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Weak and strong convergence theorems for a finite family of non-Lipschitzian nonself mappings in Banach spaces

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Abstract

In this paper, several weak and strong convergence theorems are established for a new modified iteration with errors for a finite family of nonself mappings which are asymptotically nonexpansive in the intermediate sense in Banach spaces. Mann-type, Ishikawa-type and Noor-type iterations are covered by this new iteration scheme. Our convergence theorems improve, unify and generalize many important results in the current literature.

MSC: 47H10; 47H09; 46B20

Keywords: asymptotically nonexpansive mapping in the intermediate sense; Kadec-Klee property; Opial's condition; uniformly convex Banach spaces; common fixed points

1 Introduction

Fixed-point iteration processes for nonexpansive and asymptotically nonexpansive mappings including Mann-type and Ishikawa-type iterations have been studied extensively by many authors (see [1–23] and the references cited therein). In 1991, Schu [14] considered the modified Mann iteration process for an asymptotically nonexpansive map. Later, Tan and Xu [16] studied the modified Ishikawa iteration process for an asymptotically nonexpansive map. Noor, in 2000, introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces [11]. Later, Cho *et al.* [4], Xu and Noor [18] studied weak and strong convergence theorems for the three-step Noor iterations with errors for asymptotically nonexpansive mappings in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable. Takahashi and Tamura [15], Shahzad [24] dealt with the iterative scheme for a pair of nonexpansive and asymptotically nonexpansive mappings in a uniformly convex Banach space. In 2006, Plubtieng *et al.* [12] studied a class of three-step iterative scheme, for three asymptotically nonexpansive mappings, in a uniformly convex Banach space satisfying Opial's condition. In 2007, Fukhar-ud-dina and Khan [5] studied the scheme for three nonexpansive mappings in a uniformly convex Banach space which has Opial's condition or which has a Fréchet differentiable norm or whose dual space has the Kadec-Klee property. Also in 2007, Chidume and Bashir Ali [1] introduced the iterative scheme for a finite family of asymptotically nonexpansive mappings and obtained the weak and strong convergence theorems in a Banach space whose dual space satisfies the Kadec-Klee property.

In most of these papers, the map T has been assumed to map C into itself. If, however, C is a proper subset of a Banach space X and T maps C into X (as is the case in many applications), then $\{x_n\}$ may not be well defined. One method that has been used to overcome this is to introduce a retraction $P : X \rightarrow C$. Recent results on the approximation of fixed points of nonexpansive and asymptotically nonexpansive nonself mappings can be found in [24–33] and references contained therein.

In 2003, Chidume *et al.* [2] introduced the following modified Mann iteration process and got the weak and strong convergence theorems for an asymptotically nonexpansive nonself mapping:

$$x_1 \in C, \quad x_{n+1} = P[\alpha_n x_n + (1 - \alpha_n)T(PT)^{n-1}x_n], \quad n \geq 1.$$

Recently, Wang [32] generalized the above iteration process as follows: $x_1 \in C$,

$$\begin{aligned} x_{n+1} &= P[\alpha_n x_n + (1 - \alpha_n)T_1(PT_1)^{n-1}y_n]; \\ y_n &= P[\beta_n x_n + (1 - \beta_n)T_2(PT_2)^{n-1}x_n]. \end{aligned}$$

In 2007, Chidume and Bashir Ali [27] introduced the following iteration process for a finite family of asymptotically nonexpansive nonself mappings: $x_1 \in C$ and

$$\begin{aligned} x_{n+1} &= P[\alpha_{1n}x_n + (1 - \alpha_{1n})T_1(PT_1)^{n-1}y_{n+m-2}]; \\ y_{n+m-2} &= P[\alpha_{2n}x_n + (1 - \alpha_{2n})T_2(PT_2)^{n-1}y_{n+m-3}]; \\ &\dots \\ y_n &= P[\alpha_{mn}x_n + (1 - \alpha_{mn})T_m(PT_m)^{n-1}x_n]. \end{aligned}$$

They proved strong convergence theorems in uniformly convex Banach spaces and gave the weak convergence theorem in uniformly convex Banach spaces that satisfy Opial's condition or have a Fréchet differentiable norm. They also gave the weak convergence theorem for nonexpansive nonself mappings in uniformly convex Banach spaces whose dual spaces have the Kadec-Klee property (see [27]).

The concept of asymptotically nonexpansive nonself mappings in the intermediate sense was introduced by Chidume *et al.* [28] as an important generalization of asymptotically nonexpansive self-mappings in the intermediate sense.

Definition 1.1 Let C be a nonempty subset of a Banach space X . Let $P : X \rightarrow C$ be a nonexpansive retraction of X onto C . A nonself mapping $T : C \rightarrow X$ is called asymptotically nonexpansive in the intermediate sense if T is continuous and the following inequality holds:

$$\limsup_{n \rightarrow +\infty} \sup_{x, y \in C} (\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x - y\|) \leq 0.$$

It should be noted that in [28, 31, 34], an asymptotically nonexpansive mapping in the intermediate sense is required to be uniformly continuous. In Definition 1.1, we assume the

continuity of T instead of uniform continuity. Chidume *et al.* [28], Plubtieng and Wangkeeree [31], Kim and Kim [34] gave strong convergence theorems for a uniformly continuous mapping which is asymptotically nonexpansive in the intermediate sense in uniformly convex Banach spaces if the mapping is completely continuous. Also, Chidume *et al.* [28] gave the weak convergence theorem for such a mapping in a uniformly convex Banach space whose dual space has the Kadec-Klee property. However, as we know, it remains open whether the weak convergence theorem of a multi-step iteration process with errors for a finite family of continuous nonself mappings which are asymptotically nonexpansive in the intermediate sense holds in a uniformly convex Banach space which satisfies Opial's condition or whose dual space has the Kadec-Klee property. Since the asymptotically nonexpansive mappings in the intermediate sense are non-Lipschitzian and Bruck's lemma [35] does not extend beyond Lipschitzian mappings, new techniques are needed for this more general case. It is our purpose in this paper to study the following iteration process with errors for approximating common fixed points of a finite family of nonself mappings which are asymptotically nonexpansive in the intermediate sense:

$$\begin{aligned}
 x_1 &\in C; \\
 x_{n+1} &= P[\alpha_n^{(1)}x_n + \beta_n^{(1)}T_1(PT_1)^{n-1}y_n^{(N-2)} + \gamma_n^{(1)}u_n^{(1)}]; \\
 y_n^{(N-2)} &= P[\alpha_n^{(2)}x_n + \beta_n^{(2)}T_2(PT_2)^{n-1}y_n^{(N-3)} + \gamma_n^{(2)}u_n^{(2)}]; \\
 &\dots \\
 y_n^{(1)} &= P[\alpha_n^{(N-1)}x_n + \beta_n^{(N-1)}T_{N-1}(PT_{N-1})^{n-1}y_n^{(0)} + \gamma_n^{(N-1)}u_n^{(N-1)}]; \\
 y_n^{(0)} &= P[\alpha_n^{(N)}x_n + \beta_n^{(N)}T_N(PT_N)^{n-1}x_n + \gamma_n^{(N)}u_n^{(N)}].
 \end{aligned} \tag{1.1}$$

In Section 3, using the technique established in [33], we first give some weak convergence theorems of the iterative scheme (1.1) for a finite family of nonself mappings which are asymptotically nonexpansive in the intermediate sense in a uniformly convex Banach space which satisfies Opial's condition or whose dual space has the Kadec-Klee property. We also establish some strong convergence theorems if one member of the finite family of mappings satisfies a condition weaker than complete continuity. Our results extend and improve the recently announced ones [1, 2, 4, 5, 7, 9, 12, 15, 16, 18, 24, 27, 28, 31, 32, 34] and many others.

