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# Dynamic Opial diamond- $\alpha$ integral inequalities involving the power of a function

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**Abstract**

In this paper, we present some new dynamic Opial-type diamond alpha inequalities on time scales. The obtained results are related to the function  $f^k$ .

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**Keywords:** Opial-type inequality; time scale

## 1 Introduction

A time scale  $\mathbb{T}$  is an arbitrary nonempty closed subset of real numbers. For  $t \in \mathbb{T}$ , we define the forward jump operator  $\sigma : \mathbb{T} \rightarrow \mathbb{T}$  by  $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$  and the backward jump operator  $\rho : \mathbb{T} \rightarrow \mathbb{T}$  by  $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ . If  $\sigma(t) > t$ , we say that  $t$  is right-scattered, whereas if  $\rho(t) < t$ , we say that  $t$  is left-scattered. Points that are simultaneously right-scattered and left-scattered are said to be isolated. If  $\sigma(t) = t$ , then  $t$  is called right-dense; if  $\rho(t) = t$ , then  $t$  is called left-dense. Points that are right-dense and left-dense at the same time are called dense. The mappings  $\mu, \nu : \mathbb{T} \rightarrow [0, \infty)$ , defined by  $\mu(t) := \sigma(t) - t$  and  $\nu(t) := t - \rho(t)$ , are called the forward and backward graininess function, respectively. If  $\mathbb{T}$  has a left-scattered maximum  $t_1$ , then  $\mathbb{T}^k = \mathbb{T} - \{t_1\}$ , otherwise  $\mathbb{T}^k = \mathbb{T}$ . If  $\mathbb{T}$  has a right-scattered minimum  $t_2$ , then  $\mathbb{T}_k = \mathbb{T} - \{t_2\}$ , otherwise  $\mathbb{T}_k = \mathbb{T}$ . Finally,  $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$ .

**Theorem 1.1** Assume  $f, g : \mathbb{T} \rightarrow \mathbb{R}$  are delta differentiable at  $t \in \mathbb{T}^k$ . Then:

1. The sum  $f + g : \mathbb{T} \rightarrow \mathbb{R}$  is delta differentiable at  $t$  with

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

2. For any constant  $\alpha$ ,  $\alpha f : \mathbb{T} \rightarrow \mathbb{R}$  is delta differentiable at  $t$  with

$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t).$$

3. The product  $fg : \mathbb{T} \rightarrow \mathbb{R}$  is delta differentiable at  $t$  with

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t).$$

**Theorem 1.2** Assume  $f, g : \mathbb{T} \rightarrow \mathbb{R}$  are nabla differentiable at  $t \in \mathbb{T}_k$ . Then:

1. The sum  $f + g : \mathbb{T} \rightarrow \mathbb{R}$  is nabla differentiable at  $t$  with

$$(f + g)^\nabla(t) = f^\nabla(t) + g^\nabla(t).$$

2. For any constant  $\alpha, \alpha f : \mathbb{T} \rightarrow \mathbb{R}$  is nabla differentiable at  $t$  with

$$(\alpha f)^\nabla(t) = \alpha f^\nabla(t).$$

3. The product  $fg : \mathbb{T} \rightarrow \mathbb{R}$  is nabla differentiable at  $t$  with

$$(fg)^\nabla(t) = f^\nabla(t)g(t) + f^\rho(t)g^\nabla(t) = f(t)g^\nabla(t) + f^\nabla(t)g^\rho(t).$$

The following formulas will be used in our paper:

$$(f^{l+1})^\Delta = \left\{ \sum_{k=0}^l f^k (f^\sigma)^{l-k} \right\} f^\Delta, \quad l \in \mathbb{N},$$

$$(f^{l+1})^\nabla = \left\{ \sum_{k=0}^l f^k (f^\rho)^{l-k} \right\} f^\nabla, \quad l \in \mathbb{N}.$$

**Definition 1.3** Let  $0 \leq \alpha \leq 1$  and let  $f$  be both delta and nabla differentiable at  $t \in \mathbb{T}_k^k$ . Then  $f$  is diamond- $\alpha$  differentiable at  $t$  and  $f^{\diamond_\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha)f^\nabla(t)$ .

