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Optimality and Duality Theorems in Nonsmooth Multiobjective Optimization

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Abstract

In this paper, we consider a class of nonsmooth multiobjective programming problems. Necessary and sufficient optimality conditions are obtained under higher order strongly convexity for Lipschitz functions. We formulate Mond-Weir type dual problem and establish weak and strong duality theorems for a strict minimizer of order m.

Keywords: Nonsmooth multiobjective programming, strict minimizers, optimality conditions, duality

1 Introduction

Nonlinear analysis is an important area in mathematical sciences, and has become a fundamental research tool in the field of contemporary mathematical analysis. Several nonlinear analysis problems arise from areas of optimization theory, game theory, differential equations, mathematical physics, convex analysis and nonlinear functional analysis. Park [1-3] has devoted to the study of nonlinear analysis and his results had a strong influence on the research topics of equilibrium complementarity and optimization problems. Nonsmooth phenomena in mathematics and optimization occurs naturally and frequently. Rockafellar [4] has pointed out that in many practical applications of applied mathematics the functions involved are not necessarily differentiable. Thus it is important to deal with non-differentiable mathematical programming problems.

The field of multiobjective programming, has grown remarkably in different directional in the setting of optimality conditions and duality theory since 1980s. In 1983, Vial [5] studied a class of functions depending on the sign of the constant ρ. Characteristic properties of this class of sets and related it to strong and weakly convex sets are provided.

Auslender [6] obtained necessary and sufficient conditions for a strict local minimizer of first and second order, supposing that the objective function f is locally Lipschitzian and that the feasible set S is closed. Studniarski [7] extended Auslender's results to any extended real-valued function f, any subset S and encompassing strict minimizers of order greater than 2. Necessary and sufficient conditions for strict minimizer of order m in nondifferentiable scalar programs are studied by Ward [8]. Based on this result, Jimenez [9] extended the notion of strict minimum of order m for real optimization problems to vector optimization. Jimenez and Novo [10,11] presented the first and second order sufficient conditions for strict local Pareto minima and strict local minima of first and second order to multiobjective and vector optimization problems. Subsequently, Bhatia [12] considered the notion of strict minimizer of order m for a multiobjective optimization problem and established only optimality for the concept of strict minimizer of order m under higher order strong convexity for Lipschitz functions.

In 2008, Kim and Bae [13] formulated nondifferentiable multiobjective programs involving the support functions of a compact convex sets. Also, Bae et al. [14] established duality theorems for nondifferentiable multiobjective programming problems under generalized convexity assumptions.

Very recently, Kim and Lee [15] introduce the nonsmooth multiobjective programming problems involving locally Lipschitz functions and support functions. They introduced Karush-Kuhn-Tucker optimality conditions with support functions and established duality theorems for (weak) Pareto-optimal solutions.

In this paper, we consider the nonsmooth multiobjective programming involving the support function of a compact convex set. In section 2, we introduce the concept of a strict minimizer of order m and higher order strongly convexity for Lipschitz functions. Section 3, necessary and sufficient optimality theorems are established for a strict minimizer of order m by using necessary and sufficient optimality theorems under generalized strongly convexity assumptions. Section 4, we formulate Mond-Weir type dual problem and obtained weak and strong duality theorems for a strict minimizer of order m.

2 Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and let $\mathbb{R}^n_+$ be its nonnegative orthant.

Let $x, y \in \mathbb{R}^n$. The following notation will be used for vectors in $\mathbb{R}^n$:

$$x < y \Leftrightarrow x_i < y_i, i = 1, 2, \cdots, n;$$
$$x \leqq y \Leftrightarrow x_i \leqq y_i, i = 1, 2, \cdots, n;$$
$$x \leq y \Leftrightarrow x_i \leqq y_i, i = 1, 2, \cdots, n \text{ but } x \neq y;$$
$$x \nleq y \text{ is the negation of } x \leq y;$$
$$x \nleq y \text{ is the negation of } x \leq y.$$

For $x, u \in \mathbb{R}$, $x \leqq u$ and $x < u$ have the usual meaning.