2 Preliminaries

Let C be a nonempty closed convex subset of a Banach space X . Recall that a Banach space X is said to be uniformly convex if, for each $\varepsilon \in [0, 2)$, the modulus of convexity of X given by

$$\delta(\varepsilon) = \inf \left\{ 1 - \frac{1}{2} \|x + y\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}$$

satisfies the inequality $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$. We say that X has the Kadec-Klee property if, for every sequence $\{x_n\} \subset X$, whenever $x_n \rightharpoonup x$ with $\|x_n\| \rightarrow \|x\|$, it follows that $x_n \rightarrow x$. We would like to remark that a reflexive Banach space X with a Fréchet differentiable norm implies that its dual X^* has the Kadec-Klee property, while the converse implication fails [36].

Recall that a Banach space X is said to satisfy Opial's condition [37] if $x_n \rightharpoonup x$ and $x \neq y$ implies that

$$\limsup_{n \rightarrow +\infty} \|x_n - x\| < \limsup_{n \rightarrow +\infty} \|x_n - y\|.$$

A subset C of X is said to be a retract if there exists a continuous mapping $P : X \rightarrow C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \rightarrow X$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P .

Lemma 2.1 [17] *Let the nonnegative number sequences $\{c_n\}$ and $\{w_n\}$ satisfy*

$$c_{n+1} \leq c_n + w_n, \quad n = 1, 2, \dots$$

If $\sum_{n=1}^{+\infty} w_n < +\infty$, then $\lim_{n \rightarrow +\infty} c_n$ exists.

Lemma 2.2 [14] *Suppose that X is a uniformly convex Banach space and for all positive integers n , $0 < p \leq t_n \leq q < 1$. If $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $\limsup_{n \rightarrow +\infty} \|x_n\| \leq r$, $\limsup_{n \rightarrow +\infty} \|y_n\| \leq r$ and*

$$\lim_{n \rightarrow +\infty} \|t_n x_n + (1 - t_n) y_n\| = r$$

hold for some $r \geq 0$. Then $\lim_{n \rightarrow +\infty} \|x_n - y_n\| = 0$.

Lemma 2.3 [6] *Let X be a uniformly convex Banach space. If $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon > 0$, then for all $\lambda \in [0, 1]$,*

$$\|\lambda x + (1 - \lambda)y\| \leq 1 - 2\lambda(1 - \lambda)\delta(\varepsilon).$$

Lemma 2.4 (Demiclosedness principle for a nonself-map [38]) *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $T : C \rightarrow X$ be a nonself mapping which is continuous and asymptotically nonexpansive in the intermediate sense. If $\{x_n\}$ is a sequence in C converging weakly to x and*

$$\lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \|x_n - T(PT)^{m-1} x_n\| = 0,$$

then $x \in F(T)$, i.e., $Tx = x$.

Lemma 2.5 [36] *Let X be a reflexive Banach space whose dual X^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in X and $f, g \in \omega_\omega(\{x_n\})$. Suppose that*

$$h(\lambda) = \lim_{n \rightarrow +\infty} \|\lambda x_n + (1 - \lambda)f - g\|$$

exists for all $\lambda \in [0, 1]$, then $f = g$, where $\omega_\omega(\{x_n\})$ denotes the set of weak limit points of $\{x_n\}$, i.e., $\omega_\omega(\{x_n\}) = \{p \in X : \text{there exists a subsequence } \{x_{n_i}\} \subset \{x_n\} \text{ such that } x_{n_i} \rightharpoonup p\}$.

3 Main results

In this section, let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X . Let $P : X \rightarrow C$ be a nonexpansive retraction from X onto C . Let $T_1, T_2, \dots, T_N : C \rightarrow X$ be a finite family of continuous nonself mappings which are asymptotically nonexpansive in the intermediate sense, then we can suppose that

$$r_n = \max \left\{ 0, \sup_{x, y \in C} \left(\|T_i(PT_i)^{n-1}x - T_i(PT_i)^{n-1}y\| - \|x - y\| \right) \right\}, \quad i = 1, 2, \dots, N.$$

Hence $r_n \geq 0$, $\lim_{n \rightarrow +\infty} r_n = 0$ and for all $x, y \in C$,

$$\|T_i(PT_i)^{n-1}x - T_i(PT_i)^{n-1}y\| - \|x - y\| \leq r_n, \quad n \geq 1, i = 1, 2, \dots, N.$$

For a given $x_1 \in C$, we can define the sequence $\{x_n\} \subset C$ by

$$\begin{aligned} x_{n+1} &= P[\alpha_n^{(1)}x_n + \beta_n^{(1)}T_1(PT_1)^{n-1}y_n^{(N-2)} + \gamma_n^{(1)}u_n^{(1)}]; \\ y_n^{(N-2)} &= P[\alpha_n^{(2)}x_n + \beta_n^{(2)}T_2(PT_2)^{n-1}y_n^{(N-3)} + \gamma_n^{(2)}u_n^{(2)}]; \\ &\dots \\ y_n^{(1)} &= P[\alpha_n^{(N-1)}x_n + \beta_n^{(N-1)}T_{N-1}(PT_{N-1})^{n-1}y_n^{(0)} + \gamma_n^{(N-1)}u_n^{(N-1)}]; \\ y_n^{(0)} &= P[\alpha_n^{(N)}x_n + \beta_n^{(N)}T_N(PT_N)^{n-1}x_n + \gamma_n^{(N)}u_n^{(N)}], \end{aligned} \quad (3.1)$$

where $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ are in $[0, 1]$ with $0 < p \leq \alpha_n^{(i)}, \beta_n^{(i)} \leq q < 1$, $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$, $\sum_{n=1}^{+\infty} \gamma_n^{(i)} < +\infty$ and $\{u_n^{(i)}\}$ are bounded sequences in X , $i = 1, 2, \dots, N$.

We start our investigation with the following lemmas, which are preparation for the proofs of the main results of this section. In the following, we always assume that $\sum_{n=1}^{+\infty} r_n < +\infty$ and the set of common fixed points of $\{T_1, T_2, \dots, T_N\}$ is nonempty, i.e.,

$$\bigcap_{i=1}^N F(T_i) = \bigcap_{i=1}^N \{x \in C : T_i x = x\} \neq \emptyset.$$

Lemma 3.1

$$\lim_{n \rightarrow +\infty} \|x_n - f\| = r$$

exists for each $f \in \bigcap_{i=1}^N F(T_i)$.