**Definition 1.4** Let  $a, b \in \mathbb{T}, a < b, f : \mathbb{T} \rightarrow \mathbb{R}$  and  $\alpha \in [0, 1]$ . The diamond- $\alpha$  integral of  $t$  on  $[a, b]_{\mathbb{T}}$  is defined by

$$\int_a^b f(t) \diamond_\alpha t = \alpha \int_a^b f(t) \Delta t + (1 - \alpha) \int_a^b f(t) \nabla t.$$

**Theorem 1.5** Let  $f, g : \mathbb{T} \rightarrow \mathbb{R}$  be  $\diamond_\alpha$ -differentiable at  $t \in \mathbb{T}$ . Then

1.  $f + g$  is  $\diamond_\alpha$ -differentiable at  $t \in \mathbb{T}$  with  $(f + g)^{\diamond_\alpha} = f^{\diamond_\alpha} + g^{\diamond_\alpha}$ ,
2.  $fg$  is  $\diamond_\alpha$ -differentiable at  $t \in \mathbb{T}$  with  $(fg)^{\diamond_\alpha} = f^{\diamond_\alpha}g + \alpha f^\sigma g^\Delta + (1 - \alpha)f^\rho g^\nabla$ .

Many authors have studied the theory of integral inequalities on time scales (see, for example, [1–10]). In [3], the following Opial inequality on time scales was established.

**Theorem 1.6** ([3]) For a delta differentiable  $f : [0, h] \cap \mathbb{T} \rightarrow \mathbb{R}$  with  $f(0) = 0$ , we have

$$\int_0^h |(f + f^\sigma)f^\Delta| \Delta t \leq h \int_0^h |f^\Delta|^2 \Delta t, \tag{1}$$

with equality when  $f(t) = ct$ .

In [1], the authors established the following theorem.

**Theorem 1.7** ([1]) Let  $\omega(t)$  be positive and continuous on  $(0, h)$  with  $\int_0^h \omega^{1-q} \Delta t < \infty, q > 1$ . For a differentiable  $f : [0, h] \rightarrow \mathbb{R}$  with  $f(0) = 0$ , we have

$$\int_0^h |(f + f^\sigma)f^\Delta| \Delta t \leq \left( \int_0^h \omega^{1-q} \Delta t \right)^{\frac{2}{q}} \left( \int_0^h \omega |f^\Delta|^p \Delta t \right)^{\frac{2}{p}},$$

where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , and with equality when  $f(t) = c \int_0^t \omega^{1-q} \Delta \tau$  for a constant  $c$ .

## 2 Main results

In this section, we present our results.

**Theorem 2.1** *Let  $T$  be a time scale. For  $\diamond_\alpha$  differentiable  $f : [0, h] \cap T \rightarrow R$ , with  $f(0) = 0$  we have*

$$\int_0^h |f^k|^{\diamond_\alpha}(t) \diamond_\alpha t \leq h^{k-1} \int_0^h |f^{\diamond_\alpha}|^k(t) \diamond_\alpha t. \tag{2}$$

*Proof* Starting with the left side of (2), we obtain

$$\begin{aligned} \int_0^h |f^k|^{\diamond_\alpha}(t) \diamond_\alpha(t) &= \int_0^h |f \cdot f^{k-1}|^{\diamond_\alpha}(t) \diamond_\alpha(t) \\ &= \int_0^h |f^{k-1} f^{\diamond_\alpha} + \alpha f^\sigma (f^{k-1})^\Delta + (1-\alpha) f^\rho (f^{k-1})^\nabla|(t) \diamond_\alpha(t) \\ &= \alpha \int_0^h |f^{k-1} f^{\diamond_\alpha} + \alpha f^\sigma (f^{k-1})^\Delta + (1-\alpha) f^\rho (f^{k-1})^\nabla|(t) \Delta t \\ &\quad + (1-\alpha) \int_0^h |f^{k-1} f^{\diamond_\alpha} + \alpha f^\sigma (f^{k-1})^\Delta + (1-\alpha) f^\rho (f^{k-1})^\nabla|(t) \nabla t \\ &\leq \alpha \int_0^h |f^{k-1} f^{\diamond_\alpha}|(t) \Delta t + \alpha^2 \int_0^h |f^\sigma (f^{k-1})^\Delta|(t) \Delta t \\ &\quad + \alpha(1-\alpha) \int_0^h |f^\rho (f^{k-1})^\nabla|(t) \Delta t + (1-\alpha) \int_0^h |f^{k-1} f^{\diamond_\alpha}|(t) \nabla t \\ &\quad + \alpha(1-\alpha) \int_0^h |f^\sigma (f^{k-1})^\Delta|(t) \nabla t + (1-\alpha)^2 \int_0^h |f^\rho (f^{k-1})^\nabla|(t) \nabla t. \end{aligned}$$

