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# Lipschitz and BMO norm inequalities for the composite operator on differential forms

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## Abstract

In this paper, we obtain Poincaré-type inequalities for the composite operator acting on differential forms and establish the  $L^p$ , Lipschitz, and BMO norm estimates. We also give the weighted versions of the comparison theorems for the  $L^p$ , Lipschitz, and BMO norms.

**Keywords:** norm inequality; differential form; Dirac operator

## 1 Introduction

Differential forms are a generalization of the traditional functions. In recent years, differential forms have been widely used in physics systems, differential geometry, and PDEs. In this paper, we are interested in the properties of the composite operator acting on differential forms. Operator theory plays a critical role in investigating the properties of the solutions to partial differential equations. Many questions in partial differential equations involve estimating various norms of operators. The operator theory for functions has been very well developed in recent years. However, compared to the function cases, the operator theory for differential forms is more complicated, so we need some advanced methods to deal with operators. This paper contributes to derive the properties of the composite operator  $\mathbb{M}_s^\sharp \circ D \circ G$  on differential forms, where  $\mathbb{M}_s^\sharp$  is the general sharp maximal operator defined by

$$\mathbb{M}_s^\sharp(u) = \mathbb{M}_s^\sharp u(x) = \sup_{r>0} \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(t) - u_{B(x,r)}|^s dt \right)^{1/s}$$

for any  $u(x) \in L^p(M, \Lambda^l)$ , where  $1 \leq s \leq p$ . Here  $D$  is the Dirac operator proposed by the physicist Dirac. According to the needs of practical problems, different versions of the Dirac operators have been defined. The Dirac operator we are studying is the Hodge-Dirac operator defined by  $D = d + d^*$ . Here  $d$  is the exterior differential operator on differential forms, and  $d^*$  is the formal adjoint operator of  $d$ . See [1] for more details. The operator  $G$  is the well-known Green's operator satisfying the equation

$$\Delta G(u) = u - H(u),$$

where  $H$  is the harmonic projection operator. See [2–6] for more results and applications for the sharp maximal operator, the Dirac operator, and Green's operator.

In the following,  $M$  stands for a bounded convex domain in  $\mathbb{R}^n$ ,  $n \geq 2$ . The Lebesgue measure of a measurable set  $E \subseteq \mathbb{R}^n$  is denoted by  $|E|$ . We use  $B$  and  $\sigma B$  to denote concentric balls such that  $\text{diam}(\sigma B) = \sigma \text{diam}(B)$ . By  $\Lambda^l = \Lambda^l(\mathbb{R}^n)$  we denote the linear space of all  $l$ -vectors spanned by the exterior products  $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$  for all ordered  $l$ -tuples  $I = (i_1, i_2, \dots, i_l)$ ,  $1 \leq i_1 < i_2 < \cdots < i_l \leq n$ . The  $l$ -form  $u(x) = \sum_I u_I(x) dx_I$  is a linear combination of the standard basis  $dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_l}$  for all ordered  $l$ -tuples  $I$ . If the coefficient  $u_I$  is differential, we say that  $u$  is a differential  $l$ -form. By  $D'(M, \Lambda^l)$  we denote the space of all differential  $l$ -forms. Similarly, we write  $L^s(M, \Lambda^l)$  for the  $l$ -form  $u(x)$  on  $M$  with  $u_I$  satisfying  $\int_M |u_I|^s < \infty$ .

A differential  $l$ -form  $u \in D'(M, \Lambda^l)$  is called a closed form if  $du = 0$  in  $M$ . From the Poincaré lemma  $ddu = 0$  we know that  $du$  is a closed form. The module of a differential form  $u$  is given by  $|u|^2 = *(u \wedge *u) \in D'(M, \Lambda^0)$ .