Definition 2.1 [16]*Let D be a compact convex set in* $\mathbb{R}^n$. *The support function* $s(\cdot|D)$ *is defined by*

$$s(x|D) := max\{x^T y : y \in D\}.$$

The support function $s(\cdot|D)$ *has a subdifferential. The subdifferential of* $s(\cdot|D)$ *at x is given by*

$$\partial s(x|D) := \{z \in D : z^T x = s(x|D)\}.$$

The support function $s(\cdot|D)$, *being convex and everywhere finite, that is, there exists z* $\in D$ *such that*

$$s(y|D) \geq s(x|D) + z^T(y - x) \text{ for all } y \in D.$$

Equivalently,

$$z^T x = s(x|D)$$

We consider the following multiobjective programming problem,

$$\begin{aligned}&\text{(MOP) Minimize } (f_1(x) + s(x|D_1), \dots, f_p(x) + s(x|D_p))\\&\qquad\qquad \text{subject to } g(x) \leqq 0,\end{aligned}$$

where f and g are locally Lipschitz functions from $\mathbb{R}^n \to \mathbb{R}^P$ and $\mathbb{R}^n \to \mathbb{R}^q$, respectively. D_i, for each $i \in P = \{1, 2, \dots, p\}$, is a compact convex set of $\mathbb{R}^n$. Further let, $S := \{x \in X | g_j(x) \leqq 0, j = 1, \dots, q\}$ be the feasible set of (MOP) and $B(x^0, \varepsilon) = \{x \in \mathbb{R}^n | \, ||x - x^0|| < \varepsilon\}$ denote an open ball with center x^0 and radius ε. Set $I(x^0) := \{j | g_j(x^0) = 0, j = 1, \dots, q\}$.

We introduce the following definitions due to Jimenez [9].

Definition 2.2 *A point $x^0 \in S$ is called a strict local minimizer for (MOP) if there exists an $\varepsilon > 0$, $i \in \{1, 2, \dots, p\}$ such that*

$$f_i(x) + s(x|D_i) \not< f_i(x^0) + s(x^0|D_i) \quad \textit{for all} \quad x \in B(x^0, \varepsilon) \cap S.$$

Definition 2.3 *Let $m \geqq 1$ be an integer. A point $x^0 \in S$ is called a strict local minimizer of order m for (MOP) if there exists an $\varepsilon > 0$ and a constant $c \in int\mathbb{R}^p_+$, $i \in \{1, 2, \cdots, p\}$such that*

$$f_i(x) + s(x|D_i) \not< f_i(x^0) + s(x^0|D_i) + c_i||x - x^0||^m \textit{ for all } x \in B(x^0, \varepsilon) \cap S.$$

Definition 2.4 *Let $m \geqq 1$ be an integer. A point $x^0 \in S$ is called a strict minimizer of order m for (MOP) if there exists a constant $c \in int\mathbb{R}^p_+$, $i \in \{1, 2, \cdots, p\}$such that*

$$f_i(x) + s(x|D_i) \not< f_i(x^0) + s(x^0|D_i) + c_i||x - x^0||^m \textit{ for all } x \in S.$$

Definition 2.5 [16]*Suppose that h: $X \to \mathbb{R}$ is Lipschitz on X. The Clarke's generalized directional derivative of h at $x \in X$ in the direction $v \in \mathbb{R}^n$, denoted by $h^0(x, v)$, is defined as*

$$h^0(x, v) = limsup_{y \to x\, t \downarrow 0} \frac{h(y + tv) - h(y)}{t}.$$

Definition 2.6 [16]*The Clarke's generalized gradient of h at $x \in X$, denoted by $\partial h(x)$ is defined as*

$$\partial h(x) = \{\xi \in \mathbb{R}^n : h^0(x, v) \geq \langle \xi, v \rangle \textit{ for all } v \in \mathbb{R}^n\}.$$

We recall the notion of strong convexity of order m introduced by Lin and Fukushima in [17].

Definition 2.7 *A function h: $X \to \mathbb{R}$ said to be strongly convex of order m if there exists a constant $c > 0$ such that for $x_1, x_2 \in X$ and $t \in [0, 1]$*

$$h(tx_1 + (1 - t)x_2) \leqq th(x_1) + (1 - t)h(x_2) - ct(1 - t)||x_1 - x_2||^m.$$

For $m = 2$, the function h is refered to as strongly convex in [5].

