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A rank formula for the self-commutators of rational Toeplitz tuples

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Abstract
In this paper we derive a rank formula for the self-commutators of tuples of Toeplitz operators with matrix-valued rational symbols.

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1 Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{B}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, and write $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H},\mathcal{H})$. For $A, B \in \mathcal{B}(\mathcal{H})$, we let $[A,B] := AB - BA$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be normal if $[T^*,T] = 0$, hyponormal if $[T^*,T] \geq 0$. For an operator $T \in \mathcal{B}(\mathcal{H})$, we write $\ker T$ and $\operatorname{ran} T$ for the kernel and the range of T, respectively. For a subset $\mathcal{M}$ of a Hilbert space $\mathcal{H}$, $\operatorname{cl}\mathcal{M}$ and $\mathcal{M}^\perp$ denote the closure and the orthogonal complement of $\mathcal{M}$, respectively. Also, let $\mathbb{T} \equiv \partial\mathbb{D}$ be the unit circle (where $\mathbb{D}$ denotes the open unit disk in the complex plane $\mathbb{C}$). Recall that $L^\infty \equiv L^\infty(\mathbb{T})$ is the set of bounded measurable functions on $\mathbb{T}$, that the Hilbert space $L^2 \equiv L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and that the Hardy space $H^2 \equiv H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n \geq 0\}$. An element $f \in L^2$ is said to be analytic if $f \in H^2$. Let $H^\infty := L^\infty \cap H^2$, *i.e.*, H^∞ is the set of bounded analytic functions on $\mathbb{D}$.

We review the notion of functions of bounded type and a few essential facts about Hankel and Toeplitz operators and for that we will use [1–4].

For $\varphi \in L^\infty$, we write

$$\varphi_+ \equiv P\varphi \in H^2 \quad \text{and} \quad \varphi_- \equiv \overline{P^\perp\varphi} \in zH^2,$$

where P and $P^\perp$ denote the orthogonal projection from L^2 onto H^2 and $(H^2)^\perp$, respectively. Thus we may write $\varphi = \overline{\varphi_-} + \varphi_+$. We recall that a function $\varphi \in L^\infty$ is said to be of *bounded type* (or in the Nevanlinna class $\mathcal{N}$) if there are functions $\psi_1, \psi_2 \in H^\infty$ such that

$$\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)} \quad \text{for almost all } z \in \mathbb{T}.$$

We recall [5], Lemma 3, that if $\varphi \in L^\infty$ then

$$\varphi \text{ is of bounded type} \quad \Longleftrightarrow \quad \ker H_\varphi \neq \{0\}. \tag{1.1}$$

Assume now that both φ and $\overline{\varphi}$ are of bounded type. Then from the Beurling's theorem, $\ker H_{\overline{\varphi_-}} = \theta_0 H^2$ and $\ker H_{\overline{\varphi_+}} = \theta_+ H^2$ for some inner functions θ_0, θ_+. We thus have $b := \overline{\varphi_-}\theta_0 \in H^2$, and hence we can write

$$\varphi_- = \theta_0 \overline{b} \text{ and similarly } \varphi_+ = \theta_+ \overline{a} \quad \text{for some } a \in H^2. \tag{1.2}$$

By Kronecker's lemma [3], p.183, if $f \in H^\infty$ then $\overline{f}$ is a rational function if and only if $\operatorname{rank} H_{\overline{f}} < \infty$, which implies that

$$\overline{f} \text{ is rational} \quad \Longleftrightarrow \quad f = \theta \overline{b} \text{ with a finite Blaschke product } \theta. \tag{1.3}$$

Let $M_{n\times r}$ denote the set of all $n \times r$ complex matrices and write $M_n := M_{n\times n}$. For $\mathcal{X}$ a Hilbert space, let $L^2_{\mathcal{X}} \equiv L^2_{\mathcal{X}}(\mathbb{T})$ be the Hilbert space of $\mathcal{X}$-valued norm square-integrable measurable functions on $\mathbb{T}$ and let $L^\infty_{\mathcal{X}} \equiv L^\infty_{\mathcal{X}}(\mathbb{T})$ be the set of $\mathcal{X}$-valued bounded measurable functions on $\mathbb{T}$. We also let $H^2_{\mathcal{X}} \equiv H^2_{\mathcal{X}}(\mathbb{T})$ be the corresponding Hardy space and $H^\infty_{\mathcal{X}} \equiv H^\infty_{\mathcal{X}}(\mathbb{T}) = L^\infty_{\mathcal{X}} \cap H^2_{\mathcal{X}}$. We observe that $L^2_{\mathbb{C}^n} = L^2 \otimes \mathbb{C}^n$ and $H^2_{\mathbb{C}^n} = H^2 \otimes \mathbb{C}^n$.

For a matrix-valued function $\Phi \equiv (\varphi_{ij}) \in L^\infty_{M_n}$, we say that Φ is of *bounded type* if each entry φ_{ij} is of bounded type, and we say that Φ is *rational* if each entry φ_{ij} is a rational function.

Let $\Phi \equiv (\varphi_{ij}) \in L^\infty_{M_n}$ be such that Φ^* is of bounded type. Then each $\overline{\varphi}_{ij}$ is of bounded type. Thus in view of (1.2), we may write $\varphi_{ij} = \theta_{ij}\overline{b}_{ij}$, where θ_{ij} is inner and θ_{ij} and b_{ij} are coprime, in other words, there does not exist a nonconstant inner divisor of θ_{ij} and b_{ij}. Thus if θ is the least common multiple of $\{\theta_{ij} : i, j = 1, 2, \ldots, n\}$, then we may write

$$\Phi = (\varphi_{ij}) = (\theta_{ij}\overline{b}_{ij}) = (\theta \overline{a}_{ij}) \equiv \theta A^* \quad \bigl(\text{where } A \equiv (a_{ji}) \in H^2_{M_n}\bigr). \tag{1.4}$$

In particular, $A(\alpha)$ is nonzero whenever $\theta(\alpha) = 0$ and $|\alpha| < 1$.

For $\Phi \equiv [\varphi_{ij}] \in L^\infty_{M_n}$, we write

$$\Phi_+ := \bigl[P(\varphi_{ij})\bigr] \in H^2_{M_n} \quad \text{and} \quad \Phi_- := \bigl[P^\perp(\varphi_{ij})\bigr]^* \in H^2_{M_n}.$$

Thus we may write $\Phi = \Phi_-^* + \Phi_+$. However, it will often be convenient to allow the constant term in Φ_-. Hence, if there is no confusion we may assume that Φ_- shares the constant term with Φ_+: in this case, $\Phi(0) = \Phi_+(0) + \Phi_-(0)^*$. If $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type, then in view of (1.4), we may write

$$\Phi_+ = \theta_1 A^* \quad \text{and} \quad \Phi_- = \theta_2 B^*, \tag{1.5}$$

where θ_1 and θ_2 are inner functions and $A, B \in H^2_{M_n}$. In particular, if $\Phi \in L^\infty_{M_n}$ is rational then the θ_i can be chosen as finite Blaschke products, as we observed in (1.3). For simplicity, we write H^2_0 for $zH^2_{M_n}$.

