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On hybrid split problem and its nonlinear algorithms

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Abstract

In this paper, we study a hybrid split problem (HSP for short) for equilibrium problems and fixed point problems of nonlinear operators. Some strong and weak convergence theorems are established.

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Keywords: fixed point problem; equilibrium problem; hybrid split problem; iterative algorithm; strong (weak) convergence theorem

1 Introduction

Throughout this paper, we assume that H is a real Hilbert space with zero vector θ, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty subset of H and $T : C \to H$ be a mapping. Denote by $\mathcal{F}(T)$ the set of fixed points of T. The symbols $\mathbb{N}$ and $\mathbb{R}$ are used to denote the sets of positive integers and real numbers, respectively.

Let H be a Hilbert space and C be a closed convex subset of H. Let $f : C \times C \to \mathbb{R}$ be a bi-function. The classical equilibrium problem (EP for short) is defined as follows.

$$\text{Find } p \in C \text{ such that } f(p,y) \geq 0, \quad \forall y \in C. \tag{EP}$$

The symbol $EP(f)$ is used to denote the set of all solutions of the problem (EP), that is,

$$EP(f) = \{u \in K : f(u,v) \geq 0, \forall v \in K\}.$$

It is known that the problem (EP) contains optimization problems, complementary problems, variational inequalities problems, saddle point problems, fixed point problems, bilevel problems, semi-infinite problems and others as special cases and have many applications in physics and economics problems; for detail, one can refer to [1–3] and references therein.

In last ten years or so, the problem (EP) has been generalized and improved to find a common element of the set of fixed points of a nonlinear operator and the set of solutions of the problem (EP). More precisely, many authors have studied the following problem (FTEP) (see, for instance, [4–9]):

$$\text{Find } p \in C \text{ such that } Tp = p \text{ and } f(p,y) \geq 0, \quad \forall y \in C, \tag{FTEP}$$

where C is a closed convex subset of a Hilbert space H, $f : C \times C \to \mathbb{R}$ is a bi-function and $T : C \to C$ is a nonlinear operator.

In this paper, motivated by the problems (EP) and (FTEP), we study the following mathematical model about a hybrid split problem for equilibrium problems and fixed point problems of nonlinear operators (HSP for short).

Let E_1 and E_2 be two real Banach spaces. Let C be a closed convex subset of E_1 and K be a closed convex subset of E_2. Let $f : C \times C \to \mathbb{R}$ and $g : K \times K \to \mathbb{R}$ be two bi-functions, $A : E_1 \to E_2$ be a bounded linear operator. Let $T : C \to C$ and $S : K \to K$ be two nonlinear operators with $\mathcal{F}(T) \neq \emptyset$ and $\mathcal{F}(S) \neq \emptyset$. The mathematical model about a hybrid split problem for equilibrium problems and fixed point problems of nonlinear operators (HSP for short) is defined as follows:

$$\begin{aligned} &\text{Find } p \in C \text{ such that } Tp = p, f(p,y) \geq 0, \quad \forall y \in C, \quad \text{and} \\ &u := Ap \text{ satisfying } Su = u \in K, g(u,v) \geq 0, \quad \forall v \in K. \end{aligned} \tag{HSP}$$

In fact, (HSP) contains several important problems as special cases. We give some examples to explain about it.

Example A If T is an identity operator on C, then (HSP) will reduce to the following problem (P_1):

(P_1) Find $p \in C$ such that $f(p,y) \geq 0$, $\forall y \in C$, and $u := Ap$ satisfying $Su = u \in K$, $g(u,v) \geq 0$, $\forall v \in K$.

Example B If S is an identity operator on K, then (HSP) will reduce to the following problem (P_2):

(P_2) Find $p \in C$ such that $Tp = p$, $f(p,y) \geq 0$, $\forall y \in C$, and $u := Ap \in K$ satisfying $g(u,v) \geq 0$, $\forall v \in K$.

Example C If T, S are all identity operators, then (HSP) will reduce to the following split equilibrium problem (P_3) which has been considered in [10]:

(P_3) Find $p \in C$ such that $f(p,y) \geq 0$, $\forall y \in C$, and $u := Ap \in K$ satisfying $g(u,v) \geq 0$, $\forall v \in K$.

Example D If S is an identity operator and $f(x,y) \equiv 0$ for all $(x,y) \in C \times C$, then (HSP) will reduce to the following problem (P_4) which has been studied in [11]:

(P_4) Find $p \in C$ such that $Tp = p$ and $u := Ap \in K$ satisfying $g(u,v) \geq 0$, $\forall v \in K$.

In this paper, we introduce some new iterative algorithms for (HSP) and some strong and weak convergence theorems for (HSP) will be established.

2 Preliminaries

In what follows, the symbols $\rightharpoonup$ and $\to$ will symbolize weak convergence and strong convergence as usual, respectively. A Banach space $(X, \|\cdot\|)$ is said to satisfy Opial's condition if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

It is well known that any Hilbert space satisfies Opial's condition. Let K be a nonempty subset of real Hilbert spaces H. Recall that a mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.

Let H_1 and H_2 be two Hilbert spaces. Let $A : H_1 \to H_2$ and $B : H_2 \to H_1$ be two bounded linear operators. B is called the adjoint operator (or adjoint) of A if for all $z \in H_1$, $w \in H_2$, B satisfies $\langle Az, w\rangle = \langle z, Bw\rangle$. It is known that the adjoint operator of a bounded linear operator on a Hilbert space always exists and is bounded linear and unique. Moreover, it is not hard to show that if B is an adjoint operator of A, then $\|A\| = \|B\|$.

Example 2.1 ([10]) Let $H_2 = \mathbb{R}$ with the standard norm $|\cdot|$ and $H_1 = \mathbb{R}^2$ with the norm $\|\alpha\| = (a_1^2 + a_2^2)^{\frac{1}{2}}$ for some $\alpha = (a_1, a_2) \in \mathbb{R}^2$. $\langle x, y\rangle = xy$ denotes the inner product of H_2 for some $x, y \in H_2$ and $\langle \alpha, \beta\rangle = \sum_{i=1}^{2} a_i b_i$ denotes the inner product of H_1 for some $\alpha = (a_1, a_2), \beta = (b_1, b_2) \in H_1$. Let $A\alpha = a_2 - a_1$ for $\alpha = (a_1, a_2) \in H_1$ and $Bx = (-x, x)$ for $x \in H_2$, then B is an adjoint operator of A.

Example 2.2 ([10]) Let $H_1 = \mathbb{R}^2$ with the norm $\|\alpha\| = (a_1^2 + a_2^2)^{\frac{1}{2}}$ for some $\alpha = (a_1, a_2) \in \mathbb{R}^2$ and $H_2 = \mathbb{R}^3$ with the norm $\|\gamma\| = (c_1^2 + c_2^2 + c_3^2)^{\frac{1}{2}}$ for some $\gamma = (c_1, c_2, c_3) \in \mathbb{R}^3$. Let $\langle \alpha, \beta\rangle = \sum_{i=1}^{2} a_i b_i$ and $\langle \gamma, \eta\rangle = \sum_{i=1}^{3} c_i d_i$ denote the inner product of H_1 and H_2, respectively, where $\alpha = (a_1, a_2), \beta = (b_1, b_2) \in H_1$, $\gamma = (c_1, c_2, c_3), \eta = (d_1, d_2, d_3) \in H_2$. Let $A\alpha = (a_2, a_1, a_1 - a_2)$ for $\alpha = (a_1, a_2) \in H_1$ and $B\gamma = (c_2 + c_3, c_1 - c_3)$ for $\gamma = (c_1, c_2, c_3) \in H_2$. Obviously, B is an adjoint operator of A.

Let K be a closed convex subset of a real Hilbert space H. For each point $x \in H$, there exists a unique nearest point in K, denoted by $P_K x$, such that $\|x - P_K x\| \leq \|x - y\|$ $\forall y \in K$. The mapping P_K is called the *metric projection* from H onto K. It is well known that P_K has the following characteristics:

(i) $\langle x - y, P_K x - P_K y\rangle \geq \|P_K x - P_K y\|^2$ for every $x, y \in H$;
(ii) for $x \in H$ and $z \in K$, $z = P_K(x) \Leftrightarrow \langle x - z, z - y\rangle \geq 0$, $\forall y \in K$;
(iii) for $x \in H$ and $y \in K$,

$$\|y - P_K(x)\|^2 + \|x - P_K(x)\|^2 \leq \|x - y\|^2. \tag{2.1}$$

Lemma 2.1 (see [1]) *Let K be a nonempty closed convex subset of H and F be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying the following conditions*:

(A1) $F(x, x) = 0$ *for all* $x \in K$;
(A2) *F is monotone, that is,* $F(x, y) + F(y, x) \leq 0$ *for all* $x, y \in K$;
(A3) *for each* $x, y, z \in K$, $\limsup_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
(A4) *for each* $x \in K$, $y \mapsto F(x, y)$ *is convex and lower semi-continuous.*

Let $r > 0$ *and* $x \in H$. *Then there exists* $z \in K$ *such that* $F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$ *for all* $y \in K$.