A very important operator, the homotopy operator  $T : C^\infty(M, \Lambda^l) \rightarrow C^\infty(M, \Lambda^{l-1})$ , is defined by

$$Tu = \int_M \varphi(y) K_y u \, dy$$

for differential forms  $u$ , where  $\varphi \in C_0^\infty(M)$  is normalized by  $\int_M \varphi(y) \, dy = 1$ , and  $K_y$  is the liner operator defined by

$$(K_y u)(x; \xi_1, \dots, \xi_{l-1}) = \int_0^1 t^{l-1} u(tx + y - ty; x - y; \xi_1, \dots, \xi_{l-1}) \, dt.$$

For the homotopy operator  $T$ , we have the following decomposition, which will be used repeatedly in this paper:

$$u = d(Tu) + T(du)$$

for any differential form  $u$ . A closed form  $u_M$  is defined by  $u_M = d(Tu)$ ; in particular, when  $u$  is a 0-form,  $u_M = |M|^{-1} \int_M u(y) \, dy$ . In regard to Green's operator, we need the following results in [7]:

$$\|dd^*G(u)\|_{s,B} + \|d^*dG(u)\|_{s,B} + \|dG(u)\|_{s,B} + \|d^*G(u)\|_{s,B} + \|G(u)\|_{s,B} \leq C(s)\|u\|_{s,B},$$

$$\|d^*G(u)\|_{s,B} = \|Gd^*(u)\|_{s,B}, \text{ and } \|dG(u)\|_{s,B} = \|Gd(u)\|_{s,B}$$

for any differential form  $u$  in  $M$  and  $1 < s < \infty$ .

## 2 Poincaré-type inequality

In this section, we give a Poincaré-type inequality for the composite operator  $\mathbb{M}_s^\sharp \circ D \circ G$ , which will be used in the estimates for the  $L^p$ , Lipschitz, and BMO norms. We will need the following lemmas.

The following estimate for the homotopy operator  $T$  appears in [8].

**Lemma 2.1** *Let  $u \in L^t_{\text{loc}}$ ,  $l = 1, 2, \dots, n$ ,  $1 < t < \infty$ , be a differential form in  $M$ , and  $T$  be the homotopy operator defined on differential forms. Then there exists a constant  $C$ , independent of  $u$ , such that*

$$\|Tu\|_{t,M} \leq C|M| \text{diam}(M)\|u\|_{t,M}.$$

We will use the generalized Hölder inequality repeatedly.

**Lemma 2.2** [1] *Let  $0 < q < \infty, 0 < p < \infty$ , and  $s^{-1} = q^{-1} + p^{-1}$ . If  $f$  and  $g$  are measurable functions on  $\mathbb{R}^n$ , then*

$$\|fg\|_{s,M} \leq \|f\|_{q,M} \|g\|_{p,M}$$

for any  $M \subset \mathbb{R}^n$ .

The following lemma appears in [9].

**Lemma 2.3** *Let  $\varphi : [0, +\infty)$  be a strictly increasing convex function such that  $\varphi(0) = 0$ . If  $u(x) \in D'(M, \Lambda^l)$  satisfies  $\varphi(|u|) \in L^1(M, \mu)$ , then for any  $a > 0$ , we have*

$$\int_M \varphi\left(\frac{a}{2}|u - u_M|\right) d\mu \leq \int_M \varphi(a|u|) d\mu.$$

First, we establish the boundedness for the composite operator  $\mathbb{M}_s^\sharp \circ D \circ G$ .

**Lemma 2.4** *Let  $u \in L^t(M, \Lambda^l), l = 1, 2, \dots, n, 1 \leq s < t < \infty$ , be a differential form in a domain  $M$ . Then, there exists a constant  $C$ , independent of  $u$ , such that*

$$\|\mathbb{M}_s^\sharp DG(u)\|_{t,B} \leq C|B| \text{diam}(B) \|u\|_{t,B}.$$

*Proof* For a ball  $B$  in  $M$ , using Lemma 2.1 for any  $B_{(x,r)} \subset B$  and the decomposition theorem, we have

$$\begin{aligned} & \left( \frac{1}{|B_{(x,r)}|} \int_{B_{(x,r)}} |DG(u) - (DG(u))_{B_{(x,r)}}|^s dt \right)^{1/s} \\ &= |B_{(x,r)}|^{-1/s} \|DG(u) - (DG(u))_{B_{(x,r)}}\|_{s,B_{(x,r)}} \\ &= |B_{(x,r)}|^{-1/s} \|TdDG(u)\|_{s,B_{(x,r)}} \\ &\leq C|B_{(x,r)}|^{1-1/s+1/n} \|dDG(u)\|_{s,B_{(x,r)}} \\ &= C|B_{(x,r)}|^{1+1/n-1/s} \|ddG(u) + dd^*G(u)\|_{s,B_{(x,r)}} \\ &= C|B_{(x,r)}|^{1+1/n-1/s} \|dd^*G(u)\|_{s,B_{(x,r)}} \\ &\leq C_1|B_{(x,r)}|^{1+1/n-1/s} \|u\|_{s,B_{(x,r)}}. \end{aligned} \tag{1}$$