Using Definition 1.3, we get

$$\begin{aligned} \int_0^h |f^k|^{\diamond_\alpha}(t) \diamond_\alpha(t) &\leq \alpha \int_0^h |\alpha f^{k-1} f^\Delta + (1-\alpha) f^{k-1} f^\nabla|(t) \Delta t \\ &\quad + \alpha^2 \int_0^h |f^\sigma (f^{k-1})^\Delta|(t) \Delta t + \alpha(1-\alpha) \int_0^h |f^\rho (f^{k-1})^\nabla|(t) \Delta t \\ &\quad + (1-\alpha) \int_0^h |\alpha f^{k-1} f^\Delta + (1-\alpha) f^{k-1} f^\nabla|(t) \nabla t \\ &\quad + \alpha(1-\alpha) \int_0^h |f^\sigma (f^{k-1})^\Delta|(t) \nabla t + (1-\alpha)^2 \int_0^h |f^\rho (f^{k-1})^\nabla|(t) \nabla t \\ &\leq \alpha^2 \int_0^h |f^{k-1} f^\Delta|(t) \Delta t + \alpha(1-\alpha) \int_0^h |f^{k-1} f^\nabla|(t) \Delta t \\ &\quad + \alpha^2 \int_0^h |f^\sigma (f^{k-1})^\Delta|(t) \Delta t + \alpha(1-\alpha) \int_0^h |f^\rho (f^{k-1})^\nabla|(t) \Delta t \\ &\quad + \alpha(1-\alpha) \int_0^h |f^{k-1} f^\Delta|(t) \nabla t + (1-\alpha)^2 \int_0^h |f^{k-1} f^\nabla|(t) \nabla t \\ &\quad + \alpha(1-\alpha) \int_0^h |f^\sigma (f^{k-1})^\Delta|(t) \nabla t + (1-\alpha)^2 \int_0^h |f^\rho (f^{k-1})^\nabla|(t) \nabla t. \end{aligned}$$

We find that

$$\begin{aligned}
 \int_0^h |f^\sigma (f^{k-1})^\Delta|(t)\Delta t &= \int_0^h |f^\sigma (f \cdot f^{k-2})^\Delta|(t)\Delta t \\
 &= \int_0^h |f^\sigma (f^\Delta f^{k-2} + f^\sigma (f \cdot f^{k-3})^\Delta)|(t)\Delta t \\
 &\quad \vdots \\
 &= \int_0^h |f^\sigma (f^\Delta f^{k-2} + f^\sigma f^\Delta f^{k-3} + \dots + (f^\sigma)^2 f^\Delta)|(t)\Delta t \\
 &= \int_0^h |f^\sigma f^{k-2} + (f^\sigma)^2 f^{k-3} + \dots + (f^\sigma)|f^\Delta|(t)\Delta t \\
 &= \int_0^h \left| \sum_{n=0}^{k-2} f^n (f^\sigma)^{k-1-n} \right| |f^\Delta|(t)\Delta t.
 \end{aligned}$$

Similarly,

$$\int_0^h |f^\rho (f^{k-1})^\nabla|(t)\Delta t = \int_0^h \left| \sum_{n=0}^{k-2} f^n (f^\rho)^{k-1-n} \right| |f^\nabla|(t)\nabla t.$$

Therefore,

$$\begin{aligned}
 \int_0^h |f^k|^{\diamond_\alpha}(t)\diamond_\alpha t &\leq \alpha^2 \int_0^h |f^{k-1} f^\Delta|(t)\Delta t + \alpha(1-\alpha) \int_0^h |f^{k-1} f^\nabla|(t)\Delta t \\
 &\quad + \alpha^2 \int_0^h \left| \sum_{n=0}^{k-2} f^n (f^\sigma)^{k-1-n} \right| |f^\Delta|(t)\Delta t \\
 &\quad + \alpha(1-\alpha) \int_0^h \left| \sum_{n=0}^{k-2} f^n (f^\rho)^{k-1-n} \right| |f^\nabla|(t)\Delta t \\
 &\quad + \alpha(1-\alpha) \int_0^h |f^{k-1} f^\Delta|(t)\nabla t + (1-\alpha)^2 \int_0^h |f^{k-1} f^\nabla|(t)\nabla t \\
 &\quad + \alpha(1-\alpha) \int_0^h \left| \sum_{n=0}^{k-2} f^n (f^\sigma)^{k-1-n} \right| |f^\Delta|\nabla t \\
 &\quad + (1-\alpha)^2 \int_0^h \left| \sum_{n=0}^{k-2} f^n (f^\rho)^{k-1-n} \right| |f^\nabla|\nabla t \\
 &= \alpha^2 \int_0^h \left( |f^{k-1}| + \left| \sum_{n=0}^{k-2} f^n (f^\sigma)^{k-1-n} \right| \right) |f^\Delta(t)|\Delta t \\
 &\quad + \alpha(1-\alpha) \int_0^h \left( |f^{k-1}| + \left| \sum_{n=0}^{k-2} f^n (f^\rho)^{k-1-n} \right| \right) |f^\nabla(t)|\Delta t \\
 &\quad + \alpha(1-\alpha) \int_0^h \left( |f^{k-1}| + \left| \sum_{n=0}^{k-2} f^n (f^\sigma)^{k-1-n} \right| \right) |f^\Delta(t)|\nabla t
 \end{aligned}$$

