Kim et al. *Advances in Difference Equations* 2013, **2013**:73
http://www.advancesindifferenceequations.com/content/2013/1/73

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Some new identities of Bernoulli, Euler and Hermite polynomials arising from umbral calculus

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Abstract

In this paper, we derive the identities of higher-order Bernoulli, Euler and Frobenius-Euler polynomials from the orthogonality of Hermite polynomials. Finally, we give some interesting and new identities of several special polynomials arising from umbral calculus.

MSC: 05A10; 05A19

Keywords: Bernoulli polynomial; Euler polynomial; Abel polynomial

1 Introduction

The Hermite polynomials are defined by the generating function to be

$$e^{2xt-t^2} = e^{H(x)t} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \tag{1.1}$$

with the usual convention about replacing $H^n(x)$ by $H_n(x)$ (see [1]). In the special case, $x = 0$, $H_n(0) = H_n$ are called the *nth Hermite numbers*. From (1.1) we have

$$H_n(x) = (H + 2x)^n = \sum_{l=0}^{n} \binom{n}{l} H_{n-l}x^l 2^l. \tag{1.2}$$

Thus, by (1.2), we get

$$\frac{d^k}{dx^k}H_n(x) = 2^k(n)_k H_{n-k}(x) = 2^k\frac{n!}{(n-k)!}H_{n-k}(x), \tag{1.3}$$

where $(x)_k = x(x-1)\cdots(x-k+1)$.

As is well known, the Bernoulli polynomials of order r are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!} \quad (r \in \mathbb{R}). \tag{1.4}$$

In the special case, $x = 0$, $B_n^{(r)}(0) = B_n^{(r)}$ are called the nth Bernoulli numbers of order r (see [1–4]).

The Euler polynomials of order r are also defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!} \quad (r \in \mathbb{R}). \tag{1.5}$$

In the special case, $x = 0$, $E_n^{(r)}(0) = E_n^{(r)}$ are called the *nth Euler numbers* of order r.

For $\lambda(\neq 1) \in \mathbb{C}$, the *Frobenius-Euler polynomials* of order r are given by

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!} \quad (r \in \mathbb{R}). \tag{1.6}$$

In the special case, $x = 0$, $H_n^{(r)}(0|\lambda) = H_n^{(r)}(\lambda)$ are called the *nth Frobenius-Euler numbers* of order r (see [1–16]).

The Stirling numbers of the first kind are defined by the generating function to be

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \quad \text{(see [11, 14])}, \tag{1.7}$$

and the Stirling numbers of the second kind are given by

$$\left(e^t-1\right)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!} \quad \text{(see [14])}. \tag{1.8}$$

In [1] it is known that $H_0(x), H_1(x), \ldots, H_n(x)$ from an orthogonal basis for the space

$$\mathbb{P}_n = \left\{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \le n\right\} \tag{1.9}$$

with respect to the inner product

$$\left\langle p_1(x), p_2(x)\right\rangle = \int_{-\infty}^{\infty} e^{-x^2} p_1(x)p_2(x)\,dx \quad \text{(see [1])}. \tag{1.10}$$

For $p(x) \in \mathbb{P}_n$, let us assume that

$$p(x) = \sum_{k=0}^{n} C_k H_k(x). \tag{1.11}$$

Then, from the orthogonality of Hermite polynomials and Rodrigues' formula, we have

$$\begin{aligned} C_k &= \frac{1}{2^k k!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_k(x)p(x)\,dx \\ &= \frac{(-1)^k}{2^k k!\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{d^k}{dx^k} e^{-x^2}\right)p(x)\,dx \quad \text{(see [1])}. \end{aligned} \tag{1.12}$$

In particular, for $p(x) = x^m$ $(m \geq 0)$, we easily get

$$\int_{-\infty}^{\infty}\left(\frac{d^n}{dx^n}e^{-x^2}\right)x^m\,dx = \begin{cases} 0 & \text{if } n > m \text{ or } n \leq m \text{ with } m-n \not\equiv 0 \pmod 2, \\ \frac{(-1)^n m!\sqrt{\pi}}{2^{m-n}(\frac{m-n}{2})!} & \text{if } n \leq m \text{ with } m-n \equiv 0 \pmod 2. \end{cases} \tag{1.13}$$

Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!}t^k \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{1.14}$$

Let us assume that $\mathbb{P}$ is the algebra of polynomials in the variable x over $\mathbb{C}$ and that $\mathbb{P}^*$ is the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ denotes the action of the linear functional L on polynomials $p(x)$, and we remind that the vector space structure on $\mathbb{P}^*$ is defined by

$$\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle,$$
$$\langle cL|p(x)\rangle = c\langle L|p(x)\rangle,$$

where c is a complex constant (see [2, 11, 14]).

The formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!}t^k \in \mathcal{F} \tag{1.15}$$

defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n \quad \text{for all } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \tag{1.16}$$

Thus, by (1.15) and (1.16), we get

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad (n,k \geq 0), \tag{1.17}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [2, 11, 14]).

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!}t^k$. By (1.16), we get

$$\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle, \quad n \geq 0. \tag{1.18}$$

Thus, by (1.18), we see that $f_L(t) = L$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra*. The umbral calculus is the study of umbral algebra (see [2, 11, 14]).

The order $o(f(t))$ of the nonzero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $f(t)$ having $o(f(t)) = 1$ is called a *delta series*, and a series $f(t)$ having $o(f(t)) = 0$ is called an *invertible series* (see [2, 11, 14]). By (1.16) and (1.17), we see that $\langle e^{yt}|p(x)\rangle = p(y)$. For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k\rangle}{k!} t^k, \qquad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k|p(x)\rangle}{k!} x^k. \tag{1.19}$$

Let $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we easily see that

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle. \tag{1.20}$$

From (1.19), we can derive the following equation:

$$p^{(k)}(0) = \langle t^k|p(x)\rangle \quad \text{and} \quad \langle 1|p^{(k)}(x)\rangle = p^{(k)}(0). \tag{1.21}$$

Thus, by (1.21), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{(see [2, 11, 14])}. \tag{1.22}$$

Let $f(t)$ be a delta series, and let $g(t)$ be an invertible series. Then there exists a unique sequence $S_n(x)$ of polynomials with $\langle g(t)f(t)^k|S_n(x)\rangle = n!\delta_{n,k}$, where $n, k \geq 0$ (see [2, 11, 14]). The sequence $S_n(x)$ is called *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$. For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\left\langle \frac{e^{yt}-1}{t}\middle|p(x)\right\rangle = \int_0^y p(u)\,du, \qquad \langle e^{yt}-1|p(x)\rangle = p(y) - p(0), \tag{1.23}$$

and

$$\langle f(t)|xp(x)\rangle = \langle f'(t)|p(x)\rangle. \tag{1.24}$$

In this paper, we introduce the identities of several special polynomials which are derived from the orthogonality of Hermite polynomials. Finally, we give some new and interesting identities of the higher-order Bernoulli, Euler and Frobenius-Euler polynomials arising from umbral calculus.

2 Some identities of several special polynomials

From (1.5), we note that

$$\left(\frac{2}{e^t+1}\right)^r = \left(1 + \frac{e^t-1}{2}\right)^{-r} = \sum_{j=0}^{\infty} \binom{-r}{j} \left(\frac{e^t-1}{2}\right)^j. \tag{2.1}$$

By (2.1), we get

$$\begin{aligned}\left(\frac{2}{e^t+1}\right)^r e^{xt} &= \sum_{j=0}^{\infty} \binom{-r}{j} \left(\frac{e^t-1}{2}\right)^j e^{xt} \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{-r}{j} \left(\frac{e^t-1}{2}\right)^j x^n\right) \frac{t^n}{n!}.\end{aligned} \tag{2.2}$$

From (1.5) and (2.2), we have

$$E_n^{(r)}(x) = \sum_{j=0}^{n} \binom{-r}{j} 2^{-j} \left(e^t - 1\right)^j x^n. \tag{2.3}$$