Since  $1 + 1/n - 1/s > 0$ , taking the supremum over  $r$ , we get

$$\begin{aligned} & \sup_{r>0} \left( \frac{1}{|B_{(x,r)}|} \int_{B_{(x,r)}} |DG(u) - (DG(u))_{B_{(x,r)}}|^s dt \right)^{1/s} \\ &\leq \sup_{r>0} (C_1|B_{(x,r)}|^{1+1/n-1/s} \|u\|_{s,B_{(x,r)}}) \\ &\leq \sup_{r>0} (C_1|B|^{1+1/n-1/s} \|u\|_{s,B}) \\ &= C_1|B|^{1+1/n-1/s} \|u\|_{s,B}. \end{aligned} \tag{2}$$

Using the generalized Hölder inequality, we find

$$\begin{aligned} \|u\|_{s,B} &\leq \|u\|_{t,B} \|1\|_{ts/(t-s),B} \\ &= |B|^{(t-s)/ts} \|u\|_{t,B}. \end{aligned} \tag{3}$$

Combining (2) and (3), we obtain

$$\begin{aligned} \|\mathbb{M}_s^\sharp DG(u)\|_{t,B} &\leq \|C_1 |B|^{1+1/n-1/s} \|u\|_{s,B}\|_{t,B} \\ &\leq \|C_1 |B|^{1+1/n-1/s+(t-s)/ts} \|u\|_{t,B}\|_{t,B} \\ &= C_1 |B|^{1+1/n-1/t} \|u\|_{t,B} \|1\|_{t,B} \\ &= C_1 |B|^{1+1/n} \|u\|_{t,B}. \end{aligned} \tag{4}$$

The proof of Lemma 2.4 is completed. □

**Theorem 2.5** *Let  $u \in L^t(M, \Lambda^l)$ ,  $l = 1, 2, \dots, n$ ,  $1 \leq s < t < \infty$ , be a differential form in a domain  $M$ . Then,*

$$\|\mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B\|_{t,B} \leq C |B| \text{diam}(B) \|u\|_{t,B},$$

where  $C$  is a constant independent of  $u$ .

*Proof* Choosing  $\varphi(t) = x^t$ ,  $a = 2$ , and  $\omega(x) \equiv 1$  in Lemma 2.3, we have

$$\|u - u_B\|_{t,B} \leq C_1 \|u\|_{t,B}.$$

Replacing  $u$  by  $\mathbb{M}_s^\sharp DG(u)$  and using Lemma 2.4, we get

$$\begin{aligned} \|\mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B\|_{t,B} &\leq C_1 \|\mathbb{M}_s^\sharp DG(u)\|_{t,B} \\ &\leq C |B| \text{diam}(B) \|u\|_{t,B}. \end{aligned} \tag{5}$$

The proof of Theorem 2.5 is completed. □

### 3 Lipschitz and BMO norm inequalities

In this section, we compare the  $L^p$  norm, Lipschitz norm, and BMO norm of the composite operator  $\mathbb{M}_s^\sharp \circ D \circ G$  applied to differential forms. Especially, when we estimate the Lipschitz norm in terms of the BMO norm, we need the differential form to satisfy some versions of harmonic equations. We first introduce some definitions.