$$\begin{aligned}
 & + (1 - \alpha)^2 \int_0^h \left( |f^{k-1}| + \left| \sum_{n=0}^{k-2} f^n (f^\rho)^{k-1-n} \right| \right) |f^\nabla(t)| \nabla t \\
 \leq & \alpha^2 \int_0^h \left( |f^{k-1}| + \sum_{n=0}^{k-2} |f^n (f^\sigma)^{k-1-n}| \right) |f^\Delta(t)| \Delta t \\
 & + \alpha(1 - \alpha) \int_0^h \left( |f^{k-1}| + \sum_{n=0}^{k-2} |f^n (f^\rho)^{k-1-n}| \right) |f^\nabla(t)| \Delta t \\
 & + \alpha(1 - \alpha) \int_0^h \left( |f^{k-1}| + \sum_{n=0}^{k-2} |f^n (f^\sigma)^{k-1-n}| \right) |f^\Delta(t)| \nabla t \\
 & + (1 - \alpha)^2 \int_0^h \left( |f^{k-1}| + \sum_{n=0}^{k-2} |f^n (f^\rho)^{k-1-n}| \right) |f^\nabla(t)| \nabla t \\
 = & \alpha^2 \int_0^h \left( \sum_{n=0}^{k-1} |f^n| |(f^\sigma)^{k-1-n}| \right) |f^\Delta(t)| \Delta t \\
 & + \alpha(1 - \alpha) \int_0^h \left( \sum_{n=0}^{k-1} |f^n| |(f^\rho)^{k-1-n}| \right) |f^\nabla(t)| \Delta t \\
 & + \alpha(1 - \alpha) \int_0^h \left( \sum_{n=0}^{k-1} |f^n| |(f^\sigma)^{k-1-n}| \right) |f^\Delta(t)| \nabla t \\
 & + (1 - \alpha)^2 \int_0^h \left( \sum_{n=0}^{k-1} |f^n| |(f^\rho)^{k-1-n}| \right) |f^\nabla(t)| \nabla t.
 \end{aligned}$$

Consider  $g(t) = \int_0^t |f^{\diamond\alpha}(s)| \diamond_\alpha s$ . Then we have  $g^\Delta(t) = |f^\Delta(t)|$ ,  $g^\nabla(t) = |f^\nabla(t)|$ , and  $|f| \leq g$ , so that  $g(t) = \int_0^t |f^{\diamond\alpha}(s)| \diamond_\alpha s \geq |\int_0^t f^{\diamond\alpha}(s) \diamond_\alpha s| = |f(t) - f(0)| = |f(t)|$ .

The above inequality becomes

$$\begin{aligned}
 \int_0^h |f^k|^{\diamond\alpha}(t) \diamond_\alpha & \leq \alpha^2 \int_0^h \left( \sum_{n=0}^{k-1} g^n (g^\sigma)^{k-1-n} \right) (g^\Delta)(t) \Delta t \\
 & + \alpha(1 - \alpha) \int_0^h \left( \sum_{n=0}^{k-1} g^n (g^\rho)^{k-1-n} \right) (g^\nabla)(t) \Delta t \\
 & + \alpha(1 - \alpha) \int_0^h \left( \sum_{n=0}^{k-1} g^n (g^\sigma)^{k-1-n} \right) (g^\Delta)(t) \nabla t \\
 & + (1 - \alpha)^2 \int_0^h \left( \sum_{n=0}^{k-1} g^n (g^\rho)^{k-1-n} \right) (g^\nabla)(t) \nabla t \\
 = & \alpha^2 \int_0^h (g^k)^\Delta(t) \Delta t + \alpha(1 - \alpha) \int_0^h (g^k)^\nabla(t) \Delta t \\
 & + \alpha(1 - \alpha) \int_0^h (g^k)^\Delta(t) \nabla t + (1 - \alpha)^2 \int_0^h (g^k)^\nabla(t) \nabla t \\
 = & \alpha \left[ \alpha \int_0^h (g^k)^\Delta \Delta t + (1 - \alpha) \int_0^h (g^k)^\Delta \nabla t \right]
 \end{aligned}$$