Lemma 2.2 (see [12]) *Let K be a nonempty closed convex subset of H and let F be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying* (A1)-(A4). *For* $r > 0$, *define a mapping* $T_r^F : H \to K$ *as follows*:

$$T_r^F(x) = \left\{ z \in K : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in K \right\} \tag{2.2}$$

for all $x \in H$. *Then the following hold*:

(i) T_r^F *is single-valued and* $\mathcal{F}(T_r^F) = EP(F)$ *for* $\forall r > 0$ *and* $EP(F)$ *is closed and convex*;

(ii) T_r^F *is firmly non-expansive, that is, for any* $x, y \in H$,
$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle$.

Lemma 2.3 (see, *e.g.*, [6]) *Let H be a real Hilbert space. Then the following hold*:

(a) $\|x + y\|^2 \leq \|y\|^2 + 2\langle x, x + y\rangle$;

(b) $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$ *for all* $x, y \in H$;

(c) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2$ *for all* $x, y \in H$ *and* $\alpha \in [0,1]$.

Lemma 2.4 *Let* F_r^F *be the same as in Lemma* 2.2. *If* $\mathcal{F}(T_r^F) = EP(F) \neq \emptyset$, *then for any* $x \in H$ *and* $x^* \in \mathcal{F}(T_r^F)$, $\|T_r^F x - x\|^2 \leq \|x - x^*\|^2 - \|T_r^F x - x^*\|^2$.

Proof By (ii) of Lemma 2.2 and (b) of Lemma 2.3,

$$\|T_r^F x - x^*\|^2 \leq \langle T_r^F x - x^*, x - x^*\rangle = \frac{1}{2}\left(\|T_r^F x - x^*\|^2 + \|x - x^*\|^2 - \|T_r^F x - x\|^2\right),$$

which shows that $\|T_r^F x - x\|^2 \leq \|x - x^*\|^2 - \|T_r^F x - x^*\|^2$. □

Lemma 2.5 ([10, 11]) *Let the mapping* T_r^F *be defined as in Lemma* 2.2. *Then, for* $r, s > 0$ *and* $x, y \in H$,

$$\|T_r^F(x) - T_s^F(y)\| \leq \|x - y\| + \frac{|s - r|}{s}\|T_s^F(y) - y\|.$$

In particular, $\|T_r^F(x) - T_r^F(y)\| \leq \|x - y\|$ *for any* $r > 0$ *and* $x, y \in H$, *that is*, T_r^F *is nonexpansive for any* $r > 0$.

Remark 2.1 In fact, Lemma 2.5 is motivated by a proof of [5, Theorem 3.2]. In order to the sake of convenience for proving, we restated the fact and gave its proof in Lemma 2.5 [10, 11].

Lemma 2.6 ([13]) *Let* $\{a_n\}$ *be a nonnegative real sequence satisfying the following condition*:

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n b_n, \quad \forall n \geq n_0,$$

where n_0 *is some nonnegative integer*, $\{\lambda_n\}$ *is a sequence in* $(0,1)$ *and* $\{b_n\}$ *is a sequence in* $\mathbf{R}$ *such that*

(i) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(ii) $\limsup_{n\to\infty} b_n \leq 0$ *or* $\sum_{n=0}^{\infty} \lambda_n b_n$ *is convergent. Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 ([14]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space E and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. *Suppose* $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, *then* $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

3 Weak convergence iterative algorithms for (HSP)

In this section, we will introduce some weak convergence iterative algorithms for the hybrid split problem.

Theorem 3.1 *Let H_1 and H_2 be two real Hilbert spaces. Let $C \subset H_1$ and $K \subset H_2$ be two nonempty closed convex sets. Let $T : C \to C$ and $S : K \to K$ be non-expansive mappings and $f : C \times C \to \mathbb{R}$ and $g : K \times K \to \mathbb{R}$ be bi-functions satisfying the conditions* (A1)-(A4). *Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint B. Let $x_1 \in C$, $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} u_n = T^f_{r_n} x_n, \\ y_n = (1-\alpha)u_n + \alpha T u_n, \\ w_n = T^g_{r_n} A y_n, \\ x_{n+1} = P_C(y_n + \xi B(S w_n - A y_n)), \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\alpha \in (0,1)$, $\xi \in (0, \frac{1}{\|B\|^2})$ and $\{r_n\} \subset (0,+\infty)$ with $\liminf_{n\to+\infty} r_n > 0$, P_C is a projection operator from H_1 into C. Suppose that $\Omega = \{p \in \mathcal{F}(T) \cap EP(f) : Ap \in \mathcal{F}(S) \cap EP(g)\} \neq \emptyset$, then $x_n, u_n \rightharpoonup q \in \Omega$ and $w_n \rightharpoonup Aq \in \mathcal{F}(S) \cap EP(g)$.

Proof Let $p \in \Omega$, the following several inequalities can be proved easily:

$$\|y_n - p\| \le \|u_n - p\| \le \|x_n - p\|, \qquad \|w_n - Ap\| \le \|Ay_n - Ap\|. \tag{3.2}$$

By Lemma 2.4, $\|T^g_{r_n} A y_n - A y_n\|^2 \le \|Ay_n - Ap\|^2 - \|T^g_{r_n} A y_n - Ap\|^2$, hence

$$\begin{aligned} \|Sw_n - Ap\|^2 &= \left\|ST^g_{r_n} A y_n - Ap\right\|^2 \le \left\|T^g_{r_n} A y_n - Ap\right\|^2 \\ &\le \|Ay_n - Ap\|^2 - \left\|T^g_{r_n} A y_n - A y_n\right\|^2. \end{aligned} \tag{3.3}$$

By (b) of Lemma 2.3 and (3.3), for each $n \in \mathbb{N}$, we have

$$\begin{aligned} &2\xi\langle y_n - p, B(ST^g_{r_n} - I)Ay_n\rangle \\ &\quad = 2\xi\langle A(y_n - p) + (ST^g_{r_n} - I)Ay_n - (ST^g_{r_n} - I)Ay_n, (ST^g_{r_n} - I)Ay_n\rangle \\ &\quad = 2\xi\left(\frac{1}{2}\left\|ST^g_{r_n} A y_n - Ap\right\|^2 + \frac{1}{2}\left\|(ST^g_{r_n} - I)Ay_n\right\|^2 \right. \\ &\qquad \left. - \frac{1}{2}\|Ay_n - Ap\|^2 - \left\|(ST^g_{r_n} - I)Ay_n\right\|^2\right) \\ &\quad \le 2\xi\left(-\frac{1}{2}\left\|T^g_{r_n} A y_n - A y_n\right\|^2 + \frac{1}{2}\left\|(ST^g_{r_n} - I)Ay_n\right\|^2 - \left\|(ST^g_{r_n} - I)Ay_n\right\|^2\right) \\ &\quad = -\xi\left\|(ST^g_{r_n} - I)Ay_n\right\|^2 - \xi\left\|T^g_{r_n} A y_n - A y_n\right\|^2. \end{aligned} \tag{3.4}$$

On the other hand, $\|B(ST^g_{r_n} - I)Ay_n\|^2 \le \|B\|^2\|(ST^g_{r_n} - I)Ay_n\|^2$, so from (3.1)-(3.4), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \left\|P_C(y_n + \xi B(ST^g_{r_n} - I)Ay_n) - p\right\|^2 \le \left\|y_n + \xi B(ST^g_{r_n} - I)Ay_n - p\right\|^2 \\ &= \|y_n - p\|^2 + \left\|\xi B(ST^g_{r_n} - I)Ay_n\right\|^2 + 2\xi\langle y_n - p, B(ST^g_{r_n} - I)Ay_n\rangle \end{aligned}$$

$$\begin{aligned}
&\le \|y_n - p\|^2 + \xi^2\|B\|^2\big\|(ST^g_{r_n} - I)Ay_n\big\|^2 - \xi\big\|(ST^g_{r_n} - I)Ay_n\big\|^2 \\
&\quad - \xi\big\|(T^g_{r_n} - I)Ay_n\big\|^2 \\
&= \|y_n - p\|^2 - \xi(1 - \xi\|B\|^2)\big\|(ST^f_{r_n} - I)Ay_n\big\|^2 - \xi\big\|(T^g_{r_n} - I)Ay_n\big\|^2 \\
&\le \|x_n - p\|^2 - \xi(1 - \xi\|B\|^2)\big\|(ST^f_{r_n} - I)Ay_n\big\|^2 - \xi\big\|(T^g_{r_n} - I)Ay_n\big\|^2. \qquad (3.5)
\end{aligned}$$