Proof Let $f \in \bigcap_{i=1}^N F(T_i)$. Since $\{u_n^{(i)}\}$ are bounded, we can set

$$M = \sup \{ \|u_n^{(i)} - f\| : i = 1, 2, \dots, N, n \geq 1 \} < +\infty.$$

Then

$$\begin{aligned} &\|y_n^{(0)} - f\| \\ &= \|P[\alpha_n^{(N)}x_n + \beta_n^{(N)}T_N(PT_N)^{n-1}x_n + \gamma_n^{(N)}u_n^{(N)}] - f\| \end{aligned}$$

$$\begin{aligned}
 &\leq \left\| \left[\alpha_n^{(N)} x_n + \beta_n^{(N)} T_N (PT_N)^{n-1} x_n + \gamma_n^{(N)} u_n^{(N)} \right] - f \right\| \\
 &\leq \alpha_n^{(N)} \|x_n - f\| + \beta_n^{(N)} \|T_N (PT_N)^{n-1} x_n - f\| + \gamma_n^{(N)} \|u_n^{(N)} - f\| \\
 &\leq \|x_n - f\| + r_n + M\gamma_n^{(N)}.
 \end{aligned} \tag{3.2}$$

Hence we can get

$$\begin{aligned}
 &\|y_n^{(1)} - f\| \\
 &\leq \left\| \left[\alpha_n^{(N-1)} x_n + \beta_n^{(N-1)} T_{N-1} (PT_{N-1})^{n-1} y_n^{(0)} + \gamma_n^{(N-1)} u_n^{(N-1)} \right] - f \right\| \\
 &\leq \alpha_n^{(N-1)} \|x_n - f\| + \beta_n^{(N-1)} \|T_{N-1} (PT_{N-1})^{n-1} y_n^{(0)} - f\| + \gamma_n^{(N-1)} \|u_n^{(N-1)} - f\| \\
 &\leq \alpha_n^{(N-1)} \|x_n - f\| + \beta_n^{(N-1)} (\|y_n^{(0)} - f\| + r_n) + M\gamma_n^{(N-1)} \\
 &\leq \alpha_n^{(N-1)} \|x_n - f\| + \beta_n^{(N-1)} (\|x_n - f\| + 2r_n + M\gamma_n^{(N)}) + M\gamma_n^{(N-1)} \\
 &= \|x_n - f\| + 2r_n + M(\gamma_n^{(N)} + \gamma_n^{(N-1)}) \\
 &\dots
 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
 &\|y_n^{(N-2)} - f\| \\
 &\leq \left\| \left[\alpha_n^{(2)} x_n + \beta_n^{(2)} T_2 (PT_2)^{n-1} y_n^{(N-3)} + \gamma_n^{(2)} u_n^{(2)} \right] - f \right\| \\
 &\leq \alpha_n^{(2)} \|x_n - f\| + \beta_n^{(2)} \|T_2 (PT_2)^{n-1} y_n^{(N-3)} - f\| + \gamma_n^{(2)} \|u_n^{(2)} - f\| \\
 &\leq \alpha_n^{(2)} \|x_n - f\| + \beta_n^{(2)} (\|y_n^{(N-3)} - f\| + r_n) + M\gamma_n^{(2)} \\
 &\leq \|x_n - f\| + (N-1)r_n + M(\gamma_n^{(N)} + \dots + \gamma_n^{(3)} + \gamma_n^{(2)}).
 \end{aligned} \tag{3.4}$$

Thus we obtain

$$\begin{aligned}
 &\|x_{n+1} - f\| \\
 &\leq \left\| \left[\alpha_n^{(1)} x_n + \beta_n^{(1)} T_1 (PT_1)^{n-1} y_n^{(N-2)} + \gamma_n^{(1)} u_n^{(1)} \right] - f \right\| \\
 &\leq \alpha_n^{(1)} \|x_n - f\| + \beta_n^{(1)} \|T_1 (PT_1)^{n-1} y_n^{(N-2)} - f\| + \gamma_n^{(1)} \|u_n^{(1)} - f\| \\
 &\leq \alpha_n^{(1)} \|x_n - f\| + \beta_n^{(1)} (\|y_n^{(N-2)} - f\| + r_n) + M\gamma_n^{(1)} \\
 &\leq \|x_n - f\| + Nr_n + M(\gamma_n^{(N)} + \dots + \gamma_n^{(2)} + \gamma_n^{(1)}).
 \end{aligned} \tag{3.5}$$

Set $w_n = Nr_n + M(\gamma_n^{(N)} + \gamma_n^{(N-1)} + \dots + \gamma_n^{(1)})$, then $\sum_{n=1}^{+\infty} w_n < +\infty$ and

$$\|x_{n+1} - f\| \leq \|x_n - f\| + w_n. \tag{3.6}$$

By Lemma 2.1, we have that

$$\lim_{n \rightarrow +\infty} \|x_n - f\| = r$$

exists. This completes the proof. \square

Lemma 3.2

$$\lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \|x_n - T_i(PT_i)^{m-1}x_n\| = 0, \quad i = 1, 2, \dots, N.$$

Proof By Lemma 3.1, we have that $\lim_{n \rightarrow +\infty} \|x_n - f\| = r$ exists. If $r = 0$, then it is obvious to see that the conclusion holds. In the following, we assume that $r > 0$. According to (3.2), (3.3) and (3.4), we can get

$$\limsup_{n \rightarrow +\infty} \|y_n^{(j)} - f\| \leq r, \quad j = 0, 1, \dots, N-2.$$

Then, for any $j = 0, 1, \dots, N-2$,

$$\limsup_{n \rightarrow +\infty} \|T_{N-1-j}(PT_{N-1-j})^{n-1}y_n^{(j)} - f\| \leq r,$$

and hence the sequences $\{T_{N-1-j}(PT_{N-1-j})^{n-1}y_n^{(j)}\}_{n=1}^{+\infty}$ are bounded. By (3.5), we can obtain

$$\lim_{n \rightarrow +\infty} \|\alpha_n^{(1)}x_n + \beta_n^{(1)}T_1(PT_1)^{n-1}y_n^{(N-2)} + \gamma_n^{(1)}u_n^{(1)} - f\| = r.$$

We also can see

$$\limsup_{n \rightarrow +\infty} \|T_1(PT_1)^{n-1}y_n^{(N-2)} - f + \gamma_n^{(1)}(u_n^{(1)} - x_n)\| \leq r$$

and

$$\limsup_{n \rightarrow +\infty} \|x_n - f + \gamma_n^{(1)}(u_n^{(1)} - x_n)\| \leq r.$$

It follows from Lemma 2.2 and

$$\begin{aligned} r &= \lim_{n \rightarrow +\infty} \|\alpha_n^{(1)}x_n + \beta_n^{(1)}T_1(PT_1)^{n-1}y_n^{(N-2)} + \gamma_n^{(1)}u_n^{(1)} - f\| \\ &= \lim_{n \rightarrow +\infty} \|(1 - \alpha_n^{(1)})[T_1(PT_1)^{n-1}y_n^{(N-2)} - f + \gamma_n^{(1)}(u_n^{(1)} - x_n)] \\ &\quad + \alpha_n^{(1)}[x_n - f + \gamma_n^{(1)}(u_n^{(1)} - x_n)] + \gamma_n^{(1)}[x_n - T_1(PT_1)^{n-1}y_n^{(N-2)}]\| \\ &= \lim_{n \rightarrow +\infty} \|(1 - \alpha_n^{(1)})[T_1(PT_1)^{n-1}y_n^{(N-2)} - f + \gamma_n^{(1)}(u_n^{(1)} - x_n)] \\ &\quad + \alpha_n^{(1)}[x_n - f + \gamma_n^{(1)}(u_n^{(1)} - x_n)]\| \end{aligned}$$

that

$$\lim_{n \rightarrow +\infty} \|T_1(PT_1)^{n-1}y_n^{(N-2)} - x_n\| = 0. \quad (3.7)$$