Proposition 2.1 [17]*If each h_i, $i = 1, \dots, p$ is strongly convex of order m on a convex set X, then $\sum_{i=1}^{p} t_i h_i$ and $\max_{1 \leq i \leq p} h_i$ are also strongly convex of order m on X, where $t_i \geq 0$, $i = 1, \dots, p$.*

Theorem 2.1 *Let X and S be nonempty convex subsets of $\mathbb{R}^n$ and X, respectively. Suppose that $x^0 \in S$ is a strict local minimizer of order m for (MOP) and the functions f_i: $X \to \mathbb{R}$, $i = 1, \dots, p$, are strongly convex of order m on X. Then x^0 is a strict minimizer of order m for (MOP).*

Proof. Since $x^0 \in S$ is a strict local minimizer of order m for (MOP). Therefore there exists an $\varepsilon > 0$ and a constant $c_i > 0$, $i = 1, \dots, p$ such that

$f_i(x) + s(x|D_i) \not< f_i(x^0) + s(x^0|D_i) + c_i||x - x^0||^m$ for all $x \in B(x^0, \varepsilon) \cap S$, that is, there exits no $x \in B(x^0, \varepsilon) \cap S$ such that

$$f_i(x) + s(x|D_i) < f_i(x^0) + s(x^0|D_i) + c_i||x - x^0||^m, \; i = 1, \cdots, p.$$

If x^0 is not a strict minimizer of order m for (MOP) then there exists some $z \in S$ such that

$$f_i(z) + s(z|D_i) < f_i(x^0) + s(x^0|D_i) + c_i||x - x^0||^m, \; i = 1, \cdots, p. \tag{2.1}$$

Since S is convex, $\lambda z + (1 - \lambda)x^0 \in B(x^0, \varepsilon) \cap S$, for sufficiently small $\lambda \in (0, 1)$. As f_i, $i = 1, \dots, p$, are strongly convex of order m on X, we have for $z, x^0 \in S$,

$$\begin{aligned} f_i(\lambda z + (1-\lambda)x^0) &\leqq \lambda f_i(z) + (1-\lambda) f_i(x^0) - c_i \lambda (1-\lambda) \|z - x^0\|^m \\ f_i(\lambda z + (1-\lambda)x^0) - f_i(x^0) &\leqq \lambda [f_i(z) - f_i(x^0)] - c_i \lambda (1-\lambda) \|z - x^0\|^m \\ &< \lambda[-s(z|D_i) + s(x^0|D_i) + c_i \|z - x^0\|^m] \\ &\qquad -c_i \lambda (1-\lambda) \|z - x^0\|^m, \text{ using (2.1)}, \\ &= -\lambda s(z|D_i) + \lambda s(x^0|D_i) + \lambda^2 c_i \|z - x^0\|^m \\ &< -\lambda s(z|D_i) + \lambda s(x^0|D_i) + c_i \|z - x^0\|^m \end{aligned}$$

$$f_i(\lambda z + (1-\lambda)x^0) + \lambda s(z|D_i) < f_i(x^0) + \lambda s(x^0|D_i) - s(x^0|D_i) + s(x^0|D_i) + c_i||z - x^0||^m$$

or

$$f_i(\lambda z + (1-\lambda)x^0) + \lambda s(z|D_i) + (1-\lambda) s(x^0|D_i) < f_i(x^0) + s(x^0|D_i) + c_i||z - x^0||^m$$

,

Since $s(\lambda z + (1-\lambda)x^0|D_i) \leqq \lambda s(z|D_i) + (1-\lambda)s(x^0|D_i), \; i = 1, \cdots, p$, we have

$$f_i(\lambda z + (1-\lambda)x^0) + s(\lambda z + (1-\lambda)x^0|D_i) < f_i(x^0) + s(x^0|D_i) + c_i||z - x^0||^m.$$

,

which implies that x^0 is not a strict local minimizer of order m, a contradiction. Hence, x^0 is a strict minimizer of order m for (MOP). □

Motivated by the above result, we give two obvious generalizations of strong convexity of order m which will be used to derive the optimality conditions for a strict minimizer of order m.

