Thus, $(t_u, s_u) \in (0, 1)^2$.

(ii) If $t_u > 1$, then $D_{11} > 0$ and by the condition (f_3) , we have $D_{12} \leq 0$. Therefore, $\langle J'(u), u^+ \rangle > 0$. This is a contradiction. Thus, $(t_u, s_u) \in (0, 1]^2$.

(iii) If $s_u \leq 1$, similarly $D_{21} \leq 0$ and $D_{22} \geq 0$. It follows from (2.8) that $\langle J'(u), u^- \rangle \leq 0$, which is a contradiction. Thus, $(t_u, s_u) \in (1, \infty)^2$. \square

By Lemma 2.3, we may define the minimization problem

$$m = \inf\{J(u) : u \in M\}. \tag{2.9}$$

Lemma 2.5 *Suppose that conditions (f₁)-(f₄) hold, then $m > 0$ can be achieved.*

Proof For any $u \in M$, we have $\langle J'(u), u \rangle = 0$. Then we get from conditions (f₁) and (f₄)

$$\begin{aligned} \|u\|^2 &\leq \int_{\mathbb{R}^N} (|\nabla u|^2 + V_1(x)u^2 + V_2(x)\phi_u u^2) dx + b|\nabla u|_2^4 \\ &= \int_{\mathbb{R}^N} K(x)f(u)u \\ &\leq \varepsilon \int_{\mathbb{R}^N} K(x)u^2 + C_\varepsilon \int_{\mathbb{R}^N} K(x)u^{p+1}. \end{aligned}$$

Then by Proposition 1.2, we can get

$$\|u\|^2 \leq C_1\varepsilon\|u\|^2 + C_2\|u\|^{p+1},$$

where $C_1, C_2 > 0$ are constants. So there exists a constant $\alpha > 0$ such that $\|u\| \geq \alpha$. Moreover, by (f₃), we have

$$\left[K(x)f(t)t - \theta V(x)t^2 \right] - 4 \left[K(x)F(t) - \theta \frac{1}{2} V(x)t^2 \right] \geq 0, \quad x \in \mathbb{R}^N, t \in \mathbb{R}.$$

Hence, we have

$$\begin{aligned} J(u) &= J(u) - \frac{1}{4}\langle J'(u), u \rangle \\ &= \frac{1}{4}\|u\|^2 + \frac{1}{4} \int_{\mathbb{R}^N} K(x)(f(u)u - 4F(u)) \\ &\geq \frac{1}{4}\|u\|^2 - \frac{1}{4}\theta \int_{\mathbb{R}^N} V(x)u^2 \\ &\geq \frac{1}{4}(1 - \theta)\alpha^2. \end{aligned}$$

This implies that $m \geq \frac{1}{4}(1 - \theta)\alpha^2 > 0$.

Let $\{u_n\} \subset M$ be such that $J(u_n) \rightarrow m$. Then $\{u_n\}$ is bounded in E , so there exists a subsequence of $\{u_n\}$ and $u \in E$, we may still denote the subsequence by $\{u_n\}$, such that $u_n \rightharpoonup u$ and $u_n^\pm \rightharpoonup u^\pm$ weakly in E and $u_n(x) \rightarrow u(x)$ a.e. $x \in \mathbb{R}^N$. Since $\{u_n\} \in M$, we have $\langle J'(u_n), u_n^+ \rangle = 0$. That is,

$$\|u_n^+\|^2 + b|\nabla u_n|_2^2 |\nabla u_n^+|_2^2 + \int_{\mathbb{R}^N} V_2(x)\phi_{u_n}(u_n^+)^2 = \int_{\mathbb{R}^N} Kf(u_n^+)u_n^+. \tag{2.10}$$

This implies from conditions (f_1) and (f_4) that there exists a constant $\rho > 0$ such that $\|u_n^+\| \geq \rho$ for all $n \in \mathbb{N}$. By (2.10) and Lemma 2.2, we get

$$0 < \rho^2 \leq \|u_n^+\|^2 \leq \int_{\mathbb{R}^N} Kf(u_n^+)u_n^+ = \int_{\mathbb{R}^N} Kf(u^+)u^+ + o(1),$$

where $o(1)$ denotes a quantity tending to zero as $n \rightarrow +\infty$. Thus, $u^+ \neq 0$. Similarly, we have $u^- \neq 0$.