$$\begin{aligned}
 &+ (1 - \alpha) \left[ \int_0^h \alpha (g^k)^\nabla \Delta t + (1 - \alpha) \int_0^h (g^k)^\nabla \nabla t \right] \\
 &= \alpha \int_0^h (g^k)^\Delta \diamond_\alpha + (1 - \alpha) \int_0^h (g^k)^\nabla \diamond_\alpha = \int_0^h (g^k)^\Delta \diamond_\alpha \\
 &= g^k(t)|_0^h = g^k(h) - g^k(0) = [g(h)]^k = \left[ \int_0^h |f^{\diamond_\alpha}(s)| \diamond_\alpha s \right]^k.
 \end{aligned}$$

By using Hölder’s inequality with indices  $p = \frac{k}{k-1}$  and  $q = k$ , we obtain

$$\begin{aligned}
 \left[ \int_0^h 1 \cdot |f^{\diamond_\alpha}(s)| \diamond_\alpha s \right]^k &\leq \left[ \left( \int_0^h 1^{\frac{k}{k-1}} \diamond_\alpha s \right)^{\frac{k-1}{k}} \left( \int_0^h |f^{\diamond_\alpha}(s)|^k \diamond_\alpha s \right)^{\frac{1}{k}} \right]^k \\
 &= \left( \int_0^h \diamond_\alpha s \right)^{k-1} \left( \int_0^h |f^{\diamond_\alpha}(s)|^k \diamond_\alpha s \right) \\
 &= (s|_0^h)^{k-1} \int_0^h |f^{\diamond_\alpha}(s)|^k \diamond_\alpha s \\
 &= h^{k-1} \int_0^h |f^{\diamond_\alpha}(s)|^k \diamond_\alpha s,
 \end{aligned}$$

hence the proof is complete. □

**Theorem 2.2** *Let  $\omega(t)$  be positive and continuous on  $(0, h)$ , with  $\int_0^h \omega^{1-q}(t) \Delta t < \infty$ ,  $q > 1$ . For differentiable  $f : [0, h] \rightarrow \mathbb{R}$  with  $f(0) = 0$  we have*

$$\int_0^h |f^k|^\Delta \Delta t \leq \left( \int_0^h \omega^{1-q} \Delta t \right)^{\frac{k}{q}} \left( \int_0^h \omega |f^\Delta|^p \Delta t \right)^{\frac{k}{q}}, \tag{3}$$

where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof* We take  $g(t) = \int_0^t |f^\Delta(s)| \Delta s$ . Then  $|f(t)| \leq g(t)$ ,  $g^\Delta(t) = |f^\Delta(t)|$ , so we have

$$\begin{aligned}
 \int_0^h |f^k|^\Delta \Delta t &= \int_0^h \left| \sum_{k=0}^{n-1} f^k (f^\sigma)^{n-1-k} \right| |f^\Delta|(t) \Delta t \\
 &\leq \int_0^h \left( \sum_{k=0}^{n-1} g^k (g^\sigma)^{n-1-k} \right) (g^\Delta)(t) \Delta t = \int_0^h (g^k)^\Delta \Delta t \\
 &= g^k(h) - g^k(0) = g^k(h) = \left( \int_0^h |f^\Delta|(t) \Delta t \right)^k \\
 &= \left( \int_0^h \omega^{-\frac{1}{p}} \omega^{\frac{1}{p}} |f^\Delta|(t) \Delta t \right)^k \\
 &\leq \left[ \left( \int_0^h (\omega^{-\frac{1}{p}})^q \Delta t \right)^{\frac{1}{q}} \left( \int_0^h (\omega^{\frac{1}{p}} |f^\Delta|)^p(t) \Delta t \right)^{\frac{1}{p}} \right]^k \\
 &= \left( \int_0^h \omega^{1-q} \Delta t \right)^{\frac{k}{q}} \left( \int_0^h (\omega |f^\Delta|)^p(t) \Delta t \right)^{\frac{k}{p}}.
 \end{aligned}$$