We now introduce the notion of Hankel operators and Toeplitz operators with matrix-valued symbols. If Φ is a matrix-valued function in $L^\infty_{M_n}$, then $T_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ denotes Toeplitz operator with symbol Φ defined by

$$T_\Phi f := P_n(\Phi f) \quad \text{for } f \in H^2_{\mathbb{C}^n},$$

where P_n is the orthogonal projection of $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$. A Hankel operator with symbol $\Phi \in L^\infty_{M_n}$ is an operator $H_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ defined by

$$H_\Phi f := J_n P_n^\perp(\Phi f) \quad \text{for } f \in H^2_{\mathbb{C}^n},$$

where $P_n^\perp$ is the orthogonal projection of $L^2_{\mathbb{C}^n}$ onto $(H^2_{\mathbb{C}^n})^\perp$ and J_n denotes the unitary operator from $L^2_{\mathbb{C}^n}$ onto $L^2_{\mathbb{C}^n}$ given by $J_n(f)(z) := \overline{z} f(\overline{z})$ for $f \in L^2_{\mathbb{C}^n}$. For $\Phi \in L^\infty_{M_{n\times m}}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}).$$

A matrix-valued function $\Theta \in H^\infty_{M_{n\times m}}$ is called *inner* if $\Theta^*\Theta = I_m$ almost everywhere on $\mathbb{T}$, where I_m denotes the $m \times m$ identity matrix. If there is no confusion we write simply I for I_m. The following basic relations can easily be derived:

$$T^*_\Phi = T_{\Phi^*}, \qquad H^*_\Phi = H_{\widetilde{\Phi}} \quad \bigl(\Phi \in L^\infty_{M_n}\bigr); \tag{1.6}$$

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi \quad \bigl(\Phi, \Psi \in L^\infty_{M_n}\bigr); \tag{1.7}$$

$$H_\Phi T_\Psi = H_{\Phi\Psi}, \qquad H_{\Psi\Phi} = T^*_{\widetilde{\Psi}} H_\Phi \quad \bigl(\Phi \in L^\infty_{M_n}, \Psi \in H^\infty_{M_n}\bigr). \tag{1.8}$$

In 2006, Gu *et al.* [6] have considered the hyponormality of Toeplitz operators with matrix-valued symbols and characterized it in terms of their symbols.

Lemma 1.1 (Hyponormality of block Toeplitz operators [6]) *For each $\Phi \in L^\infty_{M_n}$, let*

$$\mathcal{E}(\Phi) := \bigl\{K \in H^\infty_{M_n} : \|K\|_\infty \le 1 \text{ and } \Phi - K\Phi^* \in H^\infty_{M_n}\bigr\}.$$

Then T_Φ is hyponormal if and only if Φ is normal and $\mathcal{E}(\Phi)$ is nonempty.

For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, we say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M_{m\times r}}$. We also say that two matrix functions $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary constant, and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{m\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Two matrix functions Φ and Ψ in $H^2_{M_n}$ are said to be *coprime* if they are both left and right coprime. We note that if $\Phi \in H^2_{M_n}$ is such that $\det \Phi \neq 0$, then any left inner divisor Δ of Φ is square, *i.e.*, $\Delta \in H^2_{M_n}$ (*cf.* [7]). If $\Phi \in H^2_{M_n}$ is such that $\det \Phi \neq 0$, then we say that $\Delta \in H^2_{M_n}$ is a *right inner divisor* of Φ if $\widetilde{\Delta}$ is a left inner divisor of $\widetilde{\Phi}$.

Let $\{\Theta_i \in H^\infty_{M_n} : i \in J\}$ be a family of inner matrix functions. The greatest common left inner divisor Θ_d and the least common left inner multiple Θ_m of the family $\{\Theta_i \in H^\infty_{M_n} :$

$i \in J\}$ are the inner functions defined by

$$\Theta_d H^2_{\mathbb{C}^p} = \bigvee_{i\in J} \Theta_i H^2_{\mathbb{C}^n} \quad \text{and} \quad \Theta_m H^2_{\mathbb{C}^q} = \bigcap_{i\in J} \Theta_i H^2_{\mathbb{C}^n}.$$

Similarly, the greatest common right inner divisor Θ'_d and the least common right inner multiple Θ'_m of the family $\{\Theta_i \in H^\infty_{M_n} : i \in J\}$ are the inner functions defined by

$$\widetilde{\Theta}'_d H^2_{\mathbb{C}^r} = \bigvee_{i\in J} \widetilde{\Theta}_i H^2_{\mathbb{C}^n} \quad \text{and} \quad \widetilde{\Theta}'_m H^2_{\mathbb{C}^s} = \bigcap_{i\in J} \widetilde{\Theta}_i H^2_{\mathbb{C}^n}.$$

The Beurling-Lax-Halmos theorem guarantees that Θ_d and Θ_m exist and are unique up to a unitary constant right factor, and Θ'_d and Θ'_m are unique up to a unitary constant left factor. We write

$$\Theta_d = \text{left-g.c.d.}\{\Theta_i : i \in J\}, \qquad \Theta_m = \text{left-l.c.m.}\{\Theta_i : i \in J\},$$
$$\Theta'_d = \text{right-g.c.d.}\{\Theta_i : i \in J\}, \qquad \Theta'_m = \text{right-l.c.m.}\{\Theta_i : i \in J\}.$$

If $n = 1$, then left-g.c.d.$\{\cdot\}$ = right-g.c.d.$\{\cdot\}$ (simply denoted g.c.d.$\{\cdot\}$) and left-l.c.m.$\{\cdot\}$ = right-l.c.m.$\{\cdot\}$ (simply denoted l.c.m.$\{\cdot\}$). In general, it is not true that left-g.c.d.$\{\cdot\}$ = right-g.c.d.$\{\cdot\}$ and left-l.c.m.$\{\cdot\}$ = right-l.c.m.$\{\cdot\}$.

If θ is an inner function we write I_θ for θI_n and $\mathcal{Z}(\theta)$ for the set of all zeros of θ.