Since $u_n \rightarrow u$ in E , we have $\phi_{u_n} \rightarrow \phi_u$ in $D^{1,2}(\mathbb{R}^N)$ if $V_2 \in L^\infty(\mathbb{R}^N)$ or $\phi_{u_n} \rightarrow \phi_u$ in $D^{1,2}(\mathbb{R}^N)$ if $V_2 \in L^{(6-N)/2N}(\mathbb{R}^N)$. By the weak lower semicontinuity of the norm and Fatou's lemma, we have

$$\begin{aligned} \|u^+\|^2 + b|\nabla u|_2^2|\nabla u^+|_2^2 + \int_{\mathbb{R}^N} V_2(x)\phi_u|u^+|^2 dx \\ \leq \liminf_{n \rightarrow +\infty} \left(\|u_n^+\|^2 + b|\nabla u_n|_2^2|\nabla u_n^+|_2^2 + \int_{\mathbb{R}^N} V_2(x)\phi_{u_n}|u_n^+|^2 dx \right). \end{aligned}$$

Then from (2.10) and Lemma 2.1 we get

$$\|u^+\|^2 + b|\nabla u|_2^2|\nabla u^+|_2^2 + \int_{\mathbb{R}^N} V_2(x)\phi_u|u^+|^2 dx \leq \int_{\mathbb{R}^N} Kf(u^+)u^+. \tag{2.11}$$

Similarly, we have

$$\|u^-\|^2 + b|\nabla u|_2^2|\nabla u^-|_2^2 + \int_{\mathbb{R}^N} V_2(x)\phi_u|u^-|^2 dx \leq \int_{\mathbb{R}^N} Kf(u^-)u^-. \tag{2.12}$$

From (2.11), (2.12), and the proof of Lemma 2.4, there exists $(\bar{t}, \bar{s}) \in (0, 1] \times (0, 1]$ such that $\bar{u} = \bar{t}u^+ + \bar{s}u^- \in M$. Moreover, according to condition (f_3) we have $[K(x)f(t)t - \theta V(x)t^2] - 4[K(x)F(t) - \frac{1}{2}\theta V(x)t^2] \geq 0$ is nondecreasing on $(-\infty, 0)$ and $(0, \infty)$, respectively. Thus,

$$\begin{aligned} m &\leq J(\bar{u}) = J(\bar{u}) - \frac{1}{4}\langle J'(\bar{u}), \bar{u} \rangle \\ &= \frac{1}{4}\|\bar{u}\|^2 + \frac{1}{4} \int_{\mathbb{R}^N} K(f(\bar{u})\bar{u} - 4F(\bar{u})) dx \\ &= \frac{1}{4}(\|\bar{t}u^+\|^2 + \|\bar{s}u^-\|^2) + \frac{1}{4} \int_{\mathbb{R}^N} K(f(\bar{u})\bar{u} - 4F(\bar{u})) dx \\ &\leq \frac{1}{4}|\nabla u|_2^2 + \frac{1}{4} \int_{\mathbb{R}^N} V(x)\bar{u}^2 + \frac{1}{4} \int_{\mathbb{R}^N} K(f(u)u - 4F(u)) dx + \frac{1}{4}\theta \int_{\mathbb{R}^N} V(x)[u^2 - \bar{u}^2] \\ &\leq \frac{1}{4}|\nabla u|_2^2 + \frac{1}{4}(1-\theta) \int_{\mathbb{R}^N} V(x)u^2 + \frac{1}{4} \int_{\mathbb{R}^N} [K(f(u)u - 4F(u)) + \theta V(x)u^2] \\ &\leq \liminf_{n \rightarrow \infty} \left[J(u_n) - \frac{1}{4}\langle J'(u_n), u_n \rangle \right] = m. \end{aligned}$$

By the above inequality and Lemma 2.3 we deduce that $\bar{t} = \bar{s} = 1$. Thus $\bar{u} = u$ and $J(u) = m$. □

3 Existence of sign-changing solution

In this section, we mainly prove that the minimizer u obtained in Lemma 2.5 is indeed a sign-changing solution of (1.1), that is, $J'(u) = 0$.