□

**Theorem 2.3** *Let  $\omega(t)$  be positive and continuous on  $(0, h)$ , with  $\int_0^h \omega^{1-q}(t) \nabla t < \infty$ ,  $q > 1$ . For differentiable  $f : [0, h] \rightarrow \mathbb{R}$  with  $f(0) = 0$  we have*

$$\int_0^h |f^k|^\nabla \nabla t \leq \left( \int_0^h \omega^{1-q} \nabla t \right)^{\frac{k}{q}} \left( \int_0^h \omega |f^\nabla|^p \nabla t \right)^{\frac{k}{p}}, \tag{4}$$

where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Theorem 2.4** *Assume that  $p > 1$ ,  $q = \frac{p}{p-1}$ ,  $\alpha \in [0, 1]$ ,  $h \in (0, \infty)_{\mathbb{T}}$ ,  $\omega \in \mathbb{C}([0, h]_{\mathbb{T}}, (0, \infty))$  and  $f \in \mathbb{C}^1_{\diamond_\alpha}([0, h]_{\mathbb{T}}, \mathbb{R})$ . If  $\alpha f^\Delta \geq 0$ ,  $(1 - \alpha) f^\nabla \geq 0$  and  $f(0) = 0$  then*

$$\begin{aligned} & \alpha^k \int_0^h |(f^k)^\Delta(t)| \Delta t + (1 - \alpha)^k \int_0^h |(f^k)^\nabla(t)| \nabla t \\ & \leq \left( \int_0^h \omega^{1-q}(t) \diamond_\alpha t \right)^{\frac{k}{q}} \left( \int_0^h \omega(t) |f^{\diamond_\alpha}(t)|^p \diamond_\alpha t \right)^{\frac{k}{p}}. \end{aligned} \tag{5}$$

*Proof* By Theorems 2.2, 2.3, Hölder’s inequality and  $k = \frac{k}{q} + (1 + p)\frac{k}{p}$ , we get

$$\begin{aligned} & \alpha^k \int_0^h |(f^k)^\Delta(t)| \Delta t + (1 - \alpha)^k \int_0^h |(f^k)^\nabla(t)| \nabla t \\ & = \alpha^{\frac{k}{q} + (1+p)\frac{k}{p}} \int_0^h |(f^k)^\Delta(t)| \Delta t + (1 - \alpha)^{\frac{k}{q} + (1+p)\frac{k}{p}} \int_0^h |(f^k)^\nabla(t)| \nabla t \\ & \leq \alpha^{\frac{k}{q} + (1+p)\frac{k}{p}} \left( \int_0^h \omega^{1-q}(t) \Delta t \right)^{\frac{k}{q}} \left( \int_0^h \omega(t) |f^\Delta(t)|^p \Delta t \right)^{\frac{k}{p}} \\ & \quad + (1 - \alpha)^{\frac{k}{q} + (1+p)\frac{k}{p}} \left( \int_0^h \omega^{1-q}(t) \nabla t \right)^{\frac{k}{q}} \left( \int_0^h \omega(t) |f^\nabla(t)|^p \nabla t \right)^{\frac{k}{p}} \\ & \leq \left( \alpha \int_0^h \omega^{1-q}(t) \Delta t \right)^{\frac{k}{q}} \left( \alpha \int_0^h \omega(t) |\alpha f^\Delta(t) + (1 - \alpha) f^\nabla(t)|^p \Delta t \right)^{\frac{k}{p}} \\ & \quad + \left( (1 - \alpha) \int_0^h \omega^{1-q}(t) \nabla t \right)^{\frac{k}{q}} \\ & \quad \cdot \left( (1 - \alpha) \int_0^h \omega(t) |\alpha f^\Delta(t) + (1 - \alpha) f^\nabla(t)|^p \nabla t \right)^{\frac{k}{p}} \\ & = \left( \alpha \int_0^h \omega^{1-q}(t) \Delta t \right)^{\frac{k}{q}} \left( \alpha \int_0^h \omega(t) |f^{\diamond_\alpha}(t)|^p \Delta t \right)^{\frac{k}{p}} \\ & \quad + \left( (1 - \alpha) \int_0^h \omega^{1-q}(t) \nabla t \right)^{\frac{k}{q}} \left( (1 - \alpha) \int_0^h \omega(t) |f^{\diamond_\alpha}(t)|^p \nabla t \right)^{\frac{k}{p}} \\ & \leq \left[ \left( \alpha \int_0^h \omega^{1-q}(t) \Delta t \right)^k + \left( (1 - \alpha) \int_0^h \omega^{1-q}(t) \nabla t \right)^k \right]^{\frac{1}{q}} \\ & \quad \cdot \left[ \left( \alpha \int_0^h \omega(t) |f^{\diamond_\alpha}(t)|^p \Delta t \right)^k + \left( (1 - \alpha) \int_0^h \omega(t) |f^{\diamond_\alpha}(t)|^p \nabla t \right)^k \right]^{\frac{1}{p}} \\ & \leq \left( \alpha \int_0^h \omega^{1-q}(t) \Delta t + (1 - \alpha) \int_0^h \omega^{1-q}(t) \nabla t \right)^{\frac{k}{q}} \end{aligned}$$