Definition 2.8 *The function h is said to be strongly pseudoconvex of order m and Lipschitz on X, if there exists a constant $c > 0$ such that for $x_1, x_2, \in X$*

$$\langle \xi, x_1 - x_2 \rangle + c||x_1 - x_2||^m \geqq 0 \textit{ for all } \xi \in \partial h(x_2) \textit{ implies } h(x_1) \geqq h(x_2).$$

Definition 2.9 *The function h is said to be strongly quasiconvex of order m and Lipschitz on X, if there exists a constant c > 0 such that for* $x_1, x_2, \in X$

$$h(x_1) \leqq h(x_2) \text{ implies } \langle \xi, x_1 - x_2 \rangle + c||x_1 - x_2||^m \leqq 0 \text{ for all } \xi \in \partial h(x_2).$$

We obtain the following lemma due to the theorem 4.1 of Chankong and Haimes [18].

Lemma 2.1 x^0 *is an efficient point for (MOP) if and only if* x^0 *solves*

$$\begin{aligned} (MOP_k(x_0)) \qquad \text{Minimize} \quad & f_k(x) + s(x|D_k) \\ \text{subject to} \quad & f_i(x) + s(x|D_i) \\ & \leqq f_i(x^0) + s(x^0|D_i), \text{ for all } i \neq k, \\ & g_j(x) \leqq 0, \ j = 1, \cdots, q \end{aligned}$$

for every $k = 1, \dots, p$.

We introduce the following definition for (MOP) based on the idea of Chandra et al. [19].

Definition 2.10 *Let* x^0 *be a feasible solution for (MOP). We say that the basic regularity condition (BRC) is satisfied at* x^0 *if there exists* $r \in \{1, 2, \dots, p\}$ *such that the only scalars* $\lambda_i^0 \geqq 0$, $w_i \in D_i$, $i = 1, \dots, p$, $i \neq r$, $\mu_j^0 \geqq 0$, $j \in I(x^0)$, $\mu_j^0 = 0$, $j \notin I(x^0)$; $I(x^0) = \{j | g_j(x^0) = 0, j = 1, \dots, q\}$ *which satisfy*

$$0 \in \sum_{i=1, i \neq r}^{p} \lambda_i^0 (\partial f_i(x^0) + w_i) + \sum_{j=1}^{q} \mu_j^0 \partial g_j(x_0)$$

are $\lambda_i^0 = 0$ *for all* $i = 1, \dots, p$, $i \neq r$, $\mu_j^0 = 0$, $j = 1, \dots, q$.

3 Optimality Conditions

In this section, we establish Fritz John and Karush-Kuhn-Tucker necessary conditions and Karush-Kuhn-Tucker sufficient condition for a strict minimizer of (MOP).

Theorem 3.1 (Fritz John Necessary Optimality Conditions) *Suppose that* x^0 *is a strict minimizer of order m for (MOP) and the functions* f_i, $i = 1, \dots, p$, g_j, $j = 1, \dots, q$, *are Lipschitz at* x^0. Then there exist $\lambda^0 \in \mathbb{R}_+^p$, $w_i^0 \in D_i$, $i = 1, \dots, p$, $\mu^0 \in \mathbb{R}_+^q$ *such that*

$$\begin{aligned} & 0 \in \sum_{i=1}^{p} \lambda_i^0 (\partial f_i(x^0) + w_i^0) + \sum_{j=1}^{q} \mu_j^0 \partial g_j(x^0), \\ & \langle w_i^0, x^0 \rangle = s(x^0|D_i), \ i = 1, \cdots, p, \\ & \mu_j^0 g_j(x^0) = 0, \ j = 1, \cdots, q, \\ & (\lambda_1^0, \cdots, \lambda_p^0, \mu_1^0, \cdots, \mu_q^0) \neq (0, \cdots, 0). \end{aligned}$$