We call an equation a nonhomogeneous  $A$ -harmonic equation if

$$d^* A(x, du) = B(x, du), \tag{6}$$

where the operators  $A : M \times \Lambda^l(\mathbb{R}^n) \rightarrow \Lambda^l(\mathbb{R}^n)$  and  $B : M \times \Lambda^l(\mathbb{R}^n) \rightarrow \Lambda^{l-1}(\mathbb{R}^n)$  satisfy

$$|A(x, \xi)| \leq a |\xi|^{p-1}, \quad A(x, \xi) \cdot \xi \geq |\xi|^p \quad \text{and} \quad |B(x, \xi)| \leq b |\xi|^{p-1}$$

for almost all  $x \in M$  and all  $\xi \in \Lambda^l(\mathbb{R}^n)$ . Here  $p > 1$  is a constant related to equation (6) and  $a, b > 0$ . Now we give definitions of the BMO and Lipschitz norms. See [1] for more details.

For  $u \in L^1_{\text{loc}}(M, \Lambda^l)$ ,  $l = 0, 1, \dots, n$ , we write  $u \in \text{loc Lip}_k(M, \Lambda^l)$ ,  $0 \leq k \leq 1$ , if

$$\|u\|_{\text{loc Lip}_k, M} = \sup_{\sigma B \subset M} |B|^{-(n+k)/n} \|u - u_B\|_{1, B} < \infty$$

for some  $\sigma > 1$ .

For  $u \in L^1_{\text{loc}}(M, \Lambda^l)$ ,  $l = 0, 1, \dots, n$ , we write  $u \in \text{BMO}(M, \Lambda^l)$  if

$$\|u\|_{*, M} = \sup_{\sigma B \subset M} |B|^{-1} \|u - u_B\|_{1, B} < \infty$$

for some  $\sigma > 1$ . Similarly, we can define the weighted BMO norm and Lipschitz norm.

For  $u \in L^1_{\text{loc}}(M, \Lambda^l, w)$ ,  $l = 0, 1, 2, \dots, n$ , we write  $u \in \text{loc Lip}_k(M, \Lambda^l, w)$ ,  $0 \leq k \leq 1$ , if

$$\|u\|_{\text{loc Lip}_k, M, w} = \sup_{\sigma B \subset M} (\mu(B))^{-(n+k)/n} \|u - u_B\|_{1, B, w} < \infty$$

for some  $\sigma > 1$ , where  $w$  is a weight, and  $\mu$  is the Radon measure defined by  $d\mu = w(x) dx$ .

For  $u \in L^1_{\text{loc}}(M, \Lambda^l, w)$ ,  $l = 0, 1, 2, \dots, n$ , we write  $u \in \text{BMO}(M, \Lambda^l, w)$  if

$$\|u\|_{*, M, w} = \sup_{\sigma B \subset M} (\mu(B))^{-1} \|u - u_B\|_{1, B, w} < \infty$$

for some  $\sigma > 1$ , where  $w$  is a weight, and  $\mu$  is the Radon measure defined by  $d\mu = w(x) dx$ .

We also need the following inequality.

**Lemma 3.1** [1] *Take  $\varphi$  be a strictly increasing convex function on  $[0, +\infty)$  such that  $\varphi(0) = 0$ . If  $u(x) \in D'(M, \Lambda^l)$  satisfies  $\varphi(|u|) \in L^1(M, \mu)$  and for any constant  $c$ ,*

$$\mu\{x \in M : |u - c| > 0\} > 0,$$

where  $\mu$  is the Radon measure defined by  $d\mu(x) = \omega(x) dx$  with weight  $\omega(x)$ , then for any  $a > 0$ , we have

$$\int_M \varphi(a|u|) d\mu \leq C \int_M \varphi(2a|u - u_M|) d\mu,$$

where  $C$  is a constant independent of  $u$ .

Now, we estimate the Lipschitz norm of  $M_s^\sharp \circ D \circ G$  in terms of the  $L^t$ -norm.

**Theorem 3.2** *Let  $u \in L^t(M, \Lambda^l)$ ,  $l = 1, 2, \dots, n$ ,  $1 \leq s < t < \infty$ , be a differential form in  $M$ . Then, there exists a constant  $C$ , independent of  $u$ , such that*

$$\|M_s^\sharp DG(u)\|_{\text{loc Lip}_k, M} \leq C \|u\|_{t, M},$$

where  $k$  is a constant with  $0 \leq k \leq 1$ .