Lemma 1.2 *Let $\Theta_i := I_{\theta_i}$ for an inner function θ_i $(i \in J)$.*

(a) *left-g.c.d.$\{\Theta_i : i \in J\}$ = right-g.c.d.$\{\Theta_i : i \in J\} = I_{\theta_d}$, where θ_d = g.c.d.$\{\theta_i : i \in J\}$.*

(b) *left-l.c.m.$\{\Theta_i : i \in J\}$ = right-l.c.m.$\{\Theta_i : i \in J\} = I_{\theta_m}$, where θ_m = l.c.m.$\{\theta_i : i \in J\}$.*

Proof See [7], Lemma 2.1. □

In view of Lemma 1.2, if $\Theta_i = I_{\theta_i}$ for an inner function θ_i $(i \in J)$, we can define the greatest common inner divisor Θ_d and the least common inner multiple Θ_m of the Θ_i by

$$\Theta_d \equiv \text{g.c.d.}\{\Theta_i : i \in J\} := I_{\theta_d}, \quad \text{where } \theta_d = \text{g.c.d.}\{\theta_i : i \in J\}$$

and

$$\Theta_m \equiv \text{l.c.m.}\{\Theta_i : i \in J\} := I_{\theta_m}, \quad \text{where } \theta_m = \text{l.c.m.}\{\theta_i : i \in J\}.$$

Both Θ_d and Θ_m are *diagonal-constant* inner functions, *i.e.*, diagonal inner functions, and constant along the diagonal.

By contrast with scalar-valued functions, in (1.4), I_θ and A need not be (right) coprime. If Ω = left-g.c.d.$\{I_\theta, A\}$ in the representation (1.4), that is,

$$\Phi = \theta A^*,$$

then $I_\theta = \Omega\Omega_\ell$ and $A = \Omega A_\ell$ for some inner matrix Ω_ℓ (where $\Omega_\ell \in H^2_{M_n}$ because $\det(I_\theta) \neq 0$) and some $A_l \in H^2_{M_n}$. Therefore if $\Phi^* \in L^\infty_{M_n}$ is of bounded type then we can write

$$\Phi = A_\ell{}^* \Omega_\ell, \quad \text{where } A_\ell \text{ and } \Omega_\ell \text{ are left coprime.} \tag{1.9}$$

In this case, $A_\ell^*\Omega_\ell$ is called the *left coprime factorization* of Φ and write, briefly,

$$\Phi = A_\ell^*\Omega_\ell \quad \text{(left coprime).} \tag{1.10}$$

Similarly, we can write

$$\Phi = \Omega_r A_r^*, \quad \text{where } A_r \text{ and } \Omega_r \text{ are right coprime.} \tag{1.11}$$

In this case, $\Omega_r A_r^*$ is called the *right coprime factorization* of Φ and we write, succinctly,

$$\Phi = \Omega_r A_r^* \quad \text{(right coprime).} \tag{1.12}$$

In this case, we define the *degree* of Φ by

$$\deg(\Phi) := \dim \mathcal{H}(\Omega_r),$$

where $\mathcal{H}(\Theta) := H^2_{\mathbb{C}^n} \ominus \Theta H^2_{\mathbb{C}^n}$ for an inner function Θ. It was known (*cf.* [8], Lemma 3.3) that if θ is a finite Blaschke product then I_θ and $A \in H^2_{M_n}$ are left coprime if and only if they are right coprime. In this viewpoint, in (1.10) and (1.12), Ω_ℓ or Ω_r is I_θ (θ a finite Blaschke product) then we shall write

$$\Phi = \theta A^* \quad \text{(coprime).}$$

On the other hand, we recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be subnormal if T has a normal extension, *i.e.*, $T = N|_{\mathcal{H}}$, where N is a normal operator on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $\mathcal{H}$ is invariant for N. The Bram-Halmos criterion for subnormality [9, 10] states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if $\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$ for all finite collections $x_0, x_1, \ldots, x_k \in \mathcal{H}$. It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} [T^*, T] & [T^{*2}, T] & \ldots & [T^{*k}, T] \\ [T^*, T^2] & [T^{*2}, T^2] & \ldots & [T^{*k}, T^2] \\ \vdots & \vdots & \ddots & \vdots \\ [T^*, T^k] & [T^{*2}, T^k] & \ldots & [T^{*k}, T^k] \end{pmatrix} \geq 0 \quad (\text{all } k \geq 1). \tag{1.13}$$

Condition (1.13) provides a measure of the gap between hyponormality and subnormality. In fact the positivity condition (1.13) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (1.13) for all k. For $k \geq 1$, an operator T is said to be *k-hyponormal* if T satisfies the positivity condition (1.13) for a fixed k. Thus the Bram-Halmos criterion can be stated thus: T is subnormal if and only if T is k-hyponormal for all $k \geq 1$. The notion of k-hyponormality has been considered by many authors aiming at understanding the bridge between hyponormality and subnormality. In view of (1.13), between hyponormality and subnormality there exists a whole slew of increasingly stricter conditions, each expressible in terms of the joint hyponormality of the tuples $(I, T, T^2, \ldots, T^k)$. Given an n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ of operators on $\mathcal{H}$, we let

$[\mathbf{T}^*, \mathbf{T}] \in \mathcal{B}(\mathcal{H} \oplus \cdots \oplus \mathcal{H})$ denote the *self-commutator* of $\mathbf{T}$, defined by

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \dots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \dots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \dots & [T_n^*, T_n] \end{pmatrix}.$$

By analogy with the case $n = 1$, we shall say [11, 12] that $\mathbf{T}$ is *jointly hyponormal* (or simply, *hyponormal*) if $[\mathbf{T}^*, \mathbf{T}] \geq 0$, *i.e.*, $[\mathbf{T}^*, \mathbf{T}]$ is a positive-semidefinite operator on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$.

Tuples $\mathbf{T} \equiv (T_{\Phi_1}, \dots, T_{\Phi_m})$ of block Toeplitz operators T_{Φ_i} $(i = 1, \dots, m)$ will be called a (block) Toeplitz tuples. Moreover, if each Toeplitz operator T_{Φ_i} has a symbol Φ_i which is a matrix-valued rational function, then the tuple $\mathbf{T} \equiv (T_{\Phi_1}, \dots, T_{\Phi_m})$ is called a rational Toeplitz tuple. In this paper we will derive a rank formula for the self-commutator of a rational Topelitz tuple.