By (1.8) and (1.9), we get

$$\begin{aligned} \left(e^t - 1\right)^j x^n &= \sum_{k=0}^{n-j} \frac{\langle t^k | (e^t - 1)^j x^n \rangle}{k!} = \sum_{k=0}^{n-j} \frac{\langle (e^t - 1)^j | t^k x^n \rangle}{k!} x^k \\ &= j! \sum_{k=0}^{n-j} \binom{n}{k} \frac{\langle (e^t - 1)^j | x^{n-k} \rangle}{j!} x^k = j! \sum_{k=0}^{n-j} \binom{n}{j} S_2(n-k, j) x^k \\ &= j! \sum_{k=j}^{n} \binom{n}{k} S_2(k, j) x^{n-k}. \end{aligned} \tag{2.4}$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.1 *For $n \geq 0$, we have*

$$\begin{aligned} E_n^{(r)}(x) &= \sum_{0 \leq j \leq n} \sum_{j \leq k \leq n} \binom{n}{k} \binom{-r}{j} \frac{j!}{2^j} S_2(k, j) x^{n-k} \\ &= \sum_{0 \leq k \leq n} \binom{n}{k} \left[\sum_{0 \leq j \leq k} \binom{-r}{j} \frac{j!}{2^j} S_2(k, j) \right] x^{n-k}. \end{aligned}$$

By (1.5), we easily see that

$$E_n^{(r)}(x) = \sum_{k=0}^{n} \binom{n}{k} E_k^{(r)} x^{n-k}. \tag{2.5}$$

Therefore, by Theorem 2.1 and (2.5), we obtain the following corollary.

Corollary 2.2 *For $k \geq 0$, we have*

$$E_k^{(r)} = \sum_{j=0}^{k} \binom{-r}{j} \frac{j!}{2^j} S_2(k, j).$$

Let us take $p(x) = E_n^{(r)}(x) \in \mathbb{P}_n$. Then, by (1.11), we get

$$E_n^{(r)}(x) = \sum_{k=0}^{n} C_k H_k(x). \tag{2.6}$$

From (1.12), we can derive the computation of C_k as follows:

$$C_k = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) E_n^{(r)}(x)\, dx, \tag{2.7}$$

where

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(\frac{d^k e^{-x^2}}{dx^k}\right)E_n^{(r)}(x)\,dx \\
&= (-n)\bigl(-(n-1)\bigr)\cdots\bigl(-(n-k+1)\bigr)\int_{-\infty}^{\infty} e^{-x^2}E_{n-k}^{(r)}(x)\,dx \\
&= \frac{(-1)^k n!}{(n-k)!}\int_{-\infty}^{\infty} e^{-x^2}\sum_{l=0}^{n-k}\binom{n-k}{l}E_{n-k-l}^{(r)}x^l\,dx \\
&= \frac{(-1)^k n!}{(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}E_{n-k-l}^{(r)}\int_{-\infty}^{\infty} e^{-x^2}x^l\,dx \\
&= (-1)^k n!\sqrt{\pi}\sum_{0\le l\le n-k,l:\text{even}}\frac{1}{(n-k-l)!2^l(\frac{l}{2})!}\sum_{j=0}^{n-k-l}\binom{-r}{j}\frac{j!}{2^j}S_2(n-k-l,j).
\end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we can derive the following equation:

$$\begin{aligned}
C_k &= n!\sum_{0\le l\le n-k,l:\text{even}}\frac{E_{n-k-l}^{(r)}}{k!(n-k-l)!2^{k+l}(\frac{l}{2})!} \\
&= n!\sum_{0\le l\le n-k,l:\text{even}}\sum_{j=0}^{n-k-l}\frac{\binom{-r}{j}j!S_2(n-k-l,j)}{k!(n-k-l)!2^{k+l+j}(\frac{l}{2})!}.
\end{aligned} \tag{2.9}$$

Therefore, by Corollary 2.2, (2.6) and (2.9), we obtain the following theorem.

Theorem 2.3 *For* $n \ge 0$, *we have*

$$\begin{aligned}
E_n^{(r)}(x) &= n!\sum_{k=0}^{n}\left\{\sum_{0\le l\le n-k,l:\text{even}}\frac{E_{n-k-l}^{(r)}}{k!(n-k-l)!2^{k+l}(\frac{l}{2})!}\right\}H_k(x) \\
&= n!\sum_{k=0}^{n}\left\{\sum_{0\le l\le n-k,l:\text{even}}\sum_{j=0}^{n-k-l}\frac{\binom{-r}{j}j!S_2(n-k-l,j)}{k!(n-k-l)!2^{k+l+j}(\frac{l}{2})!}\right\}H_k(x).
\end{aligned}$$