Since $\xi \in (0, \frac{1}{\|B\|^2})$, $\xi(1 - \xi\|B\|^2) > 0$, by (3.2) and (3.5), we have

$$\|x_{n+1} - p\| \le \|y_n - p\| \le \|u_n - p\| \le \|x_n - p\| \qquad (3.6)$$

and

$$\xi(1 - \xi\|B\|^2)\big\|(ST^g_{r_n} - I)Ay_n\big\|^2 + \xi\big\|(T^g_{r_n} - I)Ay_n\big\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \qquad (3.7)$$

The inequality (3.6) implies $\lim_{n\to\infty}\|x_n - p\|$ exists. Further, from (3.6) and (3.7), we get

$$\begin{aligned}
&\lim_{n\to\infty}\|x_n - p\| = \lim_{n\to\infty}\|y_n - p\| = \lim_{n\to\infty}\|u_n - p\|, \\
&\lim_{n\to\infty}\big\|(ST^g_{r_n} - I)Ay_n\big\| = \lim_{n\to\infty}\big\|(T^g_{r_n} - I)Ay_n\big\| = \lim_{n\to\infty}\|w_n - Ay_n\| = 0.
\end{aligned} \qquad (3.8)$$

The inequality (3.8) also implies that

$$\lim_{n\to\infty}\|Sw_n - w_n\| = 0. \qquad (3.9)$$

Using Lemma 2.4 and (3.8), we have

$$\begin{aligned}
\|u_n - x_n\|^2 &= \big\|T^f_{r_n}x_n - x_n\big\|^2 \le \|x_n - p\|^2 - \big\|T^f_{r_n}x_n - p\big\|^2 \\
&= \|x_n - p\|^2 - \|u_n - p\|^2 \to 0. \qquad (3.10)
\end{aligned}$$

Notice that

$$\begin{aligned}
\|y_n - p\|^2 &= (1-\alpha)\|u_n - p\|^2 + \alpha\|Tu_n - p\|^2 - \alpha(1-\alpha)\|Tu_n - u_n\|^2 \\
&\le \|u_n - p\|^2 - \alpha(1-\alpha)\|Tu_n - u_n\|^2,
\end{aligned}$$

hence,

$$\lim_{n\to\infty}\|Tu_n - u_n\| = 0. \qquad (3.11)$$

From (3.10) and (3.11), we also have

$$\begin{aligned}
\|y_n - x_n\| &\le \|y_n - u_n\| + \|u_n - x_n\| \\
&= \alpha\|Tu_n - u_n\| + \|u_n - x_n\| \to 0 \quad \text{as } n \to \infty. \qquad (3.12)
\end{aligned}$$

The existence of $\lim_{n\to\infty}\|x_n - p\|$ implies that $\{x_n\}$ is bounded, hence $\{x_n\}$ has a weak convergence subsequence $\{x_{n_j}\}$. Assume that $x_{n_j} \rightharpoonup q$ for some $q \in C$, then $y_{n_j} \rightharpoonup q$, $Ay_{n_j} \rightharpoonup Aq \in K$ and $w_{n_j} = T^g_{r_{n_j}}Ay_{n_j} \rightharpoonup Aq$ by (3.12) and (3.8).

We say $q \in \Omega$, in other words, $q \in \mathcal{F}(T) \cap EP(f)$ and $Aq \in \mathcal{F}(S) \cap EP(g)$. By (3.10), we also obtain $u_{n_j} \rightharpoonup q$. If $Tq \neq q$, then, by Opial's condition and (3.11), we get

$$\begin{aligned}\liminf_{j\to\infty} \|u_{n_j} - q\| &< \liminf_{j\to\infty} \|u_{n_j} - Tq\| \\ &\le \liminf_{j\to\infty} \|u_{n_j} - Tu_{n_j} + Tu_{n_j} - Tq\| \\ &\le \liminf_{j\to\infty} \|u_{n_j} - q\|,\end{aligned}$$

which is a contradiction. Hence $Tq = q$ or $q \in \mathcal{F}(T)$. On the other hand, from Lemma 2.2, we know $EP(f) = \mathcal{F}(T_r^f)$ for any $r > 0$. Hence, if $T_r^f q \neq q$ for $r > 0$, then by Opial's condition and (3.10) and Lemma 2.5, we have

$$\begin{aligned}\liminf_{j\to\infty} \|x_{n_j} - q\| &< \liminf_{j\to\infty} \|x_{n_j} - T_r^f q\| \\ &= \liminf_{j\to\infty} \|x_{n_j} - T_{r_{n_j}}^f x_{n_j} + T_{r_{n_j}}^f x_{n_j} - T_r^f q\| \\ &\le \liminf_{j\to\infty} \{ \|x_{n_j} - T_{r_{n_j}}^f x_{n_j}\| + \|T_r^f q - T_{r_{n_j}}^f x_{n_j}\| \} \\ &\le \liminf_{j\to\infty} \left\{ \|x_{n_j} - T_{r_{n_j}}^f x_{n_j}\| + \|x_{n_j} - q\| + \frac{|r - r_{n_j}|}{r_{n_j}} \|T_{r_{n_j}}^f x_{n_j} - x_{n_j}\| \right\} \\ &= \liminf_{j\to\infty} \|x_{n_j} - q\|,\end{aligned}$$

which is also a contradiction. So, for each $r > 0$, $T_r^f q = q$, namely $q \in EP(f)$. Thus, we have proved $q \in \mathcal{F}(T) \cap EP(f)$. Similarly, we can also prove $Aq \in \mathcal{F}(S) \cap EP(g)$. Hence, $q \in \Omega$.

Finally, we prove $\{x_n\}$ converges weakly to $q \in \Omega$. Otherwise, if there exists another subsequence of $\{x_n\}$, which is denoted by $\{x_{n_l}\}$, such that $x_{n_l} \rightharpoonup \bar{x} \in \Omega$ with $\bar{x} \neq q$, then by Opial's condition,

$$\liminf_{l\to\infty} \|x_{n_l} - \bar{x}\| < \liminf_{l\to\infty} \|x_{n_l} - q\| = \liminf_{j\to\infty} \|x_{n_j} - q\| < \liminf_{l\to\infty} \|x_{n_l} - \bar{x}\|.$$

This is a contradiction. Hence $\{x_n\}$ converges weakly to an element $q \in \Omega$. Together with $\|u_n - x_n\| \to 0$ (see (3.10)), we also get $u_n \rightharpoonup q$.

Finally, we prove $\{w_n = T_{r_n}^g Ay_n\}$ converges weakly to $Aq \in \mathcal{F}(S) \cap EP(g)$. From (3.12), we have $y_n \rightharpoonup q$, so $Ay_n \rightharpoonup Aq$. Thus, from (3.8) we have $w_n = T_{r_n}^g Ay_n \rightharpoonup Aq \in \mathcal{F}(S) \cap EP(g)$. The proof is completed. □

If $T = I$ or $S = I$, where I denotes an identity operator, then the following corollaries follow from Theorem 3.1.

Corollary 3.1 *Let H_1 and H_2 be two real Hilbert spaces. Let $C \subset H_1$ and $K \subset H_2$ be two nonempty closed convex sets. Let $S : K \to K$ be a non-expansive mapping and $f : C \times C \to \mathbb{R}$ and $g : K \times K \to \mathbb{R}$ be bi-functions satisfying the conditions* (A1)-(A4). *Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint B. Let $x_1 \in C$, $\{x_n\}$ and $\{u_n\}$ be sequences generated* *by*

$$\begin{cases} u_n = T_{r_n}^f x_n, \\ w_n = T_{r_n}^g A u_n, \\ x_{n+1} = P_C(u_n + \xi B(Sw_n - Au_n)), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\xi \in (0, \frac{1}{\|B\|^2})$ *and* $\{r_n\} \subset (0, +\infty)$ *with* $\liminf_{n\to+\infty} r_n > 0$, P_C *is a projection operator from* H_1 *into* C. *Suppose that* $\Omega = \{p \in EP(f) : Ap \in \mathcal{F}(S) \cap EP(g)\} \neq \emptyset$, *then* $x_n, u_n \rightharpoonup q \in \Omega$ *and* $w_n \rightharpoonup Aq \in \mathcal{F}(S) \cap EP(g)$.