*Proof* From Theorem 2.5 we obtain

$$\| \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B \|_{t,B} \leq C_1 |B| \text{diam}(B) \|u\|_{t,B}$$

for all balls  $B$  with  $B \subset M$ . Using the Hölder inequality, we have

$$\begin{aligned} & \| \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B \|_{1,B} \\ & \leq \left( \int_B | \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B |^t dx \right)^{1/t} \left( \int_B 1^{t/(t-1)} dx \right)^{(t-1)/t} \\ & \leq |B|^{(t-1)/t} \| \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B \|_{t,B} \\ & = |B|^{1-1/t} \| \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B \|_{t,B} \\ & \leq |B|^{1-1/t} C_1 |B| \text{diam}(B) \|u\|_{t,B} \\ & \leq C_2 |B|^{2-1/t+1/n} \|u\|_{t,B}. \end{aligned} \tag{7}$$

From the definition of the Lipschitz norm and (7) it follows that

$$\begin{aligned} \| \mathbb{M}_s^\sharp DG(u) \|_{\text{loc Lip}_k, M} &= \sup_{\sigma B \subset M} |B|^{-(n+k)/n} \| \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B \|_{1,B} \\ &= \sup_{\sigma B \subset M} |B|^{-1-k/n} \| \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B \|_{1,B} \\ &\leq \sup_{\sigma B \subset M} |B|^{-1-k/n} C_2 |B|^{2-1/t+1/n} \|u\|_{t,B} \\ &= \sup_{\sigma B \subset M} C_2 |B|^{1-k/n-1/t+1/n} \|u\|_{t,B} \\ &\leq \sup_{\sigma B \subset M} C_2 |\mathcal{M}|^{1-k/n-1/t+1/n} \|u\|_{t,B} \\ &\leq C_3 \sup_{\sigma B \subset M} \|u\|_{t,B} \\ &\leq C_3 \|u\|_{t, M}. \end{aligned} \tag{8}$$

This ends the proof of Theorem 3.2. □

From the definitions of the Lipschitz and BMO norms we can get a simple relationship.

**Theorem 3.3** *Let  $u \in L^s(M, \Lambda^l)$ ,  $l = 1, 2, \dots, n$ ,  $1 \leq s < \infty$ , be a differential form in  $M$ . Then,*

$$\| \mathbb{M}_s^\sharp DG(u) \|_{*, M} \leq C \| \mathbb{M}_s^\sharp DG(u) \|_{\text{loc Lip}_k, M}$$

where  $k$  is a constant with  $0 \leq k \leq 1$ , and  $C$  is a constant independent of  $u$ .

*Proof* From the definition of the BMO norms we obtain

$$\begin{aligned} \| \mathbb{M}_s^\sharp DG(u) \|_{*, M} &= \sup_{\sigma B \subset M} |B|^{-1} \| \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B \|_{1,B} \\ &= \sup_{\sigma B \subset M} |B|^{k/n} |B|^{-(n+k)/n} \| \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B \|_{1,B} \end{aligned}$$

$$\begin{aligned}
 &\leq \sup_{\sigma B \subset M} |M|^{k/n} |B|^{-(n+k)/n} \left\| \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B \right\|_{1,B} \\
 &\leq |M|^{k/n} \sup_{\sigma B \subset M} |B|^{-(n+k)/n} \left\| \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B \right\|_{1,B} \\
 &\leq C \sup_{\sigma B \subset M} |B|^{-(n+k)/n} \left\| \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B \right\|_{1,B} \\
 &\leq C \left\| \mathbb{M}_s^\sharp DG(u) \right\|_{\text{loc Lip}_k, M}.
 \end{aligned} \tag{9}$$

This ends the proof of Theorem 3.3. □

**Theorem 3.4** *Let  $u \in L^s(M, \Lambda^l)$ ,  $l = 1, 2, \dots, n$ ,  $1 < s < \infty$ , satisfy equation (6) in  $M$  and suppose that  $\mu\{x \in M : |u - c| > 0\} > 0$ . Then,*

$$\left\| \mathbb{M}_s^\sharp DG(u) \right\|_{\text{loc Lip}_k, M} \leq C \|u\|_{*,M},$$

where  $k$  is a constant with  $0 \leq k \leq 1$ , and  $C$  is a constant independent of  $u$ .