2 The results and discussion

For an operator $S \in \mathcal{B}(\mathcal{H})$, $S^\sharp \in \mathcal{B}(\mathcal{H})$ is called the Moore-Penrose inverse of S if

$$SS^\sharp S = S, \qquad S^\sharp S S^\sharp = S^\sharp, \qquad (S^\sharp S)^* = S^\sharp S, \quad \text{and} \quad (SS^\sharp)^* = SS^\sharp.$$

It is well known [13], Theorem 8.7.2, that if an operator S on a Hilbert space has a closed range then S has a Moore-Penrose inverse. Moreover, the Moore-Penrose inverse is unique whenever it exists. On the other hand, it is well known that if

$$S := \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \quad \text{on } \mathcal{H}_1 \oplus \mathcal{H}_2$$

(where the $\mathcal{H}_j$ are Hilbert spaces, $A \in \mathcal{B}(\mathcal{H}_1)$, $C \in \mathcal{B}(\mathcal{H}_2)$, and $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$), then

$$S \geq 0 \quad \Longleftrightarrow \quad A \geq 0, C \geq 0, \text{ and } B = A^{\frac{1}{2}} D C^{\frac{1}{2}} \quad \text{for some contraction } D; \tag{2.1}$$

moreover, in [14], Lemma 1.2, and [15], Lemma 2.1, it was shown that if $A \geq 0$, $C \geq 0$, and $\operatorname{ran} A$ is closed then

$$S \geq 0 \quad \Longleftrightarrow \quad B^* A^\sharp B \leq C \text{ and } \operatorname{ran} B \subseteq \operatorname{ran} A, \tag{2.2}$$

or equivalently [12], Lemma 1.4,

$$\left|\langle Bg, f\rangle\right|^2 \leq \langle Af, f\rangle \langle Cg, g\rangle \quad \text{for all } f \in \mathcal{H}_1, g \in \mathcal{H}_2 \tag{2.3}$$

and furthermore, if both A and C are of finite rank then

$$\operatorname{rank} S = \operatorname{rank} A + \operatorname{rank}\bigl(C - B^* A^\sharp B\bigr). \tag{2.4}$$

In fact, if $A \geq 0$ and $\operatorname{ran} A$ is closed then we can write

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{ran} A \\ \ker A \end{bmatrix} \to \begin{bmatrix} \operatorname{ran} A \\ \ker A \end{bmatrix},$$

so that the Moore-Penrose inverse of A is given by

$$A^\sharp = \begin{bmatrix} (A_0)^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.5}$$

Proposition 2.1 *If $A \in \mathcal{B}(\mathcal{H})$ has a closed range then $A(A^*A)^\sharp A^*$ is the orthogonal projection onto* $\operatorname{ran} A$.

Proof Suppose $A \in \mathcal{B}(\mathcal{H})$ has a closed range. Then (2.5) can be written as

$$(P_{\operatorname{ran} A} A P_{\operatorname{ran} A})^{-1} = P_{\operatorname{ran} A} A^\sharp P_{\operatorname{ran} A}. \tag{2.6}$$

Since by assumption, A^*A has also a closed range, there exists the Moore-Penrose inverse $(A^*A)^\sharp$. Observe

$$\bigl(A(A^*A)^\sharp A^*\bigr)\bigl(A(A^*A)^\sharp A^*\bigr) = A(A^*A)^\sharp A^*$$

and

$$\bigl(A(A^*A)^\sharp A^*\bigr)^* = A(A^*A)^\sharp A^*,$$

which implies that $A(A^*A)^\sharp A^*$ is an orthogonal projection. Put

$$K := \operatorname{ran} A^*A = \operatorname{ran} A^* = (\ker A)^\perp.$$

We then have

$$\begin{aligned} A(A^*A)^\sharp A^* &= AP_K(A^*A)^\sharp P_K A^* \\ &= A\bigl(P_K(A^*A)P_K\bigr)^{-1}A^* \quad \bigl(\text{by (2.5)}\bigr), \end{aligned}$$

which implies that $\operatorname{ran}(A(A^*A)^\sharp A^*) = \operatorname{ran} A$. □

In the sequel we often encounter the following matrix:

$$S := \begin{bmatrix} A^*A & A^*B \\ B^*A & [B^*, B] \end{bmatrix},$$

where A has a closed range. If $S \geq 0$ and if A and $[B^*, B]$ are of finite rank then by (2.4), we have

$$\operatorname{rank} S = \operatorname{rank}\bigl(A^*A\bigr) + \operatorname{rank}\bigl(\bigl[B^*, B\bigr] - B^*A\bigl(A^*A\bigr)^\sharp A^*B\bigr). \tag{2.7}$$

Thus, if we write P_K for the orthogonal projection onto $K := \operatorname{ran} A$, then by Proposition 2.1 we have

$$\begin{aligned} \operatorname{rank} S &= \operatorname{rank}\bigl(A^*\bigr) + \operatorname{rank}\bigl(\bigl[B^*, B\bigr] - B^*P_K B\bigr) \\ &= \operatorname{rank}\bigl(A^*\bigr) + \operatorname{rank}\bigl(B^*P_{K^\perp}B - BB^*\bigr). \end{aligned} \tag{2.8}$$

If $\Phi, \Psi \in L^\infty_{M_n}$, then by (1.7),

$$[T_\Phi, T_\Psi] = H^*_{\Psi^*}H_\Phi - H^*_{\Phi^*}H_\Psi + T_{\Phi\Psi-\Psi\Phi}.$$

Since the normality of Φ is a necessary condition for the hyponormality of T_Φ (*cf.* [15]), the positivity of $H^*_{\Phi^*}H_{\Phi^*} - H^*_\Phi H_\Phi$ is an essential condition for the hyponormality of T_Φ. If $\Phi \in L^\infty_{M_n}$, the *pseudo-self-commutator* of T_Φ is defined by

$$\left[T^*_\Phi, T_\Phi\right]_p := H^*_{\Phi^*}H_{\Phi^*} - H^*_\Phi H_\Phi.$$

Then T_Φ is said to be *pseudo-hyponormal* if $[T^*_\Phi, T_\Phi]_p \geq 0$. We also see that if $\Phi \in L^\infty_{M_n}$ then $[T^*_\Phi, T_\Phi] = [T^*_\Phi, T_\Phi]_p + T_{\Phi^*\Phi-\Phi\Phi^*}$.

Proposition 2.2 *Let $\Phi \equiv \Phi^*_- + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Thus in view of* (1.4), *we may write*

$$\Phi_+ = \theta_1 A^* \quad \textit{and} \quad \Phi_- = \theta_2 B^*,$$

where θ_1 and θ_2 are inner functions and $A, B \in H^2_{M_n}$. If T_Φ is hyponormal then θ_2 is an inner divisor of θ_1, i.e., $\theta_1 = \theta_0\theta_2$ for some inner function θ_0.