Corollary 3.2 *Let* H_1 *and* H_2 *be two real Hilbert spaces. Let* $C \subset H_1$ *and* $K \subset H_2$ *be two nonempty closed convex sets. Let* $T : C \to C$ *be a non-expansive mapping and* $f : C \times C \to \mathbb{R}$ *and* $g : K \times K \to \mathbb{R}$ *be bi-functions satisfying the conditions* (A1)-(A4). *Let* $A : H_1 \to H_2$ *be a bounded linear operator with its adjoint* B. *Let* $x_1 \in C$, $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by*

$$\begin{cases} u_n = T_{r_n}^f x_n, \\ y_n = (1-\alpha)u_n + \alpha T u_n, \\ w_n = T_{r_n}^g A y_n, \\ x_{n+1} = P_C(y_n + \xi B(w_n - Ay_n)), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\alpha \in (0,1)$, $\xi \in (0, \frac{1}{\|B\|^2})$ *and* $\{r_n\} \subset (0, +\infty)$ *with* $\liminf_{n\to+\infty} r_n > 0$, P_C *is a projection operator from* H_1 *into* C. *Suppose that* $\Omega = \{p \in \mathcal{F}(T) \cap EP(f) : Ap \in EP(g)\} \neq \emptyset$, *then* $x_n, u_n \rightharpoonup q \in \Omega$ *and* $w_n \rightharpoonup Aq \in EP(g)$.

Corollary 3.3 *Let* $C \subset H_1$ *and* $K \subset H_2$ *be two nonempty closed convex sets. Let* $f : C \times C \to \mathbb{R}$ *and* $g : K \times K \to \mathbb{R}$ *be bi-functions satisfying the conditions* (A1)-(A4). *Let* $A : H_1 \to H_2$ *be a bounded linear operator with its adjoint* B. *Let* $x_1 \in C$, $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by*

$$\begin{cases} u_n = T_{r_n}^f x_n, \\ w_n = T_{r_n}^g A u_n, \\ x_{n+1} = P_C(u_n + \xi B(w_n - Au_n)), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\xi \in (0, \frac{1}{\|B\|^2})$ *and* $\{r_n\} \subset (0, +\infty)$ *with* $\liminf_{n\to+\infty} r_n > 0$, P_C *is a projection operator from* H_1 *into* C. *Suppose that* $\Omega = \{p \in EP(f) : Ap \in EP(g)\} \neq \emptyset$, *then* $x_n, u_n \rightharpoonup q \in \Omega$ *and* $w_n \rightharpoonup Aq \in EP(g)$.

4 Strong convergence iterative algorithms for (HSP)

In this section, we introduce two strong convergence algorithms for (HSP); see Theorem 4.1 and Theorem 4.2.

Theorem 4.1 *Let* H_1 *and* H_2 *be two real Hilbert spaces. Let* $C \subset H_1$ *and* $K \subset H_2$ *be two nonempty closed convex sets. Let* $T : C \to C$ *and* $S : K \to K$ *be non-expansive mappings* *and* $f : C \times C \to \mathbb{R}$ *and* $g : K \times K \to \mathbb{R}$ *be bi-functions satisfying the conditions* (A1)-(A4). *Let* $A : H_1 \to H_2$ *be a bounded linear operator with its adjoint* B. *Let* $x_1 \in C_1 := C$, $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by*

$$\begin{cases} u_n = T^f_{r_n} x_n, \\ y_n = (1-\alpha)u_n + \alpha T u_n, \\ w_n = T^g_{r_n} A y_n, \\ z_n = P_C(y_n + \xi B(S w_n - A y_n)), \\ C_{n+1} = \{v \in C_n : \|z_n - v\| \le \|y_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $\alpha \in (0,1)$, $\xi \in (0, \frac{1}{\|B\|^2})$ *and* $\{r_n\} \subset (0,+\infty)$ *with* $\liminf_{n\to+\infty} r_n > 0$, P_C *is a projection operator from* H_1 *into* C. *Suppose that* $\Omega = \{p \in \mathcal{F}(T) \cap EP(f) : Ap \in \mathcal{F}(S) \cap EP(g)\} \neq \emptyset$, *then* $x_n, u_n \to q \in \Omega$ *and* $w_n \to Aq \in \mathcal{F}(S) \cap EP(g)$.

Proof We claim that C_n is a nonempty closed convex set for $n \in \mathbb{N}$. In fact, let $p \in \Omega$, it follows from (3.4) that

$$2\xi\langle y_n - p, B(S w_n - A y_n)\rangle \le -\xi \left\| \left(T^g_{r_n} - I\right) A x_n \right\|^2 - \xi \|S w_n - A y_n\|^2. \tag{4.2}$$

By (3.2), (4.1) and (4.2), we obtain

$$\begin{aligned} \|z_n - p\|^2 &\le \left\| y_n + \xi B(S w_n - A y_n) - p \right\|^2 \\ &= \|y_n - p\|^2 + \left\|\xi B(S w_n - A y_n)\right\|^2 + 2\xi\langle y_n - p, B(S w_n - A y_n)\rangle \\ &\le \|y_n - p\|^2 + \xi^2 \|B\|^2 \|S w_n - A y_n\|^2 - \xi \left\| \left(T^g_{r_n} - I\right) A y_n \right\|^2 - \xi \|S w_n - A y_n\|^2 \\ &= \|y_n - p\|^2 - \xi\left(1 - \xi\|B\|^2\right) \left\| \left(S T^g_{r_n} - I\right) A y_n \right\|^2 - \xi \left\| \left(T^g_{r_n} - I\right) A y_n \right\|^2 \\ &\le \|u_n - p\|^2 - (1-\alpha)\alpha \|u_n - T u_n\|^2 \\ &\quad - \xi\left(1 - \xi\|B\|^2\right) \left\| \left(S T^g_{r_n} - I\right) A y_n \right\|^2 - \xi \left\| \left(T^g_{r_n} - I\right) A y_n \right\|^2 \\ &\le \|x_n - p\|^2 - \xi\left(1 - \xi\|B\|^2\right) \left\| \left(S T^g_{r_n} - I\right) A y_n \right\|^2 \\ &\quad - \xi \left\| \left(T^g_{r_n} - I\right) A y_n \right\|^2 - (1-\alpha)\alpha \|u_n - T u_n\|^2. \end{aligned} \tag{4.3}$$

Notice $\xi \in (0, \frac{1}{\|B\|^2})$, $\xi(1 - \xi\|B\|^2) > 0$. It follows from (4.3) that

$$\|z_n - p\| \le \|y_n - p\| \le \|u_n - p\| \le \|x_n - p\| \quad \text{for all } n \in \mathbb{N},$$

hence $p \in C_n$, which yields that $\Omega \subset C_n$ and $C_n \neq \emptyset$ for $n \in \mathbb{N}$.

It is not hard to verify that C_n is closed for $n \in \mathbb{N}$, so it suffices to verify C_n is convex for $n \in \mathbb{N}$. Indeed, let $w_1, w_2 \in C_{n+1}$ and $\gamma \in [0,1]$, we have

$$\begin{aligned} &\left\| z_n - \left(\gamma w_1 + (1-\gamma) w_2\right) \right\|^2 \\ &\quad = \left\| \gamma(z_n - w_1) + (1-\gamma)(z_n - w_2) \right\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \gamma\|z_n - w_1\|^2 + (1-\gamma)\|z_n - w_2\|^2 - \gamma(1-\gamma)\|w_1 - w_2\|^2 \\
&\le \gamma\|y_n - w_1\|^2 + (1-\gamma)\|y_n - w_2\|^2 - \gamma(1-\gamma)\|w_1 - w_2\|^2 \\
&= \big\|y_n - \big(\gamma w_1 + (1-\gamma)w_2\big)\big\|^2,
\end{aligned}
$$

namely $\|z_n - (\gamma w_1 + (1-\gamma)w_2)\| \le \|y_n - (\gamma w_1 + (1-\gamma)w_2)\|$. Similarly, $\|y_n - (\gamma w_1 + (1-\gamma)w_2)\| \le \|x_n - (\gamma w_1 + (1-\gamma)w_2)\|$, which implies $\gamma w_1 + (1-\gamma)w_2 \in C_{n+1}$ and C_{n+1} is a convex set, $n \in \mathbb{N}$.

Notice that $C_{n+1} \subset C_n$ and $x_{n+1} = P_{C_{n+1}}(x_1) \subset C_n$, then $\|x_{n+1} - x_1\| \le \|x_n - x_1\|$ for $n > 1$. It follows that $\lim_{n\to\infty}\|x_n - x_1\|$ exists. Hence $\{x_n\}$ is bounded, which yields that $\{z_n\}$ and $\{y_n\}$ are bounded. For some $k, n \in \mathbb{N}$ with $k > n > 1$, from $x_k = P_{C_k}(x_1) \subset C_n$ and (2.1), we have

$$
\begin{aligned}
\|x_n - x_k\|^2 + \|x_1 - x_k\|^2 &= \big\|x_n - P_{C_k}(x_1)\big\|^2 + \big\|x_1 - P_{C_k}(x_1)\big\|^2 \\
&\le \|x_n - x_1\|^2.
\end{aligned} \tag{4.4}
$$

By $\lim_{n\to\infty}\|x_n - x_1\|$ exists and (4.4), we have $\lim_{n\to\infty}\|x_n - x_k\| = 0$, so $\{x_n\}$ is a Cauchy sequence.