Proof of Theorem 1.4 Using the quantitative deformation lemma [30], Lemma 2.3, we prove that $J'(u) = 0$. It is clear that $\langle J'(u), u^+ \rangle = \langle J'(u), u^- \rangle = 0$. It follows from Lemma 2.3 that, for $(t, s) \in \mathbb{R}_+ \times \mathbb{R}_+$ and $(t, s) \neq (1, 1)$,

$$J(tu^+ + su^-) < J(u^+ + u^-) = m. \tag{3.1}$$

If $J'(u) \neq 0$, then there exist $\delta > 0$ small and $\varepsilon_0 > 0$ such that, for all $\|v - u\| \leq 3\delta$,

$$\|J'(v)\| \geq \varepsilon_0.$$

Let $D := (1 - \delta/\|u\|, 1 + \delta/\|u\|) \times (1 - \delta/\|u\|, 1 + \delta/\|u\|)$ and $h(t, s) = tu^+ + su^-$. It follows from Lemma 2.3 again that

$$\bar{m} := \max_{\partial D} J \circ h < m. \tag{3.2}$$

By [30], Lemma 2.3, for $\varepsilon := \min\{(m - \bar{m})/2, \varepsilon_0\delta/8\}$ and $S := B(u, \delta)$, there is a deformation η such that

- (a) $\eta(1, u) = u$ if $u \notin J^{-1}([m - 2\varepsilon, m + 2\varepsilon]) \cap S_{2\delta}$;
- (b) $\eta(1, J^{m+\varepsilon} \cap S) \subset J^{m-\varepsilon}$;
- (c) $J(\eta(1, u)) \leq J(u)$ for all $u \in E$.

For $(t, s) \in \bar{D}$, $\|h(t, s) - u\|^2 = (t - 1)^2\|u^+\|^2 + (s - 1)^2\|u^-\|^2 \leq \delta^2$, then $h(t, s) \in S$ for all $(t, s) \in \bar{D}$. On the other hand, $\max_{\bar{D}} J(h(t, s)) = m$, therefore, from (b) we have

$$\max_{(t,s) \in \bar{D}} J(\eta(1, h(t, s))) < m. \tag{3.3}$$

We prove that $\eta(1, h(D)) \cap M \neq \emptyset$, contradicting the definition of m . Let us define $\gamma(t, s) := \eta(1, h(t, s))$ and

$$\begin{aligned} \Psi_0(t, s) &:= (J'(h(t, s))u^+, J'(h(t, s))u^-) = (J'(tu^+ + su^-)u^+, J'(tu^+ + su^-)u^-), \\ \Psi_1(t, s) &:= \left(\frac{1}{t} J'(\gamma(t, s))\gamma^+(t, s), \frac{1}{s} J'(\gamma(t, s))\gamma^-(t, s) \right). \end{aligned}$$

Since $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $\Psi_0 : \bar{D} \rightarrow \mathbb{R}^2$ is continuous. Then we cannot directly compute the topology degree $\text{deg}(\Psi_0, D, 0)$. Now, we define the operator as follows: $\Phi(t, s) := (\phi_1(t, s), \phi_2(t, s))$ where

$$\begin{cases} \phi_1(t, s) = -a_1 t^2 (t^2 - 1), \\ \phi_2(t, s) = -b_1 s^2 (s^2 - 1). \end{cases}$$

Then Φ has unique zero $(t, s) = (1, 1)$. Let us take a homotopy operator

$$H(\tau, (t, s)) = \tau \Psi_0 + (1 - \tau) \Phi.$$

It is easy to see that $H(\tau, (1, 1)) = 0$. Then $H(1, (t, s)) = \Psi_0(t, s)$ and $H(0, (t, s)) = \Phi(t, s)$. Moreover, $H : [0, 1] \times \bar{D}$ is continuous. We will show that $H(\tau, (t, s)) \neq 0$ on $[0, 1] \times \partial D$. In fact, if $\tau = 1$, according to Lemma 2.2 we see that $(1, 1)$ is the unique zero point of

$H(0, (t, s))$. If $\tau = 0$, from the definition of Φ , the conclusion is obvious. In the following, we may assume that $\tau \in (0, 1)$. It is easy to see that for any $\tau \in (0, 1)$, $(1, 1)$ is a zero point of $H(\tau, (t, s))$. Therefore, we need only to prove that the zero point of $H(\tau, (t, s))$ is unique in \bar{D} . If $(t_0, s_0) \in \bar{D}$ is a zero point of $H(\tau, (t, s))$, that is,

$$a_1 \tau t_0^2 + a_1(1 - \tau)t_0^2(1 - t_0^2) + \tau a_2 t_0^4 + \tau A t_0^2 s_0^2 - \tau \int_{\mathbb{R}^N} [Kf(t_0 u^+)(t_0 u^+) - \theta V(x)|t_0 u^+|^2] = 0 \tag{3.4}$$

and

$$b_1 \tau s_0^2 + b_1(1 - \tau)s_0^2(1 - s_0^2) + b_2 \tau s_0^4 + A \tau t_0^2 s_0^2 - \tau \int_{\mathbb{R}^N} [Kf(s_0 u^-)(s_0 u^-) - \theta V(x)|s_0 u^-|^2] = 0. \tag{3.5}$$