$$\begin{aligned} & \cdot \left( \alpha \int_0^h \omega(t) |f^{\diamond\alpha}(t)|^p \Delta t + (1-\alpha) \int_0^h \omega(t) |f^{\diamond\alpha}(t)|^p \nabla t \right)^{\frac{k}{p}} \\ &= \left( \int_0^h \omega^{1-q}(t) \diamond_{\alpha} t \right)^{\frac{k}{q}} \left( \int_0^h \omega(t) |f^{\diamond\alpha}(t)|^p \diamond_{\alpha} t \right)^{\frac{k}{p}}. \end{aligned} \quad \square$$

**Theorem 2.5** *Assume that  $1 < p \leq 2$ ,  $q = \frac{p}{p-1}$ ,  $\alpha \in [0, 1]$ ,  $h \in (0, \infty)_{\mathbb{T}}$ ,  $\omega \in \mathbb{C}([0, h]_{\mathbb{T}}, (0, \infty))$  and  $f \in \mathbb{C}^1_{\diamond_{\alpha}}([0, h]_{\mathbb{T}}, \mathbb{R})$ . If  $\alpha f^{\Delta} \geq 0$ ,  $(1-\alpha)f^{\nabla} \geq 0$  and  $f(0) = 0$ , then*

$$\begin{aligned} & \alpha^k \int_0^u |(f^k)^{\Delta}(t)| \Delta t + (1-\alpha)^k \int_0^u |(f^k)^{\nabla}(t)| \nabla t \\ & \leq \sum_{j=0}^{k-2} \alpha^j \binom{k}{j} \gamma^j \beta^{\frac{k-j}{q}} \left[ \int_0^h \omega(t) |g^{\diamond\alpha}(t)|^p \diamond_{\alpha} t \right]^{\frac{k-1}{p}} \\ & \quad + \alpha^{k-1} \binom{k}{k-1} \gamma^{k-1} (f(h) - f(0)), \end{aligned} \tag{6}$$

where  $\beta := \min_{u \in [0, h]_{\mathbb{T}}} v(u)$ ,  $v(u) = \max\{\int_0^u \omega^{1-q}(t) \diamond_{\alpha} t, \int_u^h \omega^{1-q}(t) \diamond_{\alpha} t\}$ ,  $\gamma := \max\{|f(0)|, |f(h)|\}$ .

*Proof* We let  $u \in [0, h]_{\mathbb{T}}$  be arbitrary. By applying Theorem 2.4 to the function  $g(t) = f(t) - f(0)$ , we obtain