*Proof* From (7) we have

$$\left\| \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B \right\|_{1,B} \leq C_1 |B|^{2-1/s+1/n} \|u\|_{s,B}.$$

Using the reverse Hölder inequality and Lemma 3.1, we get

$$\|u\|_{s,B} \leq C_2 |B|^{(1-s)/s} \|u\|_{1,\sigma B} \leq C_3 |B|^{(1-s)/s} \|u - (u)_B\|_{1,\sigma B},$$

where  $\sigma > 1$ . So, we have

$$\left\| \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B \right\|_{1,B} \leq C_4 |B|^{1+1/n} \|u - (u)_B\|_{1,\sigma B}.$$

Letting  $\sigma' > \sigma$ , we have

$$\begin{aligned}
 \left\| \mathbb{M}_s^\sharp DG(u) \right\|_{\text{loc Lip}_k, M} &= \sup_{\sigma' B \subset M} |B|^{-(n+k)/n} \left\| \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B \right\|_{1,B} \\
 &\leq \sup_{\sigma' B \subset M} C_4 |B|^{1+1/n-k/n} |B|^{-1} \|u - (u)_B\|_{1,\sigma B} \\
 &\leq C_5 \sup_{\sigma' B \subset M} C_4 |B|^{-1} \|u - (u)_B\|_{1,\sigma B} \\
 &\leq C_6 \|u\|_{*,M}.
 \end{aligned} \tag{10}$$

This ends the proof of Theorem 3.4. □

By Theorem 3.2 and Theorem 3.3 we can easily estimate the BMO norm of the composite operator  $\mathbb{M}_s^\sharp \circ D \circ G$ .

**Corollary 3.5** *Let  $u \in L^t(M, \Lambda^l)$ ,  $l = 1, 2, \dots, n$ ,  $1 \leq s < t < \infty$ , be a differential form in  $M$ . Then,*

$$\left\| \mathbb{M}_s^\sharp DG(u) \right\|_{*,M} \leq C \|u\|_{t,M},$$

where  $C$  is a constant independent of  $u$ .

#### 4 The weighted norm inequalities

In this section, we consider the weighted situation. The weight function we select is  $A(\alpha, \beta, \gamma, M)$ -weight, which contains the well-known  $A_r(M)$ -weight. We will use the Radon measure to deal with the  $A(\alpha, \beta, \gamma, M)$ -weight in the proof.

**Definition 4.1** [10] We say that a measurable function  $w(x)$  defined on a subset  $M \subset \mathbb{R}^n$  satisfies the  $A(\alpha, \beta, \gamma, M)$ -condition for some positive constants  $\alpha, \beta, \gamma$  if  $w(x) > 0$  a.e. and

$$\sup_{B \subset M} \left( \frac{1}{|B|} \int_B w^\alpha dx \right) \left( \frac{1}{|B|} \int_B w^{-\beta} dx \right)^{\gamma/\beta} < \infty.$$

Now, we give estimates for the weighted Lipschitz and BMO norms.

**Theorem 4.2** Let  $u \in L^q(M, \Lambda^l, \mu)$ ,  $l = 1, 2, \dots, n$ ,  $1 \leq s < q < \infty$ . Assume that the Radon measure  $\mu$  is defined by  $d\mu = w(x) dx$  with  $w(x) \in A(\alpha, \beta, \gamma, M)$  for some  $\alpha > 1$ , where  $1 < p < \infty$ ,  $\beta = \alpha q / (\alpha p - p - \alpha q)$ ,  $\gamma = \alpha q / p$ , and  $\alpha p - p - \alpha q > 0$ . Then,

$$\| \mathbb{M}_s^\sharp DG(u) \|_{\text{loc Lip}_k, M, w} \leq C \|u\|_{p, M, w},$$

where  $C$  is a constant independent of  $u$ .