We may assume that $t_0 \leq s_0$. Then from (3.5), we have

$$b_1 \left[\frac{\tau}{s_0^2} - \frac{(1 - \tau)(s_0^2 - 1)}{s_0^2} \right] + b_2 \tau + A \tau \geq \tau \int_{\mathbb{R}^N} \frac{Kf(s_0 u^-) - \theta s_0 V(x)(u^-)^2}{s_0^3 (u^-)^3} (u^-)^4.$$

On the other hand, it follows from $u \in M$ that

$$b_1 + b_2 + A = \int_{\mathbb{R}^3} [Kf(u^-)u^- - \theta V(x)(u^-)^2].$$

Therefore,

$$b_1 \left[\frac{1}{s_0^2} - 1 \right] = b_1 \left[\frac{\tau}{s_0^2} - \frac{(1 - \tau)(s_0^2 - 1)}{s_0^2} - \tau \right] \geq \tau \int_{\mathbb{R}^N} \left\{ \frac{Kf(s_0 u^-) - \theta s_0 V(x)(u^-)^2}{s_0^3 (u^-)^3} (u^-)^4 - [Kf(u^-)u^- - \theta V(x)(u^-)^2] \right\}.$$

Hence, we have from the condition (f_3) that $s_0 \leq 1$. In the following, we prove that $t_0 \geq 1$. In fact from (3.4) and $0 < t_0 \leq s_0$, we have

$$a_1 \left[\frac{\tau}{t_0^2} - \frac{(1 - \tau)(t_0^2 - 1)}{t_0^2} \right] + a_2 \tau + A \tau \leq \tau \int_{\mathbb{R}^N} \frac{Kf(t_0 u^+) - \theta V(x)t_0 u^+}{t_0^3 (u^+)^3} (u^+)^4. \tag{3.6}$$

Since $u \in M$, we have from (3.6)

$$a_1 \left[\frac{1}{t_0^2} - 1 \right] = a_1 \left[\frac{\tau}{t_0^2} - \frac{(1 - \tau)(t_0^2 - 1)}{t_0^2} - \tau \right] \leq \tau \int_{\mathbb{R}^N} \frac{Kf(t_0 u^+) - \theta V(x)t_0 u^+}{t_0^3 (u^+)^3} (u^+)^4 - [Kf(u^+)u^+ - \theta V(x)(u^+)^2]. \tag{3.7}$$

Therefore, condition (f_3) implies that $t_0 \geq 1$ and then $s_0 = t_0 = 1$. Hence for $(s, t) \in \partial D$, we obtain $H(\tau, (t, s)) \neq 0$ for all $\tau \in [0, 1]$. Therefore, the homotopy invariance property of Brouwer's degree implies that $\deg(H(\tau, (\cdot, \cdot)), D, 0) = \text{constant}$ for $\tau \in [0, 1]$ and then

$$\deg(\Psi_0, D, 0) = \deg(H(1, (\cdot, \cdot)), D, 0) = \deg(H(0, (\cdot, \cdot)), D, 0) = \deg(\Phi, D, 0).$$

Hence, we need only to compute $\deg(\Phi, D, 0)$. Note that $\Phi \in C^1$, and Φ has an isolated zero point $(1, 1)$ in D . Since

$$\begin{aligned} \frac{\partial \phi_1}{\partial t}(1, 1) &= -2a_1, & \frac{\partial \phi_1}{\partial s}(1, 1) &= 0, \\ \frac{\partial \phi_2}{\partial t}(1, 1) &= 0, & \frac{\partial \phi_2}{\partial s}(1, 1) &= -2b_1, \end{aligned}$$

we have

$$\begin{vmatrix} -2a_1 & 0 \\ 0 & -2b_1 \end{vmatrix} = 4a_1b_1 > 0.$$

Then according to the definition of the topological degree, we have

$$\deg(\Phi, D, 0) = 1.$$

On the other hand, it follows from (3.2) that $h = \gamma$ on ∂D . Consequently, we obtain $\deg(\Psi_1, D, 0) = \deg(\Psi_0, D, 0) = 1$ from the property of the topological degree. Therefore, $\Psi_1(t_0, s_0) = 0$ for some $(t_0, s_0) \in D$, that is, $\eta(1, h(t_0, s_0)) = \gamma(t_0, s_0) \in M$, which is a contradiction with $\eta(1, h(D)) \cap M \neq \emptyset$. From this, u is a critical point of J , and then it is a sign-changing solution for problem (1.1). \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors typed, read, and approved the final manuscript.

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