Proof. Since x^0 is strict minimizer of order m for (MOP), it is strict minimizer. It can be seen that x^0 solves the following unconstrained scalar problem

$$\text{minimize } F(x)$$

where

$$\begin{aligned} F(x) = max \{ & (f_1(x) + s(x|D_1)) - (f_1(x^0) + s(x^0|D_1)), \cdots, \\ & (f_p(x) + s(x|D_p)) - (f_p(x^0) + s(x^0|D_p)), g_1(x), \cdots, g_q(x) \}. \end{aligned}$$

If it is not so then there exits $x^1 \in \mathbb{R}^n$ such that $F(x^1) < F(x^0)$. Since x^0 is strict minimizer of (MOP) then $g(x^0) \leqq 0$, for all $j = 1, \ldots, q$. Thus $F(x^0) = 0$ and hence $F(x^1) < 0$. This implies that x^1 is a feasible solution of (MOP) and contradicts the fact that x^0 is a strict minimizer of (MOP).

Since x^0 minimizes $F(x)$ it follows from Proposition 2.3.2 in Clarke[16] that $0 \in \partial F(x^0)$. Using Proposition 2.3.12 of [16], it follows that

$$\partial F(x^0) \subseteq co\{(\cup_{i=1}^{p}[\partial f_i(x^0) + \partial s(x^0 — D_i)]) \cup (\cup_{j=1}^{q}\partial g_j(x^0))\}.$$

Thus,

$$0 \in co\{(\cup_{i=1}^{p}[\partial f_i(x^0) + \partial s(x^0 — D_i)]) \cup (\cup_{j=1}^{q}\partial g_j(x^0))\}.$$

Hence there exist $\lambda_i^0 \geqq 0$, $w_i^0 \in D_i, i = 1, \cdots, p$, and $\mu_j^0 \geqq 0, j = 1, \cdots, q$, such that

$$\begin{aligned} 0 &\in \sum_{i=1}^{p}\lambda_i^0(\partial f_i(x^0) + w_i^0) + \sum_{j=1}^{q}\mu_j^0\partial g_j(x^0), \\ \langle w_i^0, x^0\rangle &= s(x^0 — D_i),\ i = 1, \cdots, p, \\ \mu_j^0 g_j(x^0) &= 0,\ j = 1, \cdots, q, \\ (\lambda_1^0, \cdots, \lambda_p^0, \mu_1^0, \cdots, \mu_q^0) &\neq (0, \cdots, 0). \end{aligned}$$

Theorem 3.2 (Karush-Kuhn-Tucker Necessary Optimality Conditions) *Suppose that x^0 is a strict minimizer of order m for (MOP) and the functions f_i, $i = 1, \ldots, p$, g_j, $j = 1, \ldots, q$, are Lipschitz at x^0. Assume that the basic regularity condition (BRC) holds at x^0, then there exist $\lambda^0 \in \mathbb{R}_+^p$, $w_i^0 \in D_i$, $i = 1, \ldots p$, $\mu^0 \in \mathbb{R}_+^q$such that*

$$0 \in \sum_{i=1}^{p}\lambda_i^0\partial f_i(x^0) + \sum_{i=1}^{p}\lambda_i^0 w_i^0 + \sum_{j=1}^{q}\mu_j^0\partial g_j(x^0), \tag{3.1}$$

$$\langle w_i^0, x^0\rangle = s(x^0 — D_i),\ i = 1, \cdots, p, \tag{3.2}$$

$$\mu_j^0 g_j(x^0) = 0,\ j = 1, \cdots, q, \tag{3.3}$$

$$(\lambda_1^0, \cdots, \lambda_p^0) \neq (0, \cdots, 0). \tag{3.4}$$

Proof. Since x^0 is a strict minimizer of order m for (MOP), by Theorem 3.1, there exist $\lambda^0 \in \mathbb{R}_+^p$, $w_i^0 \in D_i$, $i = 1, \ldots, p$ $\mu^0 \in \mathbb{R}_+^q$ such that

$$\begin{aligned} 0 &\in \sum_{i=1}^{p}\lambda_i^0(\partial f_i(x^0) + w_i^0) + \sum_{j=1}^{q}\mu_j^0\partial g_j(x^0), \\ \langle w_i^0, x^0\rangle &= s(x^0 — D_i),\ i = 1, \cdots, p, \\ \mu_j^0 g_j(x^0) &= 0,\ j = 1, \cdots, q, \\ (\lambda_1^0, \cdots, \lambda_p^0, \mu_1^0, \cdots, \mu_q^0) &\neq (0, \cdots, 0). \end{aligned}$$