By (1.4), we easily see that

$$\left(\frac{t}{e^t-1}\right)^r = \left(1+\frac{e^t-t-1}{t}\right)^{-r} = \sum_{j=0}^{\infty}\binom{-r}{j}\left(\frac{e^t-t-1}{t}\right)^j. \tag{2.10}$$

Thus, by (2.10), we get

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{-r}{j}\left(\frac{e^t-t-1}{t}\right)^j x^n\right)\frac{t^n}{n!}. \tag{2.11}$$

From (1.4) and (2.11), we have

$$B_n^{(r)}(x) = \sum_{j=0}^{n}\binom{-r}{j}\left(\frac{e^t-t-1}{t}\right)^j x^n. \tag{2.12}$$

By (1.19), we easily get

$$\begin{aligned}\left(\frac{e^t-t-1}{t}\right)^j x^n &= \sum_{k=0}^{n-j} \frac{\langle t^k | (\frac{e^t-t-1}{t})^j x^n\rangle}{k!} x^k \\ &= \sum_{k=0}^{n-j} \frac{\langle (\frac{e^t-t-1}{t})^j | t^k x^n\rangle}{k!} x^k = \sum_{k=0}^{n-j}\binom{n}{k}\sum_{l=0}^{j}\binom{j}{l}(-1)^{j-l}\left\langle \left(\frac{e^t-1}{t}\right)^l \middle| x^{n-k}\right\rangle x^k \\ &= \sum_{k=0}^{n-j}\binom{n}{k}\sum_{l=0}^{j}\binom{j}{l}(-1)^{j-l}\left\langle t^0 \middle| \left(\frac{e^t-1}{t}\right)^l x^{n-k}\right\rangle x^k. \end{aligned} \tag{2.13}$$

From (1.8), (1.21) and (2.13), we have

$$\left(\frac{e^t-t-1}{t}\right)^j x^n = \sum_{k=0}^{n-j}\sum_{l=0}^{j}\binom{n}{k}\binom{j}{l}(-1)^{j-l}\frac{(n-k)!l!}{(n-k+l)!}S_2(n-k+l,l)x^k. \tag{2.14}$$

Thus, by (2.12) and (2.14), we get

$$\begin{aligned} B_n^{(r)}(x) &= \sum_{j=0}^{n}\sum_{k=0}^{n-j}\sum_{l=0}^{j}\binom{-r}{j}\binom{n}{k}\binom{j}{l}(-1)^{j-l}\frac{S_2(n-k+l,l)}{\binom{n-k+l}{l}}x^k \\ &= \sum_{k=0}^{n}\binom{n}{k}\left[\sum_{j=0}^{k}\sum_{l=0}^{j}\binom{-r}{j}\binom{j}{l}\frac{S_2(k+l,l)}{\binom{k+l}{l}}(-1)^{j-l}\right]x^{n-k}. \end{aligned} \tag{2.15}$$

Therefore, by (2.12) and (2.15), we obtain the following theorem.

Theorem 2.4 *For $n \geq 0$, we have*

$$B_n^{(r)}(x) = \sum_{k=0}^{n}\binom{n}{k}\left[\sum_{j=0}^{k}\sum_{l=0}^{j}\binom{-r}{j}\binom{j}{l}\frac{S_2(k+l,l)}{\binom{k+l}{l}}(-1)^{j-l}\right]x^{n-k}.$$

By (1.4), we easily get

$$B_n^{(r)}(x) = \sum_{k=0}^{n}\binom{n}{k}B_k^{(r)}x^{n-k}. \tag{2.16}$$

Therefore, by Theorem 2.4 and (2.16), we obtain the following corollary.

Corollary 2.5 *For $k \geq 0$, we have*

$$B_k^{(r)} = \sum_{j=0}^{k}\sum_{l=0}^{j}(-1)^{j-l}\binom{-r}{j}\binom{j}{l}\frac{S_2(k+l,l)}{\binom{k+l}{l}}.$$