Proof See [7], Proposition 3.2. □

In view of Proposition 2.2, when we study the hyponormality of block Toeplitz operators with *bounded type symbols* Φ (*i.e.*, Φ and Φ^* are of bounded type) we may assume that the symbol $\Phi \equiv \Phi^*_- + \Phi_+ \in L^\infty_{M_n}$ is of the form

$$\Phi_+ = \theta_0\theta_1 A^* \quad \text{and} \quad \Phi_- = \theta_0 B^*,$$

where θ_0 and θ_1 are inner functions and $A, B \in H^2_{M_n}$.

We first observe that if $\mathbf{T} = (T_\varphi, T_\psi)$ then the self-commutator of $\mathbf{T}$ can be expressed as

$$\left[\mathbf{T}^*, \mathbf{T}\right] = \begin{bmatrix} [T^*_\varphi, T_\varphi] & [T^*_\psi, T_\varphi] \\ [T^*_\varphi, T_\psi] & [T^*_\psi, T_\psi] \end{bmatrix} = \begin{bmatrix} H^*_{\overline{\varphi_+}}H_{\overline{\varphi_+}} - H^*_{\overline{\varphi_-}}H_{\overline{\varphi_-}} & H^*_{\overline{\varphi_+}}H_{\overline{\psi_+}} - H^*_{\overline{\psi_-}}H_{\overline{\varphi_-}} \\ H^*_{\overline{\psi_+}}H_{\overline{\varphi_+}} - H^*_{\overline{\varphi_-}}H_{\overline{\psi_-}} & H^*_{\overline{\psi_+}}H_{\overline{\psi_+}} - H^*_{\overline{\psi_-}}H_{\overline{\psi_-}} \end{bmatrix}. \quad (2.9)$$

For a block Toeplitz pair $\mathbf{T} \equiv (T_\Phi, T_\Psi)$, the *pseudo-commutator* of $\mathbf{T}$ is defined by

$$\begin{aligned} \left[\mathbf{T}^*, \mathbf{T}\right]_p &:= \begin{bmatrix} [T^*_\Phi, T_\Phi]_p & [T^*_\Psi, T_\Phi]_p \\ [T^*_\Phi, T_\Psi]_p & [T^*_\Psi, T_\Psi]_p \end{bmatrix} \\ &= \begin{bmatrix} H^*_{\Phi^*_+}H_{\Phi^*_+} - H^*_{\Phi^*_-}H_{\Phi^*_-} & H^*_{\Phi^*_+}H_{\Psi^*_+} - H^*_{\Psi^*_-}H_{\Phi^*_-} \\ H^*_{\Psi^*_+}H_{\Phi^*_+} - H^*_{\Phi^*_-}H_{\Psi^*_-} & H^*_{\Psi^*_+}H_{\Psi^*_+} - H^*_{\Psi^*_-}H_{\Psi^*_-} \end{bmatrix}. \end{aligned}$$

Let $\Phi_i \in L^\infty_{M_n}$ $(i = 1, 2, \dots, m)$ be normal and mutually commuting and let σ be a permutation on $\{1, 2, \dots, m\}$. Then evidently,

$$\begin{aligned} &\mathbf{T} := (T_{\Phi_1}, \dots, T_{\Phi_m}) \text{ is hyponormal} \\ &\quad\Longleftrightarrow \quad \mathbf{T}_\sigma := (T_{\Phi_{\sigma(1)}}, \dots, T_{\Phi_{\sigma(m)}}) \text{ is hyponormal.} \end{aligned} \quad (2.10)$$

Moreover, we have

$$\operatorname{rank}\bigl[\mathbf{T}^*, \mathbf{T}\bigr] = \operatorname{rank}\bigl[\mathbf{T}_\sigma^*, \mathbf{T}_\sigma\bigr]. \tag{2.11}$$

For every $m_0 \le m$, let $\mathbf{T}_{m_0} := (T_{\Phi_1}, \ldots, T_{\Phi_{m_0}})$. Since

$$\bigl[\mathbf{T}^*, \mathbf{T}\bigr] = \begin{bmatrix} [\mathbf{T}_{\Phi_{m_0}}^*, \mathbf{T}_{\Phi_{m_0}}] & * \\ * & * \end{bmatrix},$$

we can see that if $\mathbf{T}$ is hyponormal then in view of (2.10), every sub-tuple of $\mathbf{T}$ is hyponormal.

We then have the following.

Lemma 2.3 *Let $\Phi_i \in L^\infty_{M_n}$ be normal and mutually commuting. Let $\mathbf{T} \equiv (T_{\Phi_1}, \ldots, T_{\Phi_m})$ and $\mathbf{S} \equiv (T_{\Lambda_1\Phi_1}, \ldots, T_{\Lambda_m\Phi_m})$, where the Λ_i are mutually commuting and are invertible constant normal matrices commuting with Φ_j and Λ_j for each $i, j = 1, 2, \ldots, m$. Then*

$$\mathbf{T} \text{ is hyponormal} \quad \Longleftrightarrow \quad \mathbf{S} \text{ is hyponormal.}$$

Furthermore, $\operatorname{rank}[\mathbf{T}^*, \mathbf{T}] = \operatorname{rank}[\mathbf{S}^*, \mathbf{S}]$.

Proof In view of equation (2.10), it suffices to prove the lemma when $\Lambda_i = I$ for all $i = 2, \ldots, m$. Put $\mathcal{T} := [\mathbf{T}^*, \mathbf{T}]$ and $\mathcal{S} := [\mathbf{S}^*, \mathbf{S}]$. Since Λ_1 is a constant normal matrix commuting with Φ_j, it follows that, for all $j > 1$,