Since BRC Condition holds at x^0. Then $(\lambda_1^0, \cdots, \lambda_p^0) \neq (0, \cdots, 0)$. If $\lambda_i^0 = 0$, $i = 1, \ldots, p$, then we have

$$0 \in \sum_{k\in P, k\neq i}\lambda_k(\partial f_k(x^0) + w_k) + \sum_{j\in I(x^0)}\mu_j\partial g_j(x^0),$$

for each $k \in P = \{1, \dots, p\}$. Since the assumptions of Basic Regularity Condition, we have $\lambda_k = 0$, $k \in P$, $k \neq i$, $\mu_j = 0$, $j \in I(x^0)$. This contradicts to the fact that λ_i, λ_k, $k \in P$, $k \neq i$, μ_j, $j \in I(x^0)$ are not all simultaneously zero. Hence $(\lambda_1, \dots, \lambda_p) \neq (0, \dots, 0)$.

Theorem 3.3 (Karush-Kuhn-Tucker Sufficient Optimality Conditions) *Let the Karush-Kuhn-Tucker Necessary Optimality Conditions be satisfied at $x^0 \in S$. Suppose that $f_i(\cdot) + (\cdot)^T w_i$, $i = 1, \cdots, p$, are strongly convex of order m on X, $g_j(\cdot)$, $j \in I(x^0)$ are strongly quasiconvex of order m on X. Then x^0 is a strict minimizer of order m for (MOP).*

Proof. As $f_i(\cdot) + (\cdot)^T w_i$, $i = 1, \dots, p$, are strongly convex of order m on X therefore there exist constants $c_i > 0$, $i = 1, \dots, p$, such that for all $x \in S$, $\xi_i \in \partial f_i(x^0)$ and $w_i \in D_i$, $i = 1, \dots, p$,

$$(f_i(x) + x^T w_i) - (f_i(x^0) + (x^0)^T w_i) \geqq \langle \xi_i + w_i, x - x^0 \rangle + c_i \|x - x^0\|^m. \tag{3.5}$$

For $\lambda_i^0 \geqq 0$, $i = 1, \dots, p$, we obtain

$$\begin{aligned} &\sum_{i=1}^{p} \lambda_i^0 (f_i(x) + x^T w_i) - \sum_{i=1}^{p} \lambda_i^0 (f_i(x^0) + (x^0)^T w_i) \\ &\geqq \sum_{i=1}^{p} \lambda_i^0 \langle \xi_i + w_i, x - x^0 \rangle + \sum_{i=1}^{p} \lambda_i^0 c_i \|x - x^0\|^m. \end{aligned} \tag{3.6}$$

Now for $x \in S$,

$$g_j(x) \leqq g_j(x^0), \ j \in I(x^0).$$

As $g_j(\cdot)$, $j \in I(x^0)$, are strongly quasiconvex of order m on X, it follows that there exist constants $c_j > 0$ and $\eta_j \in \partial g_j(x^0)$, $j \in I(x^0)$, such that

$$\langle \eta_j, x - x^0 \rangle + c_j \|x - x^0\|^m \leqq 0.$$

For $\mu_j^0 \geqq 0, j \in I(x^0)$, we obtain

$$\langle \sum_{j \in I(x^0)} \mu_j^0 \eta_j, x - x^0 \rangle + \sum_{j \in I(x^0)} \mu_j^0 c_j \|x - x^0\|^m \leqq 0.$$

As $\mu_j^0 = 0$ for $j \notin I(x^0)$, we have

$$\langle \sum_{j=1}^{m} \mu_j^0 \eta_j, x - x^0 \rangle + \sum_{j \in I(x^0)} \mu_j^0 c_j \|x - x^0\|^m \leqq 0. \tag{3.7}$$