*Proof* Using the generalized Hölder inequality with exponents satisfying  $1/q = 1/\alpha q + (\alpha - 1)/\alpha q$ , we get

$$\begin{aligned} & \left( \int_B | \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B |^q w(x) dx \right)^{1/q} \\ &= \left( \int_B | (\mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B) w(x)^{1/q} |^q dx \right)^{1/q} \\ &\leq \left( \int_B | \mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B |^{\alpha q / (\alpha - 1)} dx \right)^{(\alpha - 1) / \alpha q} \left( \int_B (w(x)^{1/q})^{\alpha q} dx \right)^{1/\alpha q} \\ &\leq C_1 |B| \text{diam}(B) \left( \int_B |u|^{\alpha q / (\alpha - 1)} dx \right)^{(\alpha - 1) / \alpha q} \left( \int_B w(x)^\alpha dx \right)^{1/\alpha q}. \end{aligned} \tag{11}$$

Using the generalized Hölder inequality with exponents satisfying  $(\alpha - 1)/\alpha q = 1/p + (\alpha p - p - \alpha q)/\alpha q p$ , we get

$$\begin{aligned} \left( \int_B |u|^{\alpha q / (\alpha - 1)} dx \right)^{(\alpha - 1) / \alpha q} &= \left( \int_B |u w(x)^{1/p} w(x)^{-1/p}|^{\alpha q / (\alpha - 1)} dx \right)^{(\alpha - 1) / \alpha q} \\ &\leq \left( \int_B |u w(x)^{1/p}|^p dx \right)^{1/p} \\ &\quad \times \left( \int_B (w(x)^{-1/p})^{\alpha q p / (\alpha p - p - \alpha q)} dx \right)^{(\alpha p - p - \alpha q) / \alpha q p} \\ &= \left( \int_B |u|^p w(x) dx \right)^{1/p} \\ &\quad \times \left( \int_B (w(x)^{-1})^{\alpha q / (\alpha p - p - \alpha q)} dx \right)^{(\alpha p - p - \alpha q) / \alpha q p}. \end{aligned} \tag{12}$$



Since  $w(x) \in A(\alpha, \frac{\alpha q}{\alpha p - p - \alpha q}, \frac{\alpha q}{p}, M)$ , we have

$$\begin{aligned} & \left( \int_B w(x)^\alpha dx \right)^{1/\alpha q} \left( \int_B (w(x)^{-1})^{\alpha q/(\alpha p - p - \alpha q)} dx \right)^{(\alpha p - p - \alpha q)/\alpha q p} \\ &= |B|^{2/\alpha q} \left( \left( \frac{1}{|B|} \int_B w(x)^\alpha dx \right) \left( \frac{1}{|B|} \int_B (w(x)^{-1})^{\alpha q/(\alpha p - p - \alpha q)} dx \right)^{(\alpha p - p - \alpha q)/p} \right)^{1/\alpha q} \\ &\leq C_2. \end{aligned} \tag{13}$$

Combining (11), (12), and (13), we obtain

$$\left( \int_B |\mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B|^q w(x) dx \right)^{1/q} \leq C_3 |B| \text{diam}(B) \left( \int_B |u|^p w(x) dx \right)^{1/p}.$$

It follows that

$$\begin{aligned} & \|\mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B\|_{1,B,w} \\ &= \int_B |\mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B| d\mu \\ &\leq \left( \int_B |\mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B|^q w(x) dx \right)^{1/q} \left( \int_B 1^{q/(q-1)} d\mu \right)^{(q-1)/q} \\ &= (\mu(B))^{(q-1)/q} \|\mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B\|_{q,B,w} \\ &\leq C_4 (\mu(B))^{(q-1)/q} |B| \text{diam}(B) \|u\|_{p,B,w}. \end{aligned} \tag{14}$$

Finally, we get

$$\begin{aligned} & \|\mathbb{M}_s^\sharp DG(u)\|_{\text{loc Lip}_k, M, w} \\ &= \sup_{\sigma B \subset M} (\mu(B))^{-(n+k)/n} \|\mathbb{M}_s^\sharp DG(u) - (\mathbb{M}_s^\sharp DG(u))_B\|_{1,B,w} \\ &\leq C_5 \sup_{\sigma B \subset M} (\mu(B))^{-(n+k)/n + (q-1)/q + 1 + 1/n} \|u\|_{p,B,w} \\ &\leq C_5 \sup_{\sigma B \subset M} (\mu(M))^{-(n+k)/n + (q-1)/q + 1 + 1/n} \|u\|_{p,B,w} \\ &\leq C \|u\|_{p, M, w}. \end{aligned} \tag{15}$$

This ends the proof of Theorem 4.2. □

Similarly to the proof of Theorem 3.3, we have the following corollary.