Let $x_n \to q$, then $q \in \Omega$. Firstly, by $x_{n+1} = P_{C_{n+1}}(x_1) \in C_{n+1} \subset C_n$, from (4.1) we have

$$
\begin{aligned}
&\|z_n - x_n\| \le \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \le 2\|x_{n+1} - x_n\| \to 0, \\
&\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \le 2\|x_{n+1} - x_n\| \to 0.
\end{aligned} \tag{4.5}
$$

Setting $\rho = \xi(1 - \xi\|B\|^2)$, by (4.3) again, we have

$$
\begin{aligned}
&\rho\big\|\big(ST^g_{r_n} - I\big)Ay_n\big\|^2 + \xi\big\|\big(T^g_{r_n} - I\big)Ay_n\big\|^2 + (1-\alpha)\alpha\|u_n - Tu_n\|^2 \\
&\quad \le \|x_n - p\|^2 - \|z_n - p\|^2 \le \|x_n - z_n\|\big\{\|x_n - p\| + \|z_n - p\|\big\} \to 0.
\end{aligned} \tag{4.6}
$$

So,

$$
\begin{aligned}
&\lim_{n\to\infty}\|Tu_n - u_n\| = 0, \qquad \lim_{n\to\infty}\|w_n - Ay_n\| = \lim_{n\to\infty}\big\|\big(T^g_{r_n} - I\big)Ay_n\big\| = 0, \\
&\lim_{n\to\infty}\|Sw_n - Ay_n\| = \lim_{n\to\infty}\big\|\big(ST^g_{r_n} - I\big)Ay_n\big\| = 0, \qquad \lim_{n\to\infty}\|Sw_n - w_n\| = 0.
\end{aligned} \tag{4.7}
$$

Notice that $\lim_{n\to\infty}\|Tu_n - u_n\| = 0$ and $\|y_n - u_n\| = \alpha\|Tu_n - u_n\|$, so

$$
\lim_{n\to\infty}\|y_n - u_n\| = 0. \tag{4.8}
$$

Further, from (4.5) and (4.8),

$$
\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{4.9}
$$

Since $x_n \to q$, we have $u_n \to q$ by (4.9). Thus

$$
\|Tq - q\| \le \|Tq - Tu_n\| + \|Tu_n - u_n\| + \|u_n - q\| \to 0,
$$

namely $Tq = q$ and $q \in \mathcal{F}(T)$. On the other hand, for $r > 0$, by Lemma 2.5, we have

$$\begin{aligned}\|T_r^f q - q\| &\leq \|T_r^f q - T_{r_n}^f x_n + T_{r_n}^f x_n - x_n + x_n - q\| \\ &\leq \|x_n - q\| + \frac{|r_n - r|}{r_n}\|T_{r_n}^f x_n - x_n\| + \|T_{r_n}^f x_n - x_n\| + \|x_n - q\| \to 0,\end{aligned}$$

which yields $q \in \mathcal{F}(T_r^f) = EP(f)$. We have verified $q \in \mathcal{F}(T) \cap EP(f)$.

Next, we prove $Aq \in \mathcal{F}(S) \cap EP(g)$. Since $x_n \to q$ and $x_n - y_n \to 0$ by (4.8) and (4.9) and $w_n - Ay_n \to 0$ by (4.7), we have $y_n \to q$ and $Ay_n \to Aq$ and $w_n \to Aq$. So,

$$\|SAq - Aq\| \leq \|SAq - Sw_n\| + \|Sw_n - w_n\| + \|w_n - Aq\| \to 0,$$

namely $SAq = Aq$ and $Aq \in \mathcal{F}(S)$. On the other hand, for $r > 0$, by Lemma 2.5 again, we have

$$\begin{aligned}\|T_r^g Aq - Aq\| &\leq \|T_r^g Aq - T_{r_n}^g Ay_n + T_{r_n}^g Ay_n - Ay_n + Ay_n - Aq\| \\ &\leq \|Ay_n - Aq\| + \frac{|r_n - r|}{r_n}\|T_{r_n}^g Ay_n - Ay_n\| \\ &\quad + \|T_{r_n}^g Ay_n - Ay_n\| + \|Ay_n - Aq\| \to 0,\end{aligned}$$

which implies that $Aq \in \mathcal{F}(T_r^g) = EP(g)$. We have verified $Aq \in \mathcal{F}(S) \cap EP(g)$.

So, we have obtained $q \in \Omega$ and $x_n, u_n \to q$ and $w_n \to Aq$, the proof is completed. □

If $T = I$ or $S = I$, where I denotes an identity operator, then the following corollaries follow from Theorem 4.1.

Corollary 4.1 *Let H_1 and H_2 be two real Hilbert spaces. Let $C \subset H_1$ and $K \subset H_2$ be two nonempty closed convex sets. Let $f : C \times C \to \mathbb{R}$ and $g : K \times K \to \mathbb{R}$ be bi-functions satisfying the conditions* (A1)-(A4) *and $S : K \to K$ be a non-expansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint B. Let $x_1 \in C_1 := C$, $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} u_n = T_{r_n}^f x_n, \\ w_n = T_{r_n}^g A u_n, \\ z_n = P_C(u_n + \xi B(Sw_n - Au_n)), \\ C_{n+1} = \{v \in C_n : \|z_n - v\| \leq \|u_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad n \in \mathbb{N}, \end{cases}$$

where $\xi \in (0, \frac{1}{\|B\|^2})$ and $\{r_n\} \subset (0, +\infty)$ with $\liminf_{n \to +\infty} r_n > 0$, P_C is a projection operator from H_1 into C. Suppose that $\Omega = \{p \in EP(f) : Ap \in \mathcal{F}(S) \cap EP(g)\} \neq \emptyset$, then $x_n, u_n \to q \in \Omega$ and $w_n \to Aq \in \mathcal{F}(S) \cap EP(g)$.

Corollary 4.2 *Let H_1 and H_2 be two real Hilbert spaces. Let $C \subset H_1$ and $K \subset H_2$ be two nonempty closed convex sets. Let $T : C \to C$ be a non-expansive mapping and $f : C \times C \to \mathbb{R}$ and $g : K \times K \to \mathbb{R}$ be bi-functions satisfying the conditions* (A1)-(A4). *Let $A : H_1 \to H_2$*

be a bounded linear operator with its adjoint B. Let $x_1 \in C_1 := C$, $\{x_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} u_n = T^f_{r_n} x_n, \\ y_n = (1-\alpha)u_n + \alpha T u_n, \\ w_n = T^g_{r_n} A y_n, \\ z_n = P_C(y_n + \xi B(w_n - A y_n)), \\ C_{n+1} = \{v \in C_n : \|z_n - v\| \le \|y_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad n \in \mathbb{N}, \end{cases}$$

where, $\alpha \in (0,1)$, $\xi \in (0, \frac{1}{\|B\|^2})$ and $\{r_n\} \subset (0,+\infty)$ with $\liminf_{n\to+\infty} r_n > 0$, P_C is a projection operator from H_1 into C. Suppose that $\Omega = \{p \in \mathcal{F}(T) \cap EP(f) : Ap \in EP(g)\} \neq \emptyset$, then $x_n, u_n \to q \in \Omega$ and $w_n \to Aq \in EP(g)$.

Corollary 4.3 *Let H_1 and H_2 be two real Hilbert spaces. Let $C \subset H_1$ and $K \subset H_2$ be two nonempty closed convex sets. Let $f : C \times C \to \mathbb{R}$ and $g : K \times K \to \mathbb{R}$ be bi-functions satisfying the conditions* (A1)-(A4). *Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint B. Let $x_1 \in C_1 := C$, $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} u_n = T^f_{r_n} x_n, \qquad w_n = T^g_{r_n} A u_n, \\ z_n = P_C(y_n + \xi B(w_n - A u_n)), \\ C_{n+1} = \{v \in C_n : \|z_n - v\| \le \|u_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad n \in \mathbb{N}, \end{cases}$$

where $\xi \in (0, \frac{1}{\|B\|^2})$ and $\{r_n\} \subset (0,+\infty)$ with $\liminf_{n\to+\infty} r_n > 0$, P_C is a projection operator from H_1 into C. Suppose that $\Omega = \{p \in EP(f) : Ap \in EP(g)\} \neq \emptyset$, then $x_n, u_n \to q \in \Omega$ and $w_n \to Aq \in EP(g)$.