Combining it with

$$\begin{aligned} \|x_n - f\| &\leq \|x_n - T_1(PT_1)^{n-1}y_n^{(N-2)}\| + \|T_1(PT_1)^{n-1}y_n^{(N-2)} - f\| \\ &\leq \|x_n - T_1(PT_1)^{n-1}y_n^{(N-2)}\| + \|y_n^{(N-2)} - f\| + r_n, \end{aligned}$$

we obtain

$$\liminf_{n \rightarrow +\infty} \|y_n^{(N-2)} - f\| \geq r.$$

Thus $\lim_{n \rightarrow +\infty} \|y_n^{(N-2)} - f\| = r$, according to (3.4), we have

$$\begin{aligned} r &= \lim_{n \rightarrow +\infty} \left\| \left[\alpha_n^{(2)} x_n + \beta_n^{(2)} T_2(P T_2)^{n-1} y_n^{(N-3)} + \gamma_n^{(2)} u_n^{(2)} \right] - f \right\| \\ &= \lim_{n \rightarrow +\infty} \left\| \left(1 - \alpha_n^{(2)} \right) \left[T_2(P T_2)^{n-1} y_n^{(N-3)} - f + \gamma_n^{(2)} (u_n^{(2)} - x_n) \right] \right. \\ &\quad \left. + \alpha_n^{(2)} [x_n - f + \gamma_n^{(2)} (u_n^{(2)} - x_n)] + \gamma_n^{(2)} [x_n - T_2(P T_2)^{n-1} y_n^{(N-3)}] \right\| \\ &= \lim_{n \rightarrow +\infty} \left\| \left(1 - \alpha_n^{(2)} \right) \left[T_2(P T_2)^{n-1} y_n^{(N-3)} - f + \gamma_n^{(2)} (u_n^{(2)} - x_n) \right] \right. \\ &\quad \left. + \alpha_n^{(2)} [x_n - f + \gamma_n^{(2)} (u_n^{(2)} - x_n)] \right\|. \end{aligned}$$

Noting

$$\limsup_{n \rightarrow +\infty} \|T_2(P T_2)^{n-1} y_n^{(N-3)} - f + \gamma_n^{(2)} (u_n^{(2)} - x_n)\| \leq r$$

and

$$\limsup_{n \rightarrow +\infty} \|x_n - f + \gamma_n^{(2)} (u_n^{(2)} - x_n)\| \leq r,$$

by Lemma 2.2 again, we have

$$\lim_{n \rightarrow +\infty} \|T_2(P T_2)^{n-1} y_n^{(N-3)} - x_n\| = 0. \quad (3.8)$$

Similarly, we can get

$$\begin{aligned} \lim_{n \rightarrow +\infty} \|T_3(P T_3)^{n-1} y_n^{(N-4)} - x_n\| &= \dots = \lim_{n \rightarrow +\infty} \|T_{N-1}(P T_{N-1})^{n-1} y_n^{(0)} - x_n\| \\ &= \lim_{n \rightarrow +\infty} \|T_N(P T_N)^{n-1} x_n - x_n\| = 0. \end{aligned}$$

Therefore, by (3.1) and (3.7),

$$\lim_{n \rightarrow +\infty} \|x_{n+1} - x_n\| \leq \lim_{n \rightarrow +\infty} \left\| \beta_n^{(1)} [T_1(P T_1)^{n-1} y_n^{(N-2)} - x_n] + \gamma_n^{(1)} (u_n^{(1)} - x_n) \right\| = 0$$

and similarly, we can have

$$\lim_{n \rightarrow +\infty} \|y_n^{(N-2)} - x_n\| = \dots = \lim_{n \rightarrow +\infty} \|y_n^{(1)} - x_n\| = \lim_{n \rightarrow +\infty} \|y_n^{(0)} - x_n\| = 0.$$

It follows from the inequality (3.7) and

$$\begin{aligned} &\|T_1(P T_1)^{n-1} x_n - x_n\| \\ &\leq \|T_1(P T_1)^{n-1} x_n - T_1(P T_1)^{n-1} y_n^{(N-2)}\| + \|T_1(P T_1)^{n-1} y_n^{(N-2)} - x_n\| \\ &\leq \|x_n - y_n^{(N-2)}\| + r_n + \|T_1(P T_1)^{n-1} y_n^{(N-2)} - x_n\| \end{aligned}$$

that

$$\lim_{n \rightarrow +\infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0. \quad (3.9)$$

Thus, for any fixed m , we have $\lim_{n \rightarrow +\infty} \|x_n - x_{n+m}\| = 0$ and

$$\begin{aligned} & \|x_n - T_1(PT_1)^{m-1}x_n\| \\ & \leq \|x_n - x_{n+m}\| + \|x_{n+m} - T_1(PT_1)^{n+m-1}x_{n+m}\| \\ & \quad + \|T_1(PT_1)^{n+m-1}x_{n+m} - T_1(PT_1)^{n+m-1}x_n\| + \|T_1(PT_1)^{n+m-1}x_n - T_1(PT_1)^{m-1}x_n\| \\ & \leq 2\|x_n - x_{n+m}\| + \|x_{n+m} - T_1(PT_1)^{n+m-1}x_{n+m}\| + r_{n+m} + \|(PT_1)^n x_n - x_n\| + r_m \\ & \leq 2\|x_n - x_{n+m}\| + \|x_{n+m} - T_1(PT_1)^{n+m-1}x_{n+m}\| + r_{n+m} + \|T_1(PT_1)^{n-1}x_n - x_n\| + r_m. \end{aligned}$$

This implies

$$\limsup_{n \rightarrow +\infty} \|x_n - T_1(PT_1)^{m-1}x_n\| \leq r_m.$$

Therefore,

$$\lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \|x_n - T_1(PT_1)^{m-1}x_n\| = 0.$$

Noting (3.8) and

$$\begin{aligned} & \|T_2(PT_2)^{n-1}x_n - x_n\| \\ & \leq \|T_2(PT_2)^{n-1}x_n - T_2(PT_2)^{n-1}y_n^{(N-3)}\| + \|T_2(PT_2)^{n-1}y_n^{(N-3)} - x_n\| \\ & \leq \|x_n - y_n^{(N-3)}\| + r_n + \|T_2(PT_2)^{n-1}y_n^{(N-3)} - x_n\|, \end{aligned}$$

we can see

$$\lim_{n \rightarrow +\infty} \|T_2(PT_2)^{n-1}x_n - x_n\| = 0.$$

Thus, for any fixed m ,

$$\begin{aligned} & \|x_n - T_2(PT_2)^{m-1}x_n\| \\ & \leq \|x_n - x_{n+m}\| + \|x_{n+m} - T_2(PT_2)^{n+m-1}x_{n+m}\| \\ & \quad + \|T_2(PT_2)^{n+m-1}x_{n+m} - T_2(PT_2)^{n+m-1}x_n\| + \|T_2(PT_2)^{n+m-1}x_n - T_2(PT_2)^{m-1}x_n\| \\ & \leq 2\|x_n - x_{n+m}\| + \|x_{n+m} - T_2(PT_2)^{n+m-1}x_{n+m}\| + r_{n+m} + \|(PT_2)^n x_n - x_n\| + r_m \\ & \leq 2\|x_n - x_{n+m}\| + \|x_{n+m} - T_2(PT_2)^{n+m-1}x_{n+m}\| + r_{n+m} + \|T_2(PT_2)^{n-1}x_n - x_n\| + r_m, \end{aligned}$$

which means

$$\limsup_{n \rightarrow +\infty} \|x_n - T_2(PT_2)^{m-1}x_n\| \leq r_m.$$

Therefore,

$$\lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \|x_n - T_2(PT_2)^{m-1}x_n\| = 0.$$