By (3.6), (3.7) and (3.1), we get

$$\sum_{i=1}^{p} \lambda_i^0 (f_i(x) + x^T w_i) - \sum_{i=1}^{p} \lambda_i^0 (f_i(x^0) + (x^0)^T w_i) \geqq a \|x - x^0\|^m,$$

where $a = \sum_{i=1}^{p} \lambda_i^0 c_i + \sum_{j \in I(x^0)} \mu_j^0 c_j$. This implies that

$$\sum_{i=1}^{p} \lambda_i^0 [(f_i(x) + x^T w_i) - (f_i(x^0) + (x^0)^T w_i) - c_i ||x - x^0||^m] \geqq 0, \tag{3.8}$$

where $c = ae$. It follows from (3.8) that there exist $c \in int\mathbb{R}^p_+$ such that for all $x \in S$

$$f_i(x) + x^T w_i \geqq f_i(x^0) + (x^0)^T w_i + c_i ||x - x^0||^m, \ i = 1, \cdots, p.$$

Since $(x^0)^T w_i = s(x^0|D_i)$, $x^T w_i \leqq s(x|D_i)$, $i = 1, \ldots, p$, we have

$$f_i(x) + s(x|D_i) \geqq f_i(x^0) + s(x^0|D_i) + c_i ||x - x^0||^m,$$

i.e.

$$f_i(x) + s(x|D_i) \not< f_i(x^0) + s(x^0|D_i) + c_i ||x - x^0||^m.$$

Thereby implying that x^0 is a strict minimizer of order m for (MOP). □

Remark 3.1 *If* $D_i = \{0\}$, $i = 1, \ldots, k$, *then our results on optimality reduces to the one of Bhatia* [12].

4 Duality Theorems

In this section, we formulate Mond-Weir type dual problem and establish duality theorems for a minima. Now we propose the following Mond-Weir type dual (MOD) to (MOP):

$$\text{(MOD)} \quad \text{Maximize} \quad (f_1(u) + u^T w_1, \cdots, f_p(u) + u^T w_p)$$
$$\text{subject to} \quad 0 \in \sum_{i=1}^{p} \lambda_i (\partial f_i(u) + w_i) + \sum_{j=1}^{q} \mu_j \partial g_j(u), \tag{4.1}$$

$$\begin{aligned} &\sum_{j=1}^{q} \mu_j g_j(u) \geqq 0, \ j = 1, \cdots, q, \\ &\mu \geq 0, \ w_i \in D_i, \ i = 1, \cdots, p, \\ &\lambda = (\lambda_1, \cdots, \lambda_p) \in \Lambda^+, \ u \in X, \end{aligned} \tag{4.2}$$

where $\Lambda^+ = \{\lambda \in \mathbb{R}^p : \lambda \geqq 0, \lambda^T e = 1, e = \{1, \ldots, 1\} \in \mathbb{R}^p\}$.

Theorem 4.1 (Weak Duality) *Let x and* (u, w, λ, μ) *be feasible solution of (MOP) and (MOD), respectively. Assume that* $f_i(\cdot) + (\cdot)^T w_i$, $i = 1, \ldots, p$, *are strongly convex of order m on X,* $g_j(\cdot)$, $j \in I(u)$; $I(u) = \{j | g_j(u) = 0\}$ *are strongly quasiconvex of order m on X. Then the following cannot hold:*

$$f(x) + s(x|D) < f(u) + u^T w. \tag{4.3}$$

Proof. Since x is feasible solution for (MOP) and (u, w, λ, μ) is feasible for (MOD), we have

$$g_j(x) \leqq g_j(u), \ j \in I(u).$$

For every $j \in I(u)$, as g_j, $j \in I(u)$, are strongly quasiconvex of order m on X, it follows that there exist constants $c_j > 0$ and $\eta_j \in \partial g_j(u)$, $j \in I(u)$ such that

$$\langle \eta_j, x - u \rangle + c_j ||x - u||^m \leqq 0.$$

This together with $\mu_j \geqq 0$, $j \in I(u)$, imply

$$\langle \sum_{j \in I(u)} \mu_j \eta_j, x - u \rangle + \sum_{j \in I(u)} \mu_j c_j \leqq 0.$$