**Corollary 4.3** *Let  $u \in L^s(M, \Lambda^l, \mu)$ ,  $l = 1, 2, \dots, n$ ,  $1 \leq s < \infty$ , be a differential form in  $M$ ,  $\mu$  and  $w(x)$  be the same as in Theorem 4.2. Then,*

$$\|\mathbb{M}_s^\sharp DG(u)\|_{*, M, w} \leq C \|\mathbb{M}_s^\sharp DG(u)\|_{\text{loc Lip}_k, M, w}$$

where  $C$  is a constant independent of  $u$ .

The following corollary can be obtained by combining Theorem 4.2 and Corollary 4.3.

**Corollary 4.4** *Let  $u, \mu, w(x)$ , and  $p$  be as in Theorem 4.2. Then,*

$$\|M_s^\sharp DG(u)\|_{*,M,w} \leq C \|u\|_{p,M,w},$$

where  $C$  is a constant independent of  $u$ .

### 5 Applications

In this section, we apply our results to some differential forms.

**Example 5.1** Let  $M = \{(x, y, z) : x_1^2 + x_2^2 + \dots + x_n^2 \leq 1\} \subset \mathbb{R}^n$  and  $u(x_1, \dots, x_n)$  be defined in  $\mathbb{R}^n$  by

$$u(x_1, \dots, x_n) = \sum_{i=1}^n e^{1+x_1^2+\dots+x_n^2} \frac{x_i}{1+(x_1^2+\dots+x_n^2)^2} dx_i.$$

So  $u(x_1, \dots, x_n)$  is a differential form in  $M$ . Now we estimate  $\|M_s^\sharp DG(u) - (M_s^\sharp DG(u))_M\|_{t,M}$ .

By simple calculation we obtain

$$\begin{aligned} \|u\|_{t,M} &= \left( \int_M |u|^t dx \right)^{1/t} \\ &\leq \left( \int_M (e^{2+2x_1^2+\dots+2x_n^2} (x_1^2 + \dots + x_n^2))^{t/2} dx \right)^{1/t} \\ &\leq \left( \int_M (e^4)^{t/2} dx \right)^{1/t} \\ &= e^2 |M|^{1/t}. \end{aligned}$$

Using Theorem 2.5, we have

$$\begin{aligned} \|M_s^\sharp DG(u) - (M_s^\sharp DG(u))_M\|_{t,\Omega} &\leq C |M| \text{diam}(M) e^2 |M|^{1/t} \\ &= 2e^2 C \left( \frac{\pi^{n/2}}{\Gamma(1+n/2)} \right)^{1+1/t}. \end{aligned}$$

**Example 5.2** Let  $u(x_1, \dots, x_n)$  be defined in  $\mathbb{R}^n$  by

$$u(x_1, \dots, x_n) = \sum_{i=1}^n \frac{1}{\sqrt{1+x_1^2+\dots+x_n^2}} dx_i.$$

For a ball  $B \subset \mathbb{R}^n$  with radius  $r$ , it is difficult to estimate the upper bound directly for  $\|M_s^\sharp DG(u)\|_{\text{loc Lip}_k, B}$ , but by Theorem 3.2 we have

$$\begin{aligned} \|M_s^\sharp DG(u)\|_{\text{loc Lip}_k, B} &\leq C \|u\|_{t,B} \\ &\leq C \left( \int_B ((x_1^2 + \dots + x_n^2)/(1+x_1^2+\dots+x_n^2))^{t/2} dx \right)^{1/t} \end{aligned}$$

$$\begin{aligned} &\leq C|B|^{1/t} \\ &= C\left(\frac{\pi^{n/2}r^n}{\Gamma(1+n/2)}\right)^{1/t}. \end{aligned}$$

Similarly, we also obtain an upper bound for the BMO norm of the composite operator  $M_s^\sharp \circ D \circ G$ .

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed to the main results. XL drafted the manuscript. YW and YX improved the final version. All authors read and approved the final manuscript.

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