By the same argument, we can get

$$\lim_{n \rightarrow +\infty} \|x_n - T_3(PT_3)^{m-1}x_n\| = \dots = \lim_{n \rightarrow +\infty} \|x_n - T_N(PT_N)^{m-1}x_n\| = 0.$$

This completes the proof. \square

Define the operator $W_n : C \rightarrow C$ by

$$\begin{aligned} W_n x &= P[\alpha_n^{(1)}x + \beta_n^{(1)}T_1(PT_1)^{n-1}x^{(N-2)} + \gamma_n^{(1)}u_n^{(1)}]; \\ x^{(N-2)} &= P[\alpha_n^{(2)}x + \beta_n^{(2)}T_2(PT_2)^{n-1}x^{(N-3)} + \gamma_n^{(2)}u_n^{(2)}]; \\ &\dots \\ x^{(1)} &= P[\alpha_n^{(N-1)}x + \beta_n^{(N-1)}T_{N-1}(PT_{N-1})^{n-1}x^{(0)} + \gamma_n^{(N-1)}u_n^{(N-1)}]; \\ x^{(0)} &= P[\alpha_n^{(N)}x + \beta_n^{(N)}T_N(PT_N)^{n-1}x + \gamma_n^{(N)}u_n^{(N)}], \end{aligned}$$

where $x \in C$. Then $x_{n+1} = W_n x_n$ and for all $x, y \in C$, we have

$$\begin{aligned} \|x^{(0)} - y^{(0)}\| &\leq \alpha_n^{(N)}\|x - y\| + \beta_n^{(N)}\|T_N(PT_N)^{n-1}x - T_N(PT_N)^{n-1}y\| \\ &\leq \|x - y\| + r_n, \\ \|x^{(1)} - y^{(1)}\| &\leq \alpha_n^{(N-1)}\|x - y\| + \beta_n^{(N-1)}\|T_{N-1}(PT_{N-1})^{n-1}x^{(0)} - T_{N-1}(PT_{N-1})^{n-1}y^{(0)}\| \\ &\leq \alpha_n^{(N-1)}\|x - y\| + \beta_n^{(N-1)}(\|x^{(0)} - y^{(0)}\| + r_n) \\ &\leq \|x - y\| + 2r_n, \\ &\dots \end{aligned}$$

and

$$\begin{aligned} \|W_n x - W_n y\| &\leq \alpha_n^{(1)}\|x - y\| + \beta_n^{(1)}\|T_1(PT_1)^{n-1}x^{(N-2)} - T_1(PT_1)^{n-1}y^{(N-2)}\| \\ &\leq \alpha_n^{(1)}\|x - y\| + \beta_n^{(1)}(\|x^{(N-2)} - y^{(N-2)}\| + r_n) \\ &\leq \alpha_n^{(1)}\|x - y\| + \beta_n^{(1)}(\|x - y\| + Nr_n) \\ &\leq \|x - y\| + w_n. \end{aligned} \tag{3.10}$$

For any $f \in \bigcap_{i=1}^N F(T_i)$, we get

$$\begin{aligned} \|f^{(0)} - f\| &\leq \|\alpha_n^{(N)}f + \beta_n^{(N)}T_N(PT_N)^{n-1}f + \gamma_n^{(N)}u_n^{(N)} - f\| \\ &= \gamma_n^{(N)}\|u_n^{(N)} - f\| \leq M\gamma_n^{(N)}, \\ \|f^{(1)} - f\| &\leq \beta_n^{(N-1)}\|f^{(0)} - f\| + r_n + \gamma_n^{(N-1)}\|u_n^{(N-1)} - f\| \\ &\leq r_n + M(\gamma_n^{(N)} + \gamma_n^{(N-1)}), \end{aligned}$$

...

$$\|f^{(N-2)} - f\| \leq (N-2)r_n + M(\gamma_n^{(N)} + \cdots + \gamma_n^{(2)})$$

and

$$\begin{aligned} \|W_n f - f\| &\leq \beta_n^{(1)} \|f^{(N-2)} - f\| + r_n + \gamma_n^{(1)} \|u_n^{(1)} - f\| \\ &\leq (N-1)r_n + M(\gamma_n^{(N)} + \cdots + \gamma_n^{(1)}) \leq w_n. \end{aligned} \quad (3.11)$$

Set $S_{n,m} = W_{n+m-1} W_{n+m-2} \cdots W_{n+1} W_n : C \rightarrow C$, then $x_{n+m} = S_{n,m} x_n$ and for any $x, y \in C$,

$$\|S_{n,m} x - S_{n,m} y\| \leq \|x - y\| + (w_{n+m-1} + \cdots + w_{n+1} + w_n). \quad (3.12)$$

We also need the following lemma, which plays a crucial role in dealing with the case of the iteration with errors. It is easy to see $S_{n,m} f \equiv f$ if $\gamma_n^{(i)} \equiv 0$ for all $i = 1, 2, \dots, N$ and all $n \geq 1$.

Lemma 3.3

$$\lim_{n \rightarrow +\infty} \limsup_{m \rightarrow +\infty} \|S_{n,m} f - f\| = 0, \quad \forall f \in \bigcap_{i=1}^N F(T_i).$$

Proof By (3.10), (3.11) and

$$\begin{aligned} \|S_{n,2} f - f\| &= \|W_{n+1} W_n f - f\| \\ &\leq \|W_{n+1} W_n f - W_{n+1} f\| + \|W_{n+1} f - f\| \\ &\leq \|W_n f - f\| + w_{n+1} + w_{n+1} \\ &\leq w_n + 2w_{n+1}, \end{aligned}$$

...

$$\begin{aligned} \|S_{n,m-1} f - f\| \\ \leq w_n + 2w_{n+1} + \cdots + 2w_{n+m-2}, \end{aligned}$$

we get

$$\begin{aligned} \|S_{n,m} f - f\| &= \|W_{n+m-1} S_{n,m-1} f - f\| \\ &\leq \|W_{n+m-1} S_{n,m-1} f - W_{n+m-1} f\| + \|W_{n+m-1} f - f\| \\ &\leq \|S_{n,m-1} f - f\| + w_{n+m-1} + w_{n+m-1} \\ &\leq w_n + 2w_{n+1} + \cdots + 2w_{n+m-1} \\ &\leq 2(w_n + w_{n+1} + \cdots + w_{n+m-1}). \end{aligned} \quad (3.13)$$

Then fixing n and taking the limsup for m , we obtain

$$\limsup_{m \rightarrow +\infty} \|S_{n,m} f - f\| \leq 2 \sum_{i=n}^{+\infty} w_i.$$

Thus,

$$\lim_{n \rightarrow +\infty} \limsup_{m \rightarrow +\infty} \|S_{n,m}f - f\| \leq \lim_{n \rightarrow +\infty} 2 \sum_{i=n}^{+\infty} w_i = 0.$$

This completes the proof. \square

Lemma 3.4 *Let $f, g \in \bigcap_{i=1}^N F(T_i)$ and $\lambda \in [0, 1]$, then*

$$h(\lambda) = \lim_{n \rightarrow +\infty} \|\lambda x_n + (1 - \lambda)f - g\|$$

exists.