It is well known that the viscosity iterative method is always applied to study the iterative solution for the fixed point problem of nonlinear operators, for example, [5, 6, 8, 15, 16]. Similarly, the viscosity iterative method can also be used to study the hybrid split problem (HSP). So, at the end of this paper, we introduce a viscosity iterative algorithm which can converge strongly to a solution of (HSP).

Theorem 4.2 *Let H_1 and H_2 be two real Hilbert spaces. Let $C \subset H_1$ and $K \subset H_2$ be two nonempty closed convex sets. Let $h : C \to C$ be a α-contraction mapping, $T : C \to C$ and $S : K \to K$ be non-expansive mappings and $f : C \times C \to \mathbb{R}$ and $g : K \times K \to \mathbb{R}$ be bi-functions satisfying the conditions* (A1)-(A4). *Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint B. Let $x_1 \in C$, $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} u_n = T^f_{r_n} x_n, \\ w_n = T^g_{r_n} A u_n, \\ y_n = P_C(u_n + \xi B(S w_n - A u_n)), \\ z_n = (1-r)x_n + r T y_n, \\ x_{n+1} = \alpha_n h(x_n) + (1-\alpha_n) z_n, \quad n \in \mathbb{N}, \end{cases} \tag{4.10}$$

where $r \in (0,1)$, $\xi \in (0, \frac{1}{\|B\|^2})$ *and* $\{r_n\} \subset (0, +\infty)$, P_C *is a projection operator from* H_1 *into* C, *and the coefficients* $\{\alpha_n\}$ *and* $\{r_n\}$ *satisfy the following conditions*:

(1) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\liminf_{n\to+\infty} r_n > 0$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Suppose that $\Omega = \{p \in \mathcal{F}(T) \cap EP(f) : Ap \in \mathcal{F}(S) \cap EP(g)\} \neq \emptyset$, *then* $x_n, u_n \to q \in \Omega$ *and* $w_n \to Aq \in \mathcal{F}(S) \cap EP(g)$, *where* $q = P_\Omega h(q)$.

Proof Let $p \in \Omega$. The following inequalities are easily verified:

$$\|u_n - p\| \le \|x_n - p\|, \qquad \|w_n - Ap\| \le \|Au_n - Ap\|. \tag{4.11}$$

By Lemma 2.4,

$$\begin{aligned} &\|u_n - p\|^2 \le \|x_n - p\|^2 - \left\| T_{r_n}^g x_n - x_n \right\|^2 = \|x_n - p\|^2 - \|u_n - x_n\|^2; \\ &\left\| Sw_n - Ap \right\|^2 = \left\| ST_{r_n}^g Au_n - Ap \right\|^2 \le \left\| T_{r_n}^g Au_n - Ap \right\|^2 \\ &\qquad \le \|Au_n - Ap\|^2 - \left\| T_{r_n}^g Au_n - Au_n \right\|^2. \end{aligned} \tag{4.12}$$

From (4.10) and (4.12), we have

$$\begin{aligned} &2\xi \langle u_n - p, B(Sw_n - Au_n) \rangle \\ &\quad = 2\xi \langle A(u_n - p) + Sw_n - Au_n - (Sw_n - Au_n), Sw_n - Au_n \rangle \\ &\quad = 2\xi \left(\frac{1}{2}\|Sw_n - Ap\|^2 + \frac{1}{2}\|Sw_n - Au_n\|^2 - \frac{1}{2}\|Au_n - Ap\|^2 - \|Sw_n - Au_n\|^2 \right) \\ &\quad \le 2\xi \left(-\frac{1}{2}\left\| T_{r_n}^g Au_n - Au_n \right\|^2 - \frac{1}{2}\|Sw_n - Au_n\|^2 \right) \\ &\quad = -\xi \|Sw_n - Au_n\|^2 - \xi \left\| T_{r_n}^g Au_n - Au_n \right\|^2 \\ &\quad = -\xi \|Sw_n - Au_n\|^2 - \xi \|w_n - Au_n\|^2 \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \|y_n - p\|^2 &= \left\| P_C(u_n + \xi B(Sw_n - Au_n) - P_C p \right\|^2 \\ &\le \left\| u_n - p + \xi B(Sw_n - Au_n) \right\|^2 \\ &= \|u_n - p\|^2 + \left\| \xi B(Sw_n - Au_n) \right\|^2 + 2\xi \langle u_n - p, B(Sw_n - Au_n) \rangle \\ &\le \|u_n - p\|^2 - \xi \left(1 - \xi \|B\|^2\right) \|Sw_n - Au_n\|^2 - \xi \left\| T_{r_n}^g Au_n - Au_n \right\|^2 \\ &\le \|x_n - p\|^2 - \xi \left(1 - \xi \|B\|^2\right) \|Sw_n - Au_n\|^2 - \xi \left\| T_{r_n}^g Au_n - Au_n \right\|^2 \\ &= \|x_n - p\|^2 - \xi \left(1 - \xi \|B\|^2\right) \|Sw_n - Au_n\|^2 - \xi \|w_n - Au_n\|^2. \end{aligned} \tag{4.14}$$

So, from (4.10)-(4.11) and (4.14), we have

$$\|y_n - p\| \le \|u_n - p\| \le \|x_n - p\|, \qquad \|z_n - p\| \le \|x_n - p\|. \tag{4.15}$$

We say $\{x_n\}$ is bounded. In fact, from (4.10) and (4.15), we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(f(x_n)-p)+(1-\alpha_n)(z_n-p)\| \le (1-\alpha_n)\|z_n-p\|+\alpha_n\|f(x_n)-p\| \\ &\le (1-\alpha_n)\|x_n-p\|+\alpha_n\alpha\|x_n-p\|+\alpha_n\|f(p)-p\| \\ &= (1-\alpha_n(1-\alpha))\|x_n-p\|+\alpha_n(1-\alpha)\frac{\|f(p)-p\|}{1-\alpha},\end{aligned}$$

which implies that

$$\|x_n-p\| \le \max\left\{\|x_1-p\|, \frac{\|f(p)-p\|}{1-\alpha}\right\}, \quad \forall n\in\mathbb{N}, \tag{4.16}$$

so $\{x_n\}$ is bounded. Further, $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ are also bounded by (4.11).

By Lemma 2.5, from (4.10) we have

$$\begin{aligned}\|u_{n+1}-u_n\|^2 &= \|T^f_{r_{n+1}}x_{n+1}-T^f_{r_n}x_n\|^2 \\ &\le \left(\|x_{n+1}-x_n\|+\frac{|r_n-r_{n+1}|}{r_n}\|T^f_{r_n}x_n-x_n\|\right)^2 \\ &\le \|x_{n+1}-x_n\|^2+\frac{|r_n-r_{n+1}|}{r_n}M_1, \\ \|w_{n+1}-w_n\|^2 &= \|T^g_{r_{n+1}}Au_{n+1}-T^g_{r_n}Au_n\|^2 \\ &\le \left(\|Au_{n+1}-Au_n\|+\frac{|r_n-r_{n+1}|}{r_n}\|T^g_{r_n}Au_n-Au_n\|\right)^2 \\ &\le \|Au_{n+1}-Au_n\|^2+\frac{|r_n-r_{n+1}|}{r_n}M_1\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\|y_{n+1}-y_n\|^2 &\le \|u_{n+1}+\xi B(Sw_{n+1}-Au_{n+1})-u_n-\xi B(Sw_n-Au_n)\|^2 \\ &= \|u_{n+1}-u_n+\xi B(Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n))\|^2 \\ &= \|u_{n+1}-u_n\|^2+\|\xi B(Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n))\|^2 \\ &\quad +2\xi\langle u_{n+1}-u_n, B(Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n))\rangle \\ &\le \|u_{n+1}-u_n\|^2+\xi^2\|B\|^2\|Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\|^2 \\ &\quad +2\xi\langle A(u_{n+1}-u_n), Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\rangle \\ &= \|u_{n+1}-u_n\|^2+\xi^2\|B\|^2\|Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\|^2 \\ &\quad +2\xi\langle A(u_{n+1}-u_n)+Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n), Sw_{n+1} \\ &\quad -Au_{n+1}-(Sw_n-Au_n)\rangle \\ &\quad -2\xi\langle Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n), Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\rangle \\ &= \|u_{n+1}-u_n\|^2+\xi^2\|B\|^2\|Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\|^2 \\ &\quad +2\xi\langle Sw_{n+1}-Sw_n, Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\rangle \\ &\quad -2\xi\|Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\|^2\end{aligned}$$