$$\begin{aligned}
\mathcal{S}_{1j} &= H^*_{(\Lambda_1\Phi_1)^*_+} H_{(\Phi_j)^*_+} - H^*_{(\Phi_j)^*_-} H_{(\Lambda_1\Phi_1)^*_-} \\
&= H^*_{(\Phi_1)^*_+\Lambda_1^*} H_{(\Phi_j)^*_+} - H^*_{(\Phi_j)^*_-} H_{\Lambda_1(\Phi_1)^*_-} \\
&= T_{\Lambda_1} H^*_{(\Phi_1)^*_+} H_{(\Phi_j)^*_+} - H^*_{(\Phi_j)^*_-} T_{\Lambda_1} H_{(\Phi_1)^*_-} \\
&= T_{\Lambda_1} H^*_{(\Phi_1)^*_+} H_{(\Phi_j)^*_+} - H^*_{(\Phi_j)^*_-\Lambda_1^*} H_{(\Phi_1)^*_-} \\
&= T_{\Lambda_1} \bigl(H^*_{(\Phi_1)^*_+} H_{(\Phi_j)^*_+} - H^*_{(\Phi_j)^*_-} H_{(\Phi_1)^*_-} \bigr) \\
&= T_{\Lambda_1} \mathcal{T}_{1j}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\mathcal{S}_{11} &= H^*_{(\Lambda_1\Phi_1)^*_+} H_{(\Lambda_1\Phi_1)^*_+} - H^*_{(\Lambda_1\Phi_1)^*_-} H_{(\Lambda_1\Phi_1)^*_-} \\
&= H^*_{(\Phi_1)^*_+\Lambda_1^*} H_{(\Phi_1)^*_+\Lambda_1^*} - H^*_{(\Phi_1)^*_-\Lambda_1^*} H_{(\Phi_1)^*_-\Lambda_1^*} \\
&= T_{\Lambda_1} H^*_{(\Phi_1)^*_+} H_{(\Phi_1)^*_+} T^*_{\Lambda_1} - T_{\Lambda_1} H^*_{(\Phi_1)^*_-} H_{(\Phi_1)^*_-} T^*_{\Lambda_1} \\
&= T_{\Lambda_1} \bigl(H^*_{(\Phi_1)^*_+} H_{(\Phi_1)^*_+} - H^*_{(\Phi_1)^*_-} H_{(\Phi_1)^*_-} \bigr) T^*_{\Lambda_1} \\
&= T_{\Lambda_1} \mathcal{T}_{11} T^*_{\Lambda_1}.
\end{aligned}$$

Let Q be the block diagonal operator with the diagonal entries $(T_{\Lambda_1}, I, \ldots, I)$. Then Q is invertible and $\mathcal{S} = Q\mathcal{T}Q^*$, which gives the result. □

Lemma 2.4 *Let* $\mathbf{T} \equiv (T_{\Phi_1}, T_{\Phi_2}, \dots T_{\Phi_m})$, *where the* $\Phi_i \in L^\infty_{M_n}$ $(i = 1, \dots, m)$ *are normal and mutually commuting. If* $\mathbf{S} := (T_{\Phi_1 - \Phi_{j_0}}, T_{\Phi_2}, \dots T_{\Phi_m})$ *for some* j_0 $(2 \le j_0 \le m)$, *then*

$$\mathbf{T} \text{ is hyponormal} \iff \mathbf{S} \text{ is hyponormal.}$$

Furthermore, $\operatorname{rank}[\mathbf{T}^*, \mathbf{T}] = \operatorname{rank}[\mathbf{S}^*, \mathbf{S}]$.

Proof Obvious. □

Corollary 2.5 *Let* $\Phi_i \in L^\infty_{M_n}$ $(i = 1, \dots, m)$ *be normal and mutually commuting. Let* $\mathbf{T} \equiv (T_{\Phi_1}, \dots T_{\Phi_m})$ *and put*

$$\mathbf{S} := (T_{\Phi_1 - \Lambda_1 \Phi_m}, T_{\Phi_2 - \Lambda_2 \Phi_m}, \dots, T_{\Phi_{m-1} - \Lambda_{m-1}\Phi_m}, T_{\Phi_m}),$$

where the Λ_i $(i = 1, \dots, m-1)$ *are mutually commuting and are invertible constant normal matrices commuting with* Φ_j *for each* $j = 1, \dots, m$. *Then*

$$\mathbf{T} \text{ is hyponormal} \iff \mathbf{S} \text{ is hyponormal.}$$

Furthermore, $\operatorname{rank}[\mathbf{T}^*, \mathbf{T}] = \operatorname{rank}[\mathbf{S}^*, \mathbf{S}]$.

Proof This follows from Lemmas 2.3 and 2.4. □

We now have the following.

Theorem 2.6 *Let* $\Phi_i \in H^\infty_{M_n}$ $(i = 1, 2, \dots, m-1)$ *be mutually commuting and normal rational functions of the form*

$$\Phi_i = A_i^* \Theta_i \quad (\textit{left coprime}),$$

where the Θ_i *are inner matrix functions and* $\Phi_m \equiv (\Phi_m)_-^* + (\Phi_m)_+ \in L^\infty_{M_n}$. *If* $\mathbf{T} := (T_{\Phi_1}, \dots, T_{\Phi_m})$ *is hyponormal then*

$$\operatorname{rank}\big[\mathbf{T}^*, \mathbf{T}\big] = \deg(\Theta) + \operatorname{rank}\big[T^*_{\Phi_m^{1,\Theta}}, T_{\Phi_m^{1,\Theta}}\big]_p, \tag{2.12}$$

where $\Theta :=$ *right-l.c.m.*$\{\Theta_i : i = 1, 2, \dots, m-1\}$ *and* $\Phi_m^{1,\Theta} := (\Phi_m)_-^* + P_{H_0^2}((\Phi_m)_+ \Theta^*)$.

Proof Let $\mathbf{H}_{\Phi^*} := (H_{\Phi_1^*}, \dots, H_{\Phi_{m-1}^*})$. Since $\Phi_i \equiv (\Phi_i)_+ \in H^\infty_{M_n} (i = 1, 2, \dots, m-1)$, $\mathbf{T}$ is hyponormal if and only if

$$\big[\mathbf{T}^*, \mathbf{T}\big] = \begin{bmatrix} \mathbf{H}^*_{\Phi^*} \mathbf{H}_{\Phi^*} & \mathbf{H}^*_{\Phi^*} H_{\Phi_m^*} \\ H^*_{\Phi_m^*} \mathbf{H}_{\Phi^*} & [T^*_{\Phi_m}, T_{\Phi_m}] \end{bmatrix} \ge 0,$$

or equivalently, for each $X \in \bigoplus_{j=1}^{m-1} H^2_{\mathbb{C}^n}$ and $Y \in H^2_{\mathbb{C}^n}$,

$$\big|\big\langle \mathbf{H}_{\Phi^*} H^*_{\Phi_m^*} Y, X\big\rangle\big|^2 \le \big\langle \mathbf{H}^*_{\Phi^*} \mathbf{H}_{\Phi^*} X, X\big\rangle \big\langle \big[T^*_{\Phi_m}, T_{\Phi_m}\big] Y, Y\big\rangle. \tag{2.13}$$