Let us consider $p(x) = B_n^{(r)}(x) \in \mathbb{P}_n$. Then, by (1.11), $B_n^{(r)}(x)$ can be written as

$$B_n^{(r)}(x) = \sum_{k=0}^{n} C_k H_k(x). \tag{2.17}$$

Now, we compute C_k's for $B_k^{(r)}(x)$ as follows:

$$C_k = \frac{(-1)^k}{2^k k!\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k}\right) B_n^{(r)}(x)\,dx, \tag{2.18}$$

where

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k}\right) B_n^{(r)}(x)\,dx \\
&\quad = (-n)\bigl(-(n-1)\bigr)\cdots\bigl(-(n-k+1)\bigr) \int_{-\infty}^{\infty} e^{-x^2} B_{n-k}^{(r)}(x)\,dx \\
&\quad = \frac{(-1)^k n!}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l}^{(r)} \int_{-\infty}^{\infty} e^{-x^2} x^l\,dx \\
&\quad = (-1)^k n!\sqrt{\pi} \sum_{0\le l\le n-k, l:\text{even}} \frac{B_{n-k-l}^{(r)}}{(n-k-l)!2^l(\frac{l}{2})!}.
\end{aligned} \tag{2.19}$$

By Corollary 2.5 and (2.19), we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k}\right) B_n^{(r)}(x)\,dx \\
&\quad = (-1)^k n!\sqrt{\pi} \sum_{0\le l\le n-k, l:\text{even}} s \sum_{j=0}^{n-k-l} \sum_{m=0}^{j} \frac{(-1)^{j-m}\binom{-r}{j}\binom{j}{m} S_2(n-k-l+m,m)}{(n-k-l)!2^l(\frac{l}{2})!\binom{n-k-l+m}{m}}.
\end{aligned} \tag{2.20}$$

From (2.18) and (2.20), we have

$$\begin{aligned}
C_k &= n! \sum_{0\le l\le n-k, l:\text{even}} \frac{B_{n-k-l}^{(r)}}{(n-k-l)!k!2^{k+l}(\frac{l}{2})!} \\
&= n! \sum_{0\le l\le n-k, l:\text{even}} \sum_{j=0}^{n-k-l} \sum_{m=0}^{j} \frac{(-1)^{j-m}\binom{-r}{j}\binom{j}{m} S_2(n-k-l+m,m)}{(n-k-l)!k!2^{k+l}(\frac{l}{2})!\binom{n-k-l+m}{m}}.
\end{aligned} \tag{2.21}$$

Therefore, by (2.17) and (2.21), we obtain the following theorem.

Theorem 2.6 *For $n \ge 0$, we have*

$$\begin{aligned}
B_n^{(r)}(x) &= n! \sum_{k=0}^{n} \left\{ \sum_{0\le l\le n-k, l:\text{even}} \frac{B_{n-k-l}^{(r)}}{(n-k-l)!k!2^{k+l}(\frac{l}{2})!} \right\} H_k(x) \\
&= n! \sum_{k=0}^{n} \left\{ \sum_{0\le l\le n-k, l:\text{even}} \sum_{j=0}^{n-k-l} \sum_{m=0}^{j} \frac{(-1)^{j-m}\binom{-r}{j}\binom{j}{m} S_2(n-k-l+m,m)}{(n-k-l)!k!2^{k+l}(\frac{l}{2})!\binom{n-k-l+m}{m}} \right\} H_k(x).
\end{aligned}$$

It is easy to show that

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r = \left(1+\frac{e^t-1}{1-\lambda}\right)^{-r} = \sum_{j=0}^{\infty} \binom{-r}{j} \left(\frac{1}{1-\lambda}\right)^j (e^t-1)^j. \tag{2.22}$$

From (1.6) and (2.22), we have

$$H_n^{(r)}(x|\lambda) = \sum_{j=0}^{n} \binom{-r}{j} (1-\lambda)^{-j} \left(e^t - 1\right)^j x^n, \tag{2.23}$$

where

$$\begin{aligned} \left(e^t - 1\right)^j x^n &= j! \sum_{k=j}^{\infty} S_2(k,j) \frac{t^k}{k!} x^n \\ &= j! \sum_{k=j}^{n} \binom{n}{k} S_2(k,j) x^{n-k}. \end{aligned} \tag{2.24}$$

Thus, by (2.24), we get

$$\left(e^t - 1\right)^j x^n = j! \sum_{k=j}^{n} \binom{n}{k} S_2(k,j) x^{n-k}. \tag{2.25}$$