$$
\begin{aligned}
&= \|u_{n+1}-u_n\|^2 + \xi^2\|B\|^2\big\|Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\big\|^2 \\
&\quad + 2\xi\frac{1}{2}\Big\{\|Sw_{n+1}-Sw_n\|^2 + \big\|Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\big\|^2 \\
&\quad - \|Au_{n+1}-Au_n\|^2\Big\} \\
&\quad - 2\xi\big\|Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\big\|^2 \\
&= \|u_{n+1}-u_n\|^2 + \xi^2\|B\|^2\big\|Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\big\|^2 \\
&\quad + \xi\big\{\|Sw_{n+1}-Sw_n\|^2 - \|Au_{n+1}-Au_n\|^2\big\} \\
&\quad - \xi\big\|Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\big\|^2 \\
&\le \|u_{n+1}-u_n\|^2 - \xi\big(1-\xi\|B\|^2\big)\big\|Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\big\|^2 \\
&\quad + \xi\big\{\|w_{n+1}-w_n\|^2 - \|Au_{n+1}-Au_n\|^2\big\} \\
&\le \|u_{n+1}-u_n\|^2 - \xi\big(1-\xi\|B\|^2\big)\big\|Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\big\|^2 \\
&\quad + \xi\left\{\|Au_{n+1}-Au_n\|^2 + \frac{|r_n-r_{n+1}|}{r_n}M_1 - \|Au_{n+1}-Au_n\|^2\right\} \\
&= \|u_{n+1}-u_n\|^2 - \xi\big(1-\xi\|B\|^2\big)\big\|Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\big\|^2 \\
&\quad + \xi\frac{|r_n-r_{n+1}|}{r_n}M_1 \\
&\le \|x_{n+1}-x_n\|^2 - \xi\big(1-\xi\|B\|^2\big)\big\|Sw_{n+1}-Au_{n+1}-(Sw_n-Au_n)\big\|^2 \\
&\quad + \frac{|r_n-r_{n+1}|}{r_n}(\xi M_1 + M_1),
\end{aligned}
\tag{4.18}
$$

where M_1 is a constant satisfying

$$
\begin{aligned}
&\sup_{n\in\mathbb{N}}\Big\{2\|x_{n+1}-x_n\|\big\|T^f_{r_n}x_n-x_n\big\| + \frac{|r_n-r_{n+1}|}{r_n}\big\|T^f_{r_n}x_n-x_n\big\|^2, \\
&\quad 2\|Au_{n+1}-Au_n\|\big\|T^g_{r_n}Au_n-Au_n\big\| + \frac{|r_n-r_{n+1}|}{r_n}\big\|T^g_{r_n}Au_n-Au_n\big\|^2\Big\} \le M_1.
\end{aligned}
$$

Proving $\|x_{n+1}-x_n\| \to 0$ as $n\to\infty$. Setting $\beta_n = 1-(1-\alpha_n)(1-r)$ and $v_n = \frac{x_{n+1}-x_n+\beta_n x_n}{\beta_n}$, namely $v_n = \frac{\alpha_n f(x_n)+(1-\alpha_n)rTy_n}{\beta_n}$. Let M_2 be a constant satisfying $\sup_{n\in\mathbb{N}}\{\|\frac{f(x_{n+1})}{\beta_{n+1}}\|, \|\frac{f(x_n)}{\beta_n}\|, \|Ty_n\|\} \le M_2$ for all $n\in\mathbb{N}$. Then

$$
\begin{aligned}
\|v_{n+1}-v_n\| &= \left\|\frac{\alpha_{n+1}f(x_{n+1})+(1-\alpha_{n+1})rTy_{n+1}}{\beta_{n+1}} - \frac{\alpha_n f(x_n)+(1-\alpha_n)rTy_n}{\beta_n}\right\| \\
&\le \alpha_{n+1}\left\|\frac{f(x_{n+1})}{\beta_{n+1}}\right\| + \alpha_n\left\|\frac{f(x_n)}{\beta_n}\right\| + r\left\|\frac{(1-\alpha_{n+1})Ty_{n+1}}{\beta_{n+1}} - \frac{(1-\alpha_n)Ty_n}{\beta_n}\right\| \\
&\le (\alpha_{n+1}+\alpha_n)M_2 + r\left\|\frac{(1-\alpha_{n+1})(Ty_{n+1}-Ty_n)}{\beta_{n+1}}\right. \\
&\quad \left. + \frac{(1-\alpha_{n+1})Ty_n}{\beta_{n+1}} - \frac{(1-\alpha_n)Ty_n}{\beta_n}\right\| \\
&\le (\alpha_{n+1}+\alpha_n)M_2 + r\frac{(1-\alpha_{n+1})\|y_{n+1}-y_n\|}{\beta_{n+1}} + \left|\frac{(1-\alpha_{n+1})}{\beta_{n+1}} - \frac{(1-\alpha_n)}{\beta_n}\right|M_2
\end{aligned}
$$

$$\begin{aligned}
&= (\alpha_{n+1} + \alpha_n)M_2 + r\frac{(1-\alpha_{n+1})\|y_{n+1} - y_n\|}{\beta_{n+1}} \\
&\quad + \left|\frac{(1-r)(\alpha_n - \alpha_{n+1}) + \beta_{n+1}\alpha_n - \beta_n\alpha_{n+1}}{\beta_n\beta_{n+1}}\right| M_2 \\
&\le (\alpha_{n+1} + \alpha_n)M_2 + r\frac{(1-\alpha_{n+1})\|y_{n+1} - y_n\|}{\beta_{n+1}} + 2\frac{\alpha_n + \alpha_{n+1}}{\beta_n\beta_{n+1}}M_2 \\
&:= \rho_n + r\frac{(1-\alpha_{n+1})\|y_{n+1} - y_n\|}{\beta_{n+1}}.
\end{aligned} \tag{4.19}$$

From (4.18) and (4.19), we have

$$\begin{aligned}
\|v_{n+1} - v_n\|^2 &\le \left(\rho_n + r\frac{(1-\alpha_{n+1})\|y_{n+1} - y_n\|}{\beta_{n+1}}\right)^2 \\
&= \rho_n^2 + 2\rho_n r\frac{(1-\alpha_{n+1})\|y_{n+1} - y_n\|}{\beta_{n+1}} + r^2\frac{(1-\alpha_{n+1})^2\|y_{n+1} - y_n\|^2}{\beta_{n+1}^2}, \\
&\le \rho_n^2 + 2\rho_n r\frac{(1-\alpha_{n+1})\|y_{n+1} - y_n\|}{\beta_{n+1}} + r^2\frac{(1-\alpha_{n+1})^2}{\beta_{n+1}^2}\|x_{n+1} - x_n\|^2 \\
&\quad + r^2\frac{(1-\alpha_{n+1})^2}{\beta_{n+1}^2}\frac{|r_n - r_{n+1}|}{r_n}(1+\xi)M_1.
\end{aligned} \tag{4.20}$$

By the conditions (1) and (2) and (4.20), we obtain

$$\limsup_{n\to\infty}\left\{\|v_{n+1} - v_n\|^2 - \|x_{n+1} - x_n\|^2\right\} \le 0. \tag{4.21}$$

Notice $\|v_{n+1} - v_n\|^2 - \|x_{n+1} - x_n\|^2 = (\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|)(\|v_{n+1} - v_n\| + \|x_{n+1} - x_n\|)$, hence from (4.21) we have

$$\limsup_{n\to\infty}\left\{\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|\right\} \le 0. \tag{4.22}$$

By Lemma 2.7 and (4.22), we have $\lim_{n\to\infty}\|v_n - x_n\| = 0$, which implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0 \tag{4.23}$$

by the definition of v_n. Since $\|x_{n+1} - z_n\| \to 0$, together with (4.23), we have

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{4.24}$$

Using (4.10), (4.12) and (4.15),

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\|\alpha_n\big(f(x_n) - p\big) + (1-\alpha_n)(z_n - p)\right\|^2 \\
&\le (1-\alpha_n)\|z_n - p\|^2 + \alpha_n\left\|f(x_n) - p\right\|^2 \\
&\le (1-r)\|x_n - p\|^2 + r\|u_n - p\|^2 + \alpha_n\left\|f(x_n) - p\right\|^2 \\
&\le \|x_n - p\|^2 - r\|u_n - x_n\|^2 + \alpha_n\left\|f(x_n) - p\right\|^2,
\end{aligned} \tag{4.25}$$

which yields

$$\begin{aligned} r\|u_n - x_n\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n \|f(x_n) - p\|^2 \\ &= (\|x_n - p\| + \|x_{n+1} - p\|)(\|x_n - p\| - \|x_{n+1} - p\|) + \alpha_n \|f(x_n) - p\|^2 \\ &\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - p\|^2. \end{aligned} \tag{4.26}$$

From (4.26) we have

$$\lim_{n\to\infty} \|T_{r_n}^f x_n - x_n\| = \lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{4.27}$$