Since $\mathrm{cl\,ran}\, H_{\Phi_i^*} = \mathcal{H}(\widetilde{\Theta}_i)$ $(i = 1, 2, \ldots, n-1)$, it follows that

$$\begin{aligned}\mathrm{cl\,ran}\,\mathbf{H}_{\Phi^*} &= \bigvee_{i=1}^{m-1} \mathrm{cl\,ran}\, H_{\Phi_i^*} = \bigvee_{i=1}^{m-1} \mathcal{H}(\widetilde{\Theta}_i) = \left(\bigcap_{i=1}^{m-1} \widetilde{\Theta}_i H^2_{\mathbb{C}^n}\right)^{\perp} \\ &= \big(\widetilde{\Theta} H^2_{\mathbb{C}^n}\big)^{\perp} = \mathcal{H}(\widetilde{\Theta}) = \mathrm{cl\,ran}\, H_{\Theta^*}, \end{aligned} \tag{2.14}$$

where $\mathcal{H}(\Delta) := H^2_{\mathbb{C}^n} \ominus \Delta H^2_{\mathbb{C}^n}$. If the Φ_i are rational functions then, by (1.3) and (1.4), we can write

$$\Phi_i = \theta_i A_i^* \quad (\theta_i, \text{ finite Blaschke product}).$$

Since Θ_i is a right inner divisor of I_{θ_i}, we have $\deg(\Theta_i) \le \deg(I_{\theta_i}) = n \deg(\theta_i) < \infty$. Thus since by (2.14), $\mathrm{cl\,ran}\,\mathbf{H}_{\Phi^*} = \mathcal{H}(\widetilde{\Theta})$ and

$$\deg(\Theta) = \mathrm{rank}\, H^*_{\Theta^*} = \mathrm{rank}\, H_{\Theta^*} = \deg(\widetilde{\Theta}) < \infty.$$

Therefore $\mathbf{H}_{\Phi^*}$ is of finite rank and hence, so is $\mathbf{H}^*_{\Phi^*}\mathbf{H}_{\Phi^*}$ and, moreover,

$$\mathrm{rank}\big(\mathbf{H}^*_{\Phi^*}\mathbf{H}_{\Phi^*}\big) = \mathrm{rank}\big(\mathbf{H}^*_{\Phi^*}\big) = \mathrm{rank}(\mathbf{H}_{\Phi^*}) = \deg(\Theta).$$

Thus by (2.7), we have

$$\begin{aligned}\mathrm{rank}\big[\mathbf{T}^*, \mathbf{T}\big] &= \mathrm{rank}\begin{bmatrix} \mathbf{H}^*_{\Phi^*}\mathbf{H}_{\Phi^*} & \mathbf{H}^*_{\Phi^*} H_{\Phi^*_m} \\ H^*_{\Phi^*_m}\mathbf{H}_{\Phi^*} & [T^*_{\Phi_m}, T_{\Phi_m}] \end{bmatrix} \\ &= \mathrm{rank}\big(\mathbf{H}^*_{\Phi^*}\mathbf{H}_{\Phi^*}\big) + \mathrm{rank}\big(\big[T^*_{\Phi_m}, T_{\Phi_m}\big] - H^*_{\Phi^*_m}\mathbf{H}_{\Phi^*}\big(\mathbf{H}^*_{\Phi^*}\mathbf{H}_{\Phi^*}\big)^{\sharp}\mathbf{H}^*_{\Phi^*} H_{\Phi^*_m}\big) \\ &= \deg(\Theta) + \mathrm{rank}\big(\big[T^*_{\Phi_m}, T_{\Phi_m}\big] - H^*_{\Phi^*_m}\mathbf{H}_{\Phi^*}\big(\mathbf{H}^*_{\Phi^*}\mathbf{H}_{\Phi^*}\big)^{\sharp}\mathbf{H}^*_{\Phi^*} H_{\Phi^*_m}\big). \end{aligned}$$

On the other hand, by Proposition 2.1, $\mathbf{H}_{\Phi^*}(\mathbf{H}^*_{\Phi^*}\mathbf{H}_{\Phi^*})^{\sharp}\mathbf{H}^*_{\Phi^*}$ is the projection $P_{\mathcal{H}(\widetilde{\Theta})}$. Therefore it follows from (1.7) and (1.8) that

$$\begin{aligned}&\big[T^*_{\Phi_m}, T_{\Phi_m}\big] - H^*_{\Phi^*_m}\mathbf{H}_{\Phi^*}\big(\mathbf{H}^*_{\Phi^*}\mathbf{H}_{\Phi^*}\big)^{\sharp}\mathbf{H}^*_{\Phi^*} H_{\Phi^*_m} \\ &\quad = \big[T^*_{\Phi_m}, T_{\Phi_m}\big] - H^*_{\Phi^*_m} H_{\Theta^*} H^*_{\Theta^*} H_{\Phi^*_m} \\ &\quad = H^*_{\Phi^*_{m+}}\big(I - H_{\Theta^*}H^*_{\Theta^*}\big)H_{\Phi^*_{m+}} - H^*_{\Phi^*_{m-}} H_{\Phi^*_{m-}} \\ &\quad = \big(H^*_{\Phi^*_{m+}} T_{\widetilde{\Theta}}\big)\big(T_{\widetilde{\Theta}^*} H_{\Phi^*_{m+}}\big) - H^*_{\Phi^*_{m-}} H_{\Phi^*_{m-}} \\ &\quad = H^*_{\Theta\Phi^*_{m+}} H_{\Theta\Phi^*_{m+}} - H^*_{\Phi^*_{m-}} H_{\Phi^*_{m-}} \\ &\quad = \big[T^*_{\Phi_m^{1,\Theta}}, T_{\Phi_m^{1,\Theta}}\big]_p, \end{aligned}$$

which gives the result. □

Very recently, the hyponormality of rational Toeplitz pairs was characterized in [16].

Lemma 2.7 (Hyponormality of rational Toeplitz pairs) [16] *Let $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ be a Toeplitz pair with rational symbols $\Phi, \Psi \in L^\infty_{M_n}$ of the form*

$$\Phi_+ = \theta_0\theta_1 A^*, \qquad \Phi_- = \theta_0 B^*, \qquad \Psi_+ = \theta_2\theta_3 C^*, \qquad \Psi_- = \theta_2 D^* \quad (coprime). \tag{2.15}$$

Assume that θ_0 and θ_2 are not coprime. Assume also that $B(\gamma_0)$ and $D(\gamma_0)$ are diagonal-constant for some $\gamma_0 \in \mathcal{Z}(\theta_0)$. Then the pair $\mathbf{T}$ is hyponormal if and only if

(i) *Φ and Ψ are normal and $\Phi\Psi = \Psi\Phi$;*
(ii) *$\Phi_- = \Lambda^*\Psi_-$ (with $\Lambda := B(\gamma_0)D(\gamma_0)^{-1}$);*
(iii) *$T_{\Psi^{1,\Omega}}$ is pseudo-hyponormal with $\Omega := \theta_0\theta_1\theta_3\overline{\theta}\Delta^*$,*

where $\theta :=$ g.c.d.(θ_1, θ_3) and $\Delta :=$ left-g.c.d.$(I_{\theta_0\theta}, \overline{\theta}(\theta_3 A - \theta_1 C\Lambda^*))$.