From (2.23) and (2.25), we can derive the following equation:

$$\begin{aligned} H_n^{(r)}(x|\lambda) &= \sum_{j=0}^{n} \sum_{k=j}^{n} \binom{n}{k} \binom{-r}{j} \frac{j!}{(1-\lambda)^j} S_2(k,j) x^{n-k} \\ &= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{-r}{j} \frac{j!}{(1-\lambda)^j} S_2(k,j) x^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} \left[\sum_{j=0}^{k} \binom{-r}{j} \frac{j!}{(1-\lambda)^j} S_2(k,j) \right] x^{n-k}. \end{aligned} \tag{2.26}$$

By (1.6), we easily see that

$$H_n^{(r)}(x|\lambda) = \sum_{k=0}^{n} \binom{n}{k} H_k^{(r)}(\lambda) x^{n-k}. \tag{2.27}$$

Therefore, by (2.26) and (2.27), we obtain the following theorem.

Theorem 2.7 *For $k \geq 0$, we have*

$$H_k^{(r)}(\lambda) = \sum_{j=0}^{k} \binom{-r}{j} \frac{j!}{(1-\lambda)^j} S_2(k,j).$$

Let us take $p(x) = H_n^{(r)}(x|\lambda) \in \mathbb{P}_n$. Then, by (1.11), $H_n^{(r)}(x|\lambda)$ is given by

$$H_n^{(r)}(x|\lambda) = \sum_{k=0}^{n} C_k H_k(x). \tag{2.28}$$

By (1.12), we get

$$C_k = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) H_n^{(r)}(x|\lambda)\, dx, \tag{2.29}$$

where

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) H_n^{(r)}(x|\lambda)\, dx \\
&\quad = \frac{(-1)^k n!}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} H_{n-k-l}^{(r)}(\lambda) \int_{-\infty}^{\infty} e^{-x^2} x^l \, dx \\
&\quad = (-1)^k n! \sqrt{\pi} \sum_{0 \le l \le n-k, l:\text{even}} \frac{H_{n-k-l}^{(r)}(\lambda)}{(n-k-l)! 2^l (\frac{l}{2})!} \\
&\quad = (-1)^k n! \sqrt{\pi} \sum_{0 \le l \le n-k, l:\text{even}} \sum_{j=0}^{n-k-l} \frac{\binom{-r}{j} j! S_2(n-k-l, j)}{(n-k-l)! 2^l (1-\lambda)^j (\frac{l}{2})!}.
\end{aligned} \tag{2.30}$$

By (2.29) and (2.30), we get

$$\begin{aligned}
C_k &= n! \sum_{0 \le l \le n-k, l:\text{even}} \frac{H_{n-k-l}^{(r)}(\lambda)}{(n-k-l)! k! 2^{l+k} (\frac{l}{2})!} \\
&= n! \sum_{0 \le l \le n-k, l:\text{even}} \sum_{j=0}^{n-k-l} \frac{\binom{-r}{j} j! S_2(n-k-l, j)}{(n-k-l)! k! 2^{k+l} (1-\lambda)^j (\frac{l}{2})!}.
\end{aligned} \tag{2.31}$$

Therefore, by (2.28) and (2.31), we obtain the following theorem.

Corollary 2.8 *For $n \ge 0$, we have*

$$\begin{aligned}
H_n^{(r)}(x|\lambda) &= n! \sum_{k=0}^{n} \left\{ \sum_{0 \le l \le n-k, l:\text{even}} \frac{H_{n-k-l}^{(r)}(\lambda)}{(n-k-l)! k! 2^{l+k} (\frac{l}{2})!} \right\} H_k(x) \\
&= n! \sum_{k=0}^{n} \left\{ \sum_{0 \le l \le n-k, l:\text{even}} \sum_{j=0}^{n-k-l} \frac{\binom{-r}{j} j! S_2(n-k-l, j)}{(n-k-l)! k! 2^{k+l} (1-\lambda)^j (\frac{l}{2})!} \right\} H_k(x).
\end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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Acknowledgements
The authors would like to express their sincere gratitude to referees for their valuable comments. This paper is supported in part by the Research Grant of Kwangwoon University in 2013.

Received: 3 January 2013 Accepted: 6 March 2013 Published: 26 March 2013

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doi:10.1186/1687-1847-2013-73
Cite this article as: Kim et al.: Some new identities of Bernoulli, Euler and Hermite polynomials arising from umbral calculus. *Advances in Difference Equations* 2013 **2013**:73.