Proof It follows from Lemma 3.1 that $r = \lim_{n \rightarrow +\infty} \|x_n - f\|$ exists. If $\lambda = 0, 1$ or $r = 0$, then the conclusion holds. In the following, we assume that $r > 0$ and $\lambda \in (0, 1)$. Then, for any $\varepsilon > 0$, there exists $d > 0$ ($d < \varepsilon$) such that

$$(r + d) \left[1 - 2\lambda(1 - \lambda)\delta \left(\frac{\varepsilon}{r + d} \right) \right] < r - d, \quad (3.14)$$

where δ is the modulus of convexity of the norm. Hence there exists a positive integer n_0 such that for all $n > n_0$,

$$r - \frac{d}{4} \leq \|x_n - f\| \leq r + \frac{d}{4} \quad (3.15)$$

and

$$\sum_{i=n}^{+\infty} w_i \leq \lambda(1 - \lambda) \frac{d}{4} < \frac{\varepsilon}{4}. \quad (3.16)$$

Now we claim that for all $n > n_0$,

$$\|S_{n,m}[\lambda x_n + (1 - \lambda)f] - [\lambda S_{n,m}x_n + (1 - \lambda)S_{n,m}f]\| \leq \varepsilon, \quad \forall m = 1, 2, \dots \quad (3.17)$$

Otherwise, we can suppose that

$$\|S_{n,m}[\lambda x_n + (1 - \lambda)f] - [\lambda S_{n,m}x_n + (1 - \lambda)S_{n,m}f]\| \geq \varepsilon$$

for some m . Put $z = \lambda x_n + (1 - \lambda)f$, $x = (1 - \lambda)(S_{n,m}z - S_{n,m}f)$ and $y = \lambda(S_{n,m}x_n - S_{n,m}z)$, then by (3.12), (3.14) and (3.15),

$$\begin{aligned} \|x\| &\leq (1 - \lambda)(\|S_{n,m}z - S_{n,m}f\|) \\ &\leq (1 - \lambda)[\|z - f\| + (w_{n+m-1} + \dots + w_{n+1} + w_n)] \\ &\leq \lambda(1 - \lambda) \left(\|x_n - f\| + \frac{d}{4} \right) \\ &\leq \lambda(1 - \lambda)(r + d) \end{aligned}$$

and

$$\begin{aligned}\|y\| &= \lambda \|S_{n,m}x_n - S_{n,m}z\| \\ &\leq \lambda [\|x_n - z\| + (w_{n+m-1} + \cdots + w_{n+1} + w_n)] \\ &\leq \lambda(1-\lambda) \left(\|x_n - f\| + \frac{d}{4} \right) \\ &\leq \lambda(1-\lambda)(r+d).\end{aligned}$$

We also have

$$\|x - y\| = \|S_{n,m}[\lambda x_n + (1-\lambda)f] - [\lambda S_{n,m}x_n + (1-\lambda)S_{n,m}f]\| \geq \varepsilon$$

and

$$\lambda x + (1-\lambda)y = \lambda(1-\lambda)(S_{n,m}x_n - S_{n,m}f).$$

So, by using Lemma 2.3, we get

$$\begin{aligned}&\lambda(1-\lambda)\|S_{n,m}x_n - S_{n,m}f\| \\ &= \|\lambda x + (1-\lambda)y\| \\ &\leq \lambda(1-\lambda)(r+d) \left[1 - 2\lambda(1-\lambda)\delta \left(\frac{\varepsilon}{\lambda(1-\lambda)(r+d)} \right) \right] \\ &\leq \lambda(1-\lambda)(r+d) \left[1 - 2\lambda(1-\lambda)\delta \left(\frac{\varepsilon}{r+d} \right) \right],\end{aligned}$$

and then by (3.13), (3.15) and (3.16), we have

$$\begin{aligned}r-d &\leq \|x_{n+m} - f\| - \|S_{n,m}f - f\| \\ &\leq \|S_{n,m}x_n - f\| - \|S_{n,m}f - f\| \\ &\leq \|S_{n,m}x_n - S_{n,m}f\| \\ &\leq (r+d) \left[1 - 2\lambda(1-\lambda)\delta \left(\frac{\varepsilon}{r+d} \right) \right].\end{aligned}$$

This contradicts (3.14). Thus we can conclude that for all $n > n_0$, (3.17) holds. Hence, for all $n > n_0$,

$$\begin{aligned}&\|\lambda x_{n+m} + (1-\lambda)f - g\| \\ &= \|\lambda S_{n,m}x_n + (1-\lambda)f - g\| \\ &\leq \|\lambda S_{n,m}x_n + (1-\lambda)S_{n,m}f - S_{n,m}[\lambda x_n + (1-\lambda)f]\| + (1-\lambda)\|S_{n,m}f - f\| \\ &\quad + \|S_{n,m}[\lambda x_n + (1-\lambda)f] - S_{n,m}g\| + \|S_{n,m}g - g\| \\ &\leq 2\varepsilon + \|S_{n,m}f - f\| + \|\lambda x_n + (1-\lambda)f - g\| + \|S_{n,m}g - g\|.\end{aligned}$$

For any fixed $n > n_0$, we can take the limsup for m and obtain

$$\begin{aligned} & \limsup_{m \rightarrow +\infty} \|\lambda x_m + (1 - \lambda)f - g\| \\ & \leq \|\lambda x_n + (1 - \lambda)f - g\| + 2\varepsilon + \limsup_{m \rightarrow +\infty} \|S_{n,m}f - f\| + \limsup_{m \rightarrow +\infty} \|S_{n,m}g - g\|. \end{aligned}$$

Hence we have

$$\limsup_{m \rightarrow +\infty} \|\lambda x_m + (1 - \lambda)f - g\| \leq \liminf_{n \rightarrow +\infty} \|\lambda x_n + (1 - \lambda)f - g\| + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this implies that

$$h(\lambda) = \lim_{n \rightarrow +\infty} \|\lambda x_n + (1 - \lambda)f - g\|$$

exists. This completes the proof. \square

Remark 3.1 If the mappings T_1, T_2, \dots, T_N are asymptotically nonexpansive, we can use Bruck's lemma [35] to prove Lemma 3.4. While Bruck's lemma is not valid for non-Lipschitzian mappings, we must introduce some new techniques to establish a similar inequality. As we have seen, we use mainly the technique of the modulus of convexity and our proof is straightforward.