We now get a rank formula for the self-commutators of Toeplitz m-tuples.

Corollary 2.8 *For each $i = 1, 2, \dots, m$, suppose that $\Phi_i = (\Phi_i)_-^* + (\Phi_i)_+ \in L^\infty_{M_n}$ is a matrix-valued normal rational function of the form*

$$(\Phi_i)_+ = \theta_i\delta_i A_i^* \quad and \quad (\Phi_i)_- = \theta_i B_i^* \quad (coprime),$$

where the θ_i and the δ_i are finite Blaschke products and there exists j_0 $(1 \le j_0 \le m)$ such that θ_{j_0} and θ_i are not coprime for each $i = 1, 2, \dots, m$. Suppose $\Phi_i\Phi_j = \Phi_j\Phi_i$ for all $i, j = 1, \dots, m$. Assume that each $B_i(\gamma_0)$ is diagonal-constant for some $\gamma_0 \in \mathcal{Z}(\theta_i)$. If $\mathbf{T} \equiv (T_{\Phi_1}, T_{\Phi_2}, \dots, T_{\Phi_m})$ is hyponormal then

$$\operatorname{rank}\big[\mathbf{T}^*, \mathbf{T}\big] = \deg(\Omega) + \operatorname{rank}\big[T^*_{\Phi^{1,\Omega}_{j_0}}, T_{\Phi^{1,\Omega}_{j_0}}\big]_p,$$

where $\Omega :=$ right-l.c.m.$\{\theta_i\delta_i\delta_{j_0}\overline{\delta(i)}\Theta(i)^* : i = 1, 2, \dots, m\}$. Here $\delta(i) :=$ g.c.d.$\{\delta_i, \delta_{j_0}\}$ and $\Theta(i) :=$ left-g.c.d.$\{\theta_i\delta(i), \overline{\delta(i)}(\delta_{j_0}A_i - \delta_i A_{j_0}\Lambda(i)^*)\}$ with $\Lambda(i) := B_i(\gamma_0)B_{j_0}(\gamma_0)^{-1}$.

Proof Suppose $\mathbf{T}$ is hyponormal. Since every sub-tuple of $\mathbf{T}$ is hyponormal, we can see that (T_{Φ_i}, T_{Φ_j}) is hyponormal for all $i, j = 1, 2, \dots, m$. In view of (2.10), we may assume that $j_0 = m$. Put

$$\mathbf{S} := (T_{\Phi_1 - \Lambda(1)\Phi_m}, T_{\Phi_2 - \Lambda(2)\Phi_m}, \dots, T_{\Phi_{m-1} - \Lambda(m-1)\Phi_m}, T_{\Phi_m}).$$

It follows from Corollary 2.5 that

$$\mathbf{T} \text{ is hyponormal} \quad \Longleftrightarrow \quad \mathbf{S} \text{ is hyponormal}.$$

Since $\delta(i) =$ g.c.d.$\{\delta_i, \delta_m\}$, we can write

$$\delta_i = \delta(i)\omega_i \quad \text{and} \quad \delta_m = \delta(i)\omega_m,$$

where ω_i is a finite Blaschke product for $i = 1, 2, \dots, m$. Since $\Theta(i) =$ left-g.c.d.$\{\theta_i\delta(i), \overline{\delta(i)}(\delta_m A_i - \delta_1 A_m\Lambda(i)^*)\}$, we get the following left coprime factorization:

$$\Phi_i - \Lambda(i)\Phi_m = \big[\big(\overline{\omega_m}A_i^* - \overline{\omega_i}\Lambda(i)A_m^*\big)\Theta(i)\big]\theta_i\delta_i\delta_m\overline{\delta(i)}\Theta(i)^*.$$

Thus the result follows at once from Theorem 2.6. □

We conclude with the following.

Corollary 2.9 *For each $i = 1, 2, \dots, m$, suppose that $\phi_i = \overline{(\phi_i)_-} + (\phi_i)_+ \in L^\infty$ is a rational function of the form*

$$(\phi_i)_+ = \theta_i \overline{a_i} \quad \text{and} \quad (\phi_i)_- = \theta_i \overline{b_i} \quad (\text{coprime}).$$

If there exists j_0 $(1 \le j_0 \le m)$ such that θ_{j_0} and θ_i are not coprime for each $i = 1, 2, \dots, m$ and $\mathbf{T} \equiv (T_{\phi_1}, T_{\phi_2}, \dots, T_{\phi_m})$ is hyponormal then

$$\operatorname{rank}[\mathbf{T}^*, \mathbf{T}] = \operatorname{rank}[T^*_{\Phi_{j_0}}, T_{\Phi_{j_0}}].$$

Proof For each $i = 1, 2, \dots, m$, let $\lambda(i) := b_i(\gamma_0) b_{j_0}(\gamma_0)^{-1}$ for some $\gamma_0 \in \mathcal{Z}(\theta_i)$. Write $\theta(i) \equiv$ g.c.d.$\{\theta_i, (a_i - a_{j_0}\overline{\lambda(i)})\}$. Since $\mathbf{T} \equiv (T_{\phi_1}, T_{\phi_2}, \dots, T_{\phi_n})$ is hyponormal, $(T_{\phi_i}, T_{\phi_{j_0}})$ is hyponormal for all $i = 1, 2, \dots, n$. Thus it follows from Lemma 2.7 that $T_{\phi_{j_0}^{1,\omega(i)}}$ is hyponormal with $\omega(i) := \theta_i \overline{\theta(i)}$. Observe that

$$\left(\phi_{j_0}^{1,\omega(i)}\right)_+ = \theta(i)\overline{c_i} \quad \text{and} \quad \left(\phi_{j_0}^{1,\omega(i)}\right)_- = \theta_i \overline{b_i} \quad (\text{coprime}),$$

where $c_i := P_{\mathcal{H}(\theta(i))}(a_i)$. Since $T_{\phi_{j_0}^{1,\omega(i)}}$ is hyponormal, it follows from Proposition 2.2 that θ_i is an inner divisor of $\theta(i)$ and hence $\theta(i) = \theta_i$. Thus the result follows from Corollary 2.8. □

3 Conclusions

The self-commutators of bounded linear operators play an important role in the study of hyponormal and subnormal operators. The main result of this paper is to derive a rank formula for the self-commutators of tuples of Toeplitz operators with matrix-valued rational symbols. This result will contribute to the study of Toeplitz operators and the bridge theory of operators.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper.

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