Again, applying (4.25), (4.15) and (4.14), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le (1-\alpha_n)\|z_n - p\|^2 + \alpha_n \|f(x_n) - p\|^2 \\ &\le (1-r)\|x_n - p\|^2 + r\|y_n - p\|^2 + \alpha_n \|f(x_n) - p\|^2 \\ &\le \|x_n - p\|^2 - r\xi(1 - \xi\|B\|^2)\|Sw_n - Au_n\|^2 \\ &\quad - r\xi\|w_n - Au_n\|^2 + \alpha_n \|f(x_n) - p\|^2, \end{aligned} \tag{4.28}$$

which implies that

$$\begin{aligned} &r\xi(1 - \xi\|B\|^2)\|Sw_n - Au_n\|^2 + r\xi\|w_n - Au_n\|^2 \\ &\quad \le \{\|x_n - p\| + \|x_{n+1} - p\|\}\|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - p\|^2. \end{aligned} \tag{4.29}$$

From (4.29) we have

$$\lim_{n\to\infty} \|T_{r_n}^g Au_n - Au_n\| = \lim_{n\to\infty} \|w_n - Au_n\| = 0, \qquad \lim_{n\to\infty} \|Sw_n - Au_n\| = 0 \tag{4.30}$$

and

$$\lim_{n\to\infty} \|Sw_n - w_n\| = 0. \tag{4.31}$$

Notice $y_n = P_C(u_n + \xi B(Sw_n - Au_n))$ and $u_n \in C$ for all $n \in \mathbb{N}$, so

$$\begin{aligned} \|y_n - u_n\| &= \|P_C(u_n + \xi B(Sw_n - Au_n)) - P_C u_n\| \le \|\xi B(Sw_n - Au_n)\| \\ &\le \xi\|B\|\|Sw_n - Au_n\|, \end{aligned}$$

so

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{4.32}$$

Further, from (4.27), (4.32) and (4.24), we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0, \qquad \lim_{n\to\infty} \|y_n - z_n\| = 0 \tag{4.33}$$

and

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0 \quad \text{by (4.10), (4.24) and (4.33).} \tag{4.34}$$

Let $q = P_\Omega f(q)$. Choose a subsequence $\{x_{n_k}\}$ such that

$$\limsup_{n\to\infty} \langle f(q) - q, x_n - q\rangle = \lim_{k\to\infty} \langle f(q) - q, x_{n_k} - q\rangle. \tag{4.35}$$

Since $\{x_n\}$ is bounded, $\{\langle f(q) - q, x_n - q\rangle\}$ is bounded. Hence $\limsup_{n\to\infty} \langle f(q) - q, x_n - q\rangle$ is a constant, namely $\lim_{n\to\infty} \langle f(q) - q, x_{n_k} - q\rangle$ exists, which implies (4.35) is well defined. Because $\{x_n\}$ is bounded, $\{x_{n_k}\}$ has a weak convergence subsequence which is still denoted by $\{x_{n_k}\}$. Suppose $x_{n_k} \rightharpoonup x^*$, we say $x^* \in \Omega$. When $x_{n_k} \rightharpoonup x^*$, from (4.30), (4.32) and (4.33), we have

$$u_{n_k} \rightharpoonup x^*, \qquad y_{n_k} \rightharpoonup x^*, \qquad z_{n_k} \rightharpoonup x^*, \qquad Au_{n_k} \rightharpoonup Ax^*, \qquad w_{n_k} \rightharpoonup Ax^*. \tag{4.36}$$

If $Tx^* \neq x^*$, then by (4.34) and (4.36) and Opial's condition, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|y_{n_k} - x^*\| &< \liminf_{k\to\infty} \|y_{n_k} - Tx^*\| \\
&\leq \liminf_{k\to\infty} \{\|y_{n_k} - Ty_{n_k}\| + \|Ty_{n_k} - Tx^*\|\} \\
&\leq \liminf_{k\to\infty} \{\|y_{n_k} - Ty_{n_k}\| + \|y_{n_k} - x^*\|\} = \liminf_{k\to\infty} \|y_{n_k} - x^*\|,
\end{aligned} \tag{4.37}$$

which is a contradiction, so $Tx^* = x^*$ and $x^* \in \mathcal{F}(T)$. Since for each $r > 0$, $EP(f) = \mathcal{F}(T_r^f)$ by Lemma 2.2, we have $x^* \in \mathcal{F}(T_r^f)$. Otherwise, if there exists $r > 0$ such that $T_r^f x^* \neq x^*$, then by (4.27) and Lemma 2.5 and Opial's condition, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - x^*\| &< \liminf_{k\to\infty} \|x_{n_k} - T_r^f x^*\| \\
&\leq \liminf_{k\to\infty} \{\|x_{n_k} - T_{n_k}^f x_{n_k}\| + \|T_{n_k}^f x_{n_k} - T_r^f x^*\|\} \\
&= \liminf_{k\to\infty} \|T_{n_k}^f x_{n_k} - T_r^f x^*\| \\
&\leq \liminf_{k\to\infty} \left\{\|x_{n_k} - x^*\| + \frac{|r_{n_k} - r|}{r_{n_k}} \|T_{n_k}^f x_{n_k} - x_{n_k}\|\right\} \\
&= \liminf_{k\to\infty} \|x_{n_k} - x^*\|,
\end{aligned} \tag{4.38}$$

which is also a contradiction, so $T_r^f x^* = x^*$ and $x^* \in \mathcal{F}(T_r^f) = EP(f)$. Up to now, we have proved $x^* \in \mathcal{F}(T) \cap EP(f)$. Similarly, we can also prove $Ax^* \in \mathcal{F}(S) \cap EP(g)$. Hence $x^* \in \Omega$, because of this, we can also obtain

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(q) - q, x_n - q\rangle &= \lim_{k\to\infty} \langle f(q) - q, x_{n_k} - q\rangle \\
&= \langle f(q) - q, x^* - q\rangle \leq 0, \quad \text{where } q = P_C f(q).
\end{aligned} \tag{4.39}$$

Finally, we prove the conclusion of this theorem is right. For $q = P_\Omega f(q)$, from (4.10) we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \big\|\alpha_n\big(h(x_n) - q\big) + (1 - \alpha_n)(z_n - q)\big\|^2 \\
&\le (1 - \alpha_n)^2\|z_n - q\|^2 + 2\alpha_n\big\langle h(x_n) - q, x_{n+1} - q\big\rangle \\
&\le (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\big\langle h(x_n) - h(q) + h(q) - q, x_{n+1} - q\big\rangle \\
&\le (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\alpha\|x_n - q\|\|x_{n+1} - q\| + 2\alpha_n\big\langle h(q) - q, x_{n+1} - q\big\rangle \\
&\le (1 - \alpha_n)^2\|x_n - q\|^2 + \alpha_n\alpha\|x_n - q\|^2 + \alpha_n\alpha\|x_{n+1} - q\|^2 \\
&\quad + 2\alpha_n\big\langle h(q) - q, x_{n+1} - q\big\rangle \\
&= (1 - 2\alpha_n)\|x_n - q\|^2 + \alpha_n^2\|x_n - q\|^2 + \alpha_n\alpha\|x_n - q\|^2 + \alpha_n\alpha\|x_{n+1} - q\|^2 \\
&\quad + 2\alpha_n\big\langle h(q) - q, x_{n+1} - q\big\rangle. \qquad (4.40)
\end{aligned}$$

From (4.40) we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \left(1 - \alpha_n\frac{2 - 2\alpha}{1 - \alpha_n\alpha}\right)\|x_n - q\|^2 + \frac{\alpha_n^2}{1 - \alpha_n\alpha}\|x_n - q\|^2 \\
&\quad + 2\frac{\alpha_n}{1 - \alpha_n\alpha}\big\langle h(q) - q, x_{n+1} - q\big\rangle, \qquad (4.41)
\end{aligned}$$

by (4.41) and Lemma 2.6, we have $x_n \to q \in \Omega$. Again, from (4.27) and (4.30), we have $u_n \to q \in \Omega$ and $w_n \to Aq \in F(S) \cap EP(f)$, respectively. The proof is completed. □

Remark

(1) In this paper, the iterative coefficient α or r can be replaced with the sequence $\{\zeta_n\}$ if $\{\zeta_n\}$ satisfies $\{\zeta_n\} \subset [\varrho, \vartheta]$, where $\varrho, \vartheta \in (0, 1)$;
(2) Obviously, if $H_1 = H_2$ in this paper, these weak and strong convergence theorems are also true;
(3) In this paper, if T is a nonexpansive mapping from H_1 into H_1 and $f(x, y)$ is a bi-function from $H_1 \times H_1$ into $\mathbb{R}$ with the conditions (A1)-(A4), S is a nonexpansive mapping from H_2 into H_2 and $g(u, v)$ is a bi-function from $H_2 \times H_2$ into $\mathbb{R}$ with the conditions (A1)-(A4), then we may obtain a series of similar algorithms.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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