Now we can prove the weak convergence theorem of the iterative scheme (3.1).

Theorem 3.1 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition or whose dual X^* has the Kadec-Klee property. Let $P : X \rightarrow C$ be a nonexpansive retraction from X onto C . Let $T_1, T_2, \dots, T_N : C \rightarrow X$ be a finite family of nonself mappings which are asymptotically nonexpansive in the intermediate sense with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and the sequences $\{r_n\} \subset [0, +\infty)$ satisfying $\sum_{n=1}^{+\infty} r_n < +\infty$. Let $\{x_n\}$ be defined by*

$$\begin{aligned} & x_1 \in C; \\ & x_{n+1} = P[\alpha_n^{(1)} x_n + \beta_n^{(1)} T_1(P T_1)^{n-1} y_n^{(N-2)} + \gamma_n^{(1)} u_n^{(1)}]; \\ & y_n^{(N-2)} = P[\alpha_n^{(2)} x_n + \beta_n^{(2)} T_2(P T_2)^{n-1} y_n^{(N-3)} + \gamma_n^{(2)} u_n^{(2)}]; \\ & \dots \\ & y_n^{(1)} = P[\alpha_n^{(N-1)} x_n + \beta_n^{(N-1)} T_{N-1}(P T_{N-1})^{n-1} y_n^{(0)} + \gamma_n^{(N-1)} u_n^{(N-1)}]; \\ & y_n^{(0)} = P[\alpha_n^{(N)} x_n + \beta_n^{(N)} T_N(P T_N)^{n-1} x_n + \gamma_n^{(N)} u_n^{(N)}], \end{aligned}$$

where $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ are in $[0, 1]$ with $0 < p \leq \alpha_n^{(i)}, \beta_n^{(i)} \leq q < 1$, $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$, $\sum_{n=1}^{+\infty} \gamma_n^{(i)} < +\infty$ and $\{u_n^{(i)}\}$ are bounded sequences in X , $i = 1, 2, \dots, N$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.

Proof It suffices to prove that the set $\omega_\omega(\{x_n\})$ is a singleton. Since X is reflexive and $\{x_n\}$ is bounded, we obtain $\omega_\omega(\{x_n\}) \neq \emptyset$. Assuming that $f, g \in \omega_\omega(\{x_n\})$, in the following, we need

to show $f = g$. First, by Lemma 2.4 and Lemma 3.2, we know $f, g \in \bigcap_{i=1}^N F(T_i)$. Second, on the one hand, if X^* has the Kadec-Klee property, then from Lemma 3.4 and Lemma 2.5, we can get $f = g$. On the other hand, if X satisfies Opial's condition, we assume that $f \neq g$ and two subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ in $\{x_n\}$ such that $x_{n_i} \rightharpoonup f$ and $x_{n_j} \rightharpoonup g$. Hence by Opial's condition and Lemma 3.1, we get

$$\begin{aligned} \lim_{n \rightarrow +\infty} \|x_n - f\| &= \lim_{i \rightarrow +\infty} \|x_{n_i} - f\| \\ &< \lim_{i \rightarrow +\infty} \|x_{n_i} - g\| = \lim_{n \rightarrow +\infty} \|x_n - g\| = \lim_{j \rightarrow +\infty} \|x_{n_j} - g\| \\ &< \lim_{i \rightarrow +\infty} \|x_{n_j} - f\| = \lim_{n \rightarrow +\infty} \|x_n - f\|. \end{aligned}$$

This contraction implies $f = g$. This completes the proof. \square

Remark 3.2 Theorem 3.1 generalizes and improves many recent important results. For instance, if $N = 1$ and $T_1 : C \rightarrow X$ is a uniformly continuous mapping which is asymptotically nonexpansive in the intermediate sense, then we can get Theorem 3.13 in [28]. If $T_1, T_2, \dots, T_N : C \rightarrow X$ are asymptotically nonexpansive nonself mappings and (1) $\gamma_n^{(i)} \equiv 0$, then we can obtain Theorem 3.6 in [27]; (2) $\gamma_n^{(i)} \equiv 0$, $T_i : C \rightarrow C$, then we can get Theorem 3.4 in [1]; (3) $T_i : C \rightarrow C$ and $N \leq 3$, then we can get Theorem 3.10 in [2], Theorem 2.1 in [4], Theorems 3.1-3.2 in [7], Theorem 1 in [9], Theorem 2.9 in [12], Theorem 3.3 in [15], Theorems 3.1-3.2 in [16], Theorem 3.5 in [32] and many others.

If $\{T_i\}_{i=1}^N$ is a family of nonexpansive mappings, we can have the following theorem, which is an extension of Theorem 4.1 in [5], Theorem 1 in [9], Theorem 3.2 in [15], Theorem 3.5 and Theorem 4.1 in [24], Theorem 3.9 and Theorem 4.2 in [27] and others. The proof is immediate corollaries of our lemmas and Theorem 3.1.

Theorem 3.2 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition or whose dual X^* has the Kadec-Klee property. Let $P : X \rightarrow C$ be a nonexpansive retraction from X onto C . Let $T_1, T_2, \dots, T_N : C \rightarrow X$ be a finite family of nonself nonexpansive mappings with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined by*

$$\begin{aligned} x_1 &\in C; \\ x_{n+1} &= P[\alpha_n^{(1)} x_n + \beta_n^{(1)} T_1 y_n^{(N-2)} + \gamma_n^{(1)} u_n^{(1)}]; \\ y_n^{(N-2)} &= P[\alpha_n^{(2)} x_n + \beta_n^{(2)} T_2 y_n^{(N-3)} + \gamma_n^{(2)} u_n^{(2)}]; \\ &\dots \\ y_n^{(1)} &= P[\alpha_n^{(N-1)} x_n + \beta_n^{(N-1)} T_{N-1} y_n^{(0)} + \gamma_n^{(N-1)} u_n^{(N-1)}]; \\ y_n^{(0)} &= P[\alpha_n^{(N)} x_n + \beta_n^{(N)} T_N x_n + \gamma_n^{(N)} u_n^{(N)}], \end{aligned}$$

where $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ are in $[0, 1]$ with $0 < p \leq \alpha_n^{(i)}, \beta_n^{(i)} \leq q < 1$, $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$, $\sum_{n=1}^{+\infty} \gamma_n^{(i)} < +\infty$ and $\{u_n^{(i)}\}$ are bounded sequences in X , $i = 1, 2, \dots, N$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.

Now we can give the strong convergence theorem of the scheme (3.1).

Theorem 3.3 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $P : X \rightarrow C$ be a nonexpansive retraction from X onto C . Let $T_1, T_2, \dots, T_N : C \rightarrow X$ be a finite family of nonself mappings which are asymptotically nonexpansive in the intermediate sense and $\{x_n\}$ be as in Theorem 3.1. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$ if and only if*

$$\lim_{n \rightarrow +\infty} d(x_n, F) = 0,$$

where F denotes the set of common fixed points of $\{T_i\}_{i=1}^N$, i.e., $F = \bigcap_{i=1}^N F(T_i)$.