$$\begin{aligned} & \alpha^k \int_0^u |(f^k)^{\Delta}(t)| \Delta t + (1-\alpha)^k \int_0^u |(f^k)^{\nabla}(t)| \nabla t \\ &= \alpha^k \int_0^u \left| \sum_{j=0}^{k-1} \binom{k}{j} (g^{k-j})^{\Delta} f^j(0) \right| \Delta t \\ & \quad + (1-\alpha)^k \int_0^u \left| \sum_{j=0}^{k-1} \binom{k}{j} (g^{k-j})^{\nabla} f^j(0) \right| \nabla t \\ & \leq \binom{k}{0} \left[ \alpha^k \int_0^u |g^k|^{\Delta} \Delta t + (1-\alpha)^k \int_0^u |g^k|^{\nabla} \nabla t \right] \\ & \quad + \alpha \binom{k}{1} |f(0)| \left[ \alpha^{k-1} \int_0^u |g^{k-1}|^{\Delta} \Delta t + (1-\alpha)^{k-1} \int_0^u |g^{k-1}|^{\nabla} \nabla t \right] \\ & \quad + \alpha^2 \binom{k}{2} |f^2(0)| \left[ \alpha^{k-2} \int_0^u |g^{k-2}|^{\Delta} \Delta t + (1-\alpha)^{k-2} \int_0^u |g^{k-2}|^{\nabla} \nabla t \right] \\ & \quad \vdots \\ & \quad + \alpha^{k-2} \binom{k}{k-2} |f^{k-2}(0)| \left[ \alpha^2 \int_0^u |g^2|^{\Delta} \Delta t + (1-\alpha)^2 \int_0^u |g^2|^{\nabla} \nabla t \right] \\ & \quad + \alpha^{k-1} \binom{k}{k-1} |f^{k-1}(0)| \left[ \alpha \int_0^u |f|^{\Delta} \Delta t + (1-\alpha) \int_0^u |f|^{\nabla} \nabla t \right] \\ & \leq \binom{k}{0} \left[ \int_0^u \omega^{1-q}(t) \diamond_{\alpha} t \right]^{\frac{k}{q}} \left[ \int_0^u \omega(t) |g^{\diamond\alpha}(t)|^p \diamond_{\alpha} t \right]^{\frac{k}{p}} \end{aligned}$$



$$\begin{aligned}
 & + \alpha \binom{k}{1} |f(0)| \left[ \int_0^u \omega^{1-q}(t) \diamond_{\alpha} t \right]^{\frac{k-1}{q}} \left[ \int_0^u \omega(t) |g^{\diamond_{\alpha}}(t)|^p \diamond_{\alpha} t \right]^{\frac{k-1}{p}} \\
 & + \alpha^2 \binom{k}{2} |f^2(0)| \left[ \int_0^u \omega^{1-q}(t) \diamond_{\alpha} t \right]^{\frac{k-2}{q}} \left[ \int_0^u \omega(t) |g^{\diamond_{\alpha}}(t)|^p \diamond_{\alpha} t \right]^{\frac{k-2}{p}} \\
 & \vdots \\
 & + \alpha^{k-2} \binom{k}{k-2} |f^{k-2}(0)| \left[ \int_0^u \omega^{1-q}(t) \diamond_{\alpha} t \right]^{\frac{2}{q}} \\
 & \cdot \left[ \int_0^u \omega(t) |g^{\diamond_{\alpha}}(t)|^p \diamond_{\alpha} t \right]^{\frac{2}{p}} \\
 & + \alpha^{k-1} \binom{k}{k-1} |f^{k-1}(0)| \alpha \int_0^u |f^{\Delta}(t)| \Delta t \\
 & + \alpha^{k-1} \binom{k}{k-1} |f^{k-1}(0)| (1-\alpha) \int_0^u |f^{\nabla}(t)| \nabla t \\
 & \leq \sum_{j=0}^{k-2} \alpha^j \binom{k}{j} |f^j(0)| \left[ \int_0^u \omega^{1-q}(t) \diamond_{\alpha} t \right]^{\frac{k-j}{q}} \left[ \int_0^u \omega(t) |g^{\diamond_{\alpha}}(t)|^p \diamond_{\alpha} t \right]^{\frac{k-j}{p}} \\
 & + \alpha^{k-1} \binom{k}{k-1} \gamma^{k-1} (f(u) - f(0)).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 & \alpha^k \int_u^h |(f^k)^{\Delta}(t)| \Delta t + (1-\alpha)^k \int_u^h |(f^k)^{\nabla}(t)| \nabla t \\
 & \leq \sum_{j=0}^{k-2} \alpha^j \binom{k}{j} |f^j(0)| \left[ \int_u^h \omega^{1-q}(t) \diamond_{\alpha} t \right]^{\frac{k-j}{q}} \left[ \int_u^h \omega(t) |g^{\diamond_{\alpha}}(t)|^p \diamond_{\alpha} t \right]^{\frac{k-j}{p}} \\
 & + \alpha^{k-1} \binom{k}{k-1} \gamma^{k-1} (f(h) - f(u)).
 \end{aligned}$$

Adding these two inequalities and taking into account that  $a^r + b^r \leq (a + b)^r$  holds, for  $a, b \geq 0$  and  $r \geq 1$ , yield the desired inequality. □

### 3 Conclusion

In this paper, we have obtained several Opial-type integral inequalities on time scales via the notion of the diamond-alpha derivative. These inequalities are related to the function  $f^k$ .

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**Author's contributions**

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