Proof We only need to show the sufficiency. If $\lim_{n \rightarrow +\infty} d(x_n, F) = 0$, then for any $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n \geq n_0$,

$$d(x_n, F) < \frac{\varepsilon}{4} \quad \text{and} \quad \sum_{i=n}^{+\infty} w_i < \frac{\varepsilon}{3}.$$

Hence, for any $n \geq n_0$, there exists $f \in F$ such that $\|x_n - f\| < \frac{\varepsilon}{3}$. Therefore, for any $m \in \mathbb{N}$, by (3.6),

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - f\| + \|x_n - f\| \\ &\leq \|x_{n+m-1} - f\| + w_{n+m-1} + \|x_n - f\| \\ &\dots \\ &\leq \|x_n - f\| + (w_n + \dots + w_{n+m-2} + w_{n+m-1}) + \|x_n - f\| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence in C and then it must converge to some point in C . Set $\lim_{n \rightarrow +\infty} x_n = p$, since $\lim_{n \rightarrow +\infty} d(x_n, F) = 0$ and F is closed, we get $p \in F$. This completes the proof. \square

In the following, we shall give a sufficient condition to ensure the strong convergence of the iterative sequence (3.1). We need the following notions. Recall that a finite family of nonself mappings $T_1, T_2, \dots, T_N : C \rightarrow X$ with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ satisfies Condition (\tilde{C}) if there exists a nondecreasing function $f : [0, +\infty) \rightarrow [0, +\infty)$ with $f(0) = 0, f(r) > 0$, for $r \in (0, +\infty)$, such that at least one of the $\{T_i\}_{i=1}^N$ satisfies condition (\tilde{I}) , i.e.,

$$\|T_i x - x\| \geq f(d(x, F)), \quad \forall x \in C,$$

for at least one $T_i, 1 \leq i \leq N$, where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

A mapping $T : C \rightarrow X$ is said to be demicompact if, for any bounded sequence $\{x_n\}$ in C such that $x_n - Tx_n$ converges, there exists a subsequence, say $\{x_{n_j}\}$ of $\{x_n\}$, such that $\{x_{n_j}\}$ converges strongly to some point in C . T is said to be completely continuous if it is continuous and for every bounded sequence $\{x_n\}$, there exists a subsequence, say $\{x_{n_j}\}$ of $\{x_n\}$, such that the sequence Tx_{n_j} converges to some element of the range of T .

It is well known that every continuous and demicompact mapping must satisfy condition (\tilde{I}) since every completely continuous mapping $T : C \rightarrow C$ is continuous and demicompact so that it satisfies condition (\tilde{I}) . Therefore, the condition (\tilde{I}) is weaker than the demicompactness and complete continuity (see [27]). Next we shall give several strong convergence theorems in uniformly convex Banach spaces if one member of the finite family of asymptotically nonexpansive in the intermediate sense mappings $\{T_i\}_{i=1}^N$ satisfies condition (\tilde{I}) .

Theorem 3.4 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $P : X \rightarrow C$ be a nonexpansive retraction from X onto C . Let $T_1, T_2, \dots, T_N : C \rightarrow X$ be a finite family of uniformly continuous nonself mappings which are asymptotically nonexpansive in the intermediate sense and let $\{x_n\}$ be as in Theorem 3.1. If the family $\{T_1, T_2, \dots, T_N\}$ satisfies condition (\tilde{C}) , then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$.*

Proof Without loss of generality, we assume that T_1 satisfies condition (\tilde{I}) , i.e.,

$$\|T_1x - x\| \geq f(d(x, F)), \quad \forall x \in C.$$

Hence we have

$$\|T_1x_n - x_n\| \geq f(d(x_n, F)), \quad n = 1, 2, \dots \quad (3.18)$$

Then by (3.9), the uniform continuity of T_1 and

$$\begin{aligned} \|x_n - T_1x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_1(PT_1)^n x_{n+1}\| \\ &\quad + \|T_1(PT_1)^n x_{n+1} - T_1(PT_1)^n x_n\| + \|T_1(PT_1)^n x_n - T_1x_n\| \\ &\leq 2\|x_n - x_{n+1}\| + \|x_{n+1} - T_1(PT_1)^n x_{n+1}\| + \|T_1(PT_1)^n x_n - T_1x_n\| + r_{n+1}, \end{aligned}$$

we derive

$$\lim_{n \rightarrow +\infty} \|x_n - T_1x_n\| = 0. \quad (3.19)$$

By (3.6), we have for all $f \in F$, $\|x_{n+1} - f\| \leq \|x_n - f\| + w_n$, where $\sum_{n=1}^{+\infty} w_n < +\infty$. Hence

$$d(x_{n+1}, F) \leq d(x_n, F) + w_n.$$

Then it follows from Lemma 2.1 that $\lim_{n \rightarrow +\infty} d(x_n, F)$ exists. Hence, by (3.18) and (3.19), we see $\lim_{n \rightarrow +\infty} f(d(x_n, F)) = 0$ and therefore,

$$\lim_{n \rightarrow +\infty} d(x_n, F) = 0.$$

By Theorem 3.3, we can get what we desired. This completes the proof. \square

Remark 3.3 From Theorem 3.4, we can get Theorem 3.8 and Theorem 3.10 in [28], Theorem 3.5 in [31], Theorem 1 and Theorem 2 in [34].

For completeness, we conclude with the following strong convergence theorem for a finite family of nonexpansive and asymptotically nonexpansive nonself mappings.

Theorem 3.5 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $P : X \rightarrow C$ be a nonexpansive retraction from X onto C . Let $T_1, T_2, \dots, T_N : C \rightarrow X$ be a finite family of asymptotically nonexpansive nonself mappings and $\{x_n\}$ be as in Theorem 3.1. If the family $\{T_1, T_2, \dots, T_N\}$ satisfies condition (\tilde{C}) , then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$.*

Theorem 3.6 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $P : X \rightarrow C$ be a nonexpansive retraction from X onto C . Let $T_1, T_2, \dots, T_N : C \rightarrow X$ be a finite family of nonexpansive nonself mappings and $\{x_n\}$ be as in Theorem 3.2. If the family $\{T_1, T_2, \dots, T_N\}$ satisfies condition (\tilde{C}) , then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$.*

Remark 3.4 Theorem 3.5 and Theorem 3.6 generalize and improve many recent important results such as Theorem 3.5 in [1], Theorem 3.7 in [2], Theorem 2.4 in [4], Theorem 4.2 in [5], Theorem 2 in [9], Theorem 2.4 in [12], Theorems 2.1-2.3 in [18], Theorem 3.6 in [24], Theorem 3.4 and Theorem 4.1 in [27], Theorems 3.3-3.4 in [32] and others.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Acknowledgements

The authors are grateful to the referees for their constructive comments and helpful suggestions. This research is supported by the National Science Foundation of China (11201410, 11271316 and 11101353), the Natural Science Foundation of Jiangsu Province (BK2012260) and the Natural Science Foundation of Jiangsu Education Committee (10KJB110012 and 11KJB110018).

Received: 16 March 2013 Accepted: 8 June 2013 Published: 1 July 2013

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doi:10.1186/1687-1812-2013-170

Cite this article as: Huang and Zhu: Weak and strong convergence theorems for a finite family of non-Lipschitzian nonself mappings in Banach spaces. *Fixed Point Theory and Applications* 2013 **2013**:170.