As $\mu_j = 0$, for $j \notin I(u)$, we have

$$\langle \sum_{j=1}^{m} \mu_j \eta_j, x - u \rangle + \sum_{j \in I(u)} \mu_j c_j ||x - u||^m \leqq 0. \tag{4.4}$$

Now, suppose contrary to the result that (4.3) holds. Since $x^T w_i \leqq s(x|D)$, $i = 1, \dots, p$, we obtain

$$f_i(x) + x^T w_i < f_i(u) + u^T w_i, \; i = 1, \cdots, p.$$

As $f_i(\cdot) + (\cdot)^T w_i$, $i = 1, \dots, p$, are strongly convex of order m on X, therefore there exist constants $c_i > 0$, $i = 1, \dots, p$, such that for all $x \in S$, $\xi_i \in \partial f_i(u)$, $i = 1, \dots, p$,

$$(f_i(x) + x^T w_i) - (f_i(u) + u^T w_i) \geqq \langle \xi_i + w_i, x - u \rangle + c_i ||x - u||^m. \tag{4.5}$$

For $\lambda_i \geqq 0$, $i = 1, \dots, p$, (4.5) yields

$$\begin{aligned} &\sum_{i=1}^{p} \lambda_i (f_i(x) + x^T w_i) - \sum_{i=1}^{p} \lambda_i (f_i(u) + u^T w_i) \\ &\quad \geqq \langle \sum_{i=1}^{p} \lambda_i (\xi_i + w_i), x - u \rangle + \sum_{i=1}^{p} \lambda_i c_i ||x - u||^m. \end{aligned} \tag{4.6}$$

By (4.4),(4.6) and (4.1), we get

$$\sum_{i=1}^{p} \lambda_i (f_i(x) + x^T w_i) - \sum_{i=1}^{p} \lambda_i (f_i(u) + u^T w_i) \geqq a ||x - u||^m, \tag{4.7}$$

where $a = \sum_{i=1}^{p} \lambda_i c_i + \sum_{j \in I(u)} \mu_j c_j$. This implies that

$$\sum_{i=1}^{p} \lambda_i [(f_i(x) + x^T w_i) - (f_i(u) + u^T w_i) - c_i ||x - u||^m] \geqq 0, \tag{4.8}$$

where $c = ae$, since $\lambda^T e = 1$. It follows from (4.8) that there exist $c \in int\ \mathbb{R}^p$ such that for all $x \in S$

$$f_i(x) + x^T w_i \geqq f_i(u) + u^T w_i + c_i ||x - u|^m, \; i = 1, \cdots, p.$$

Since $x^T w_i \leqq s(x|D_i)$, $i = 1, \dots, p$, and $c \in int\ \mathbb{R}^p$, we have

$$\begin{aligned} f_i(x) + s(x|D_i) &\geqq f_i(x) + x^T w_i \\ &\geqq f_i(u) + u^T w_i + c_i ||x - u||^m \\ &> f_i(u) + u^T w_i, \; i = 1, \cdots, p. \end{aligned}$$

which contradicts to the fact that (4.3)holds. □

Theorem 4.2 (Strong Duality) *If x^0 is a strictly minimizer of order m for (MOP), and assume that the basic regularity condition (BRC) holds at x^0, then there exists $\lambda^0 \in \mathbb{R}^p$, $w_i^0 \in D_i$, $i = 1, \dots, p$, $\mu^0 \in \mathbb{R}^q$ such that $(x^0, w^0, \lambda^0, \mu^0)$ is feasible solution for (MOD) and $(x^0)^T w_i^0 = s(x^0|D_i)$, $i = 1, \cdots, p$. Moreover, if the assumptions of weak duality are satisfied, then $(x^0, w^0, \lambda^0, \mu^0)$ is a strictly minimizer of order m for (MOD).*

Proof. By Theorem 3.2, there exists $\lambda^0 \in \mathbb{R}^p$, $w_i^0 \in D_i$, $i = 1, \dots, p$, and $\mu^0 \in \mathbb{R}^q$ such that

$$0 \in \sum_{i=1}^{p} \lambda_i^0 (\partial f_i(x^0) + w_i^0) + \sum_{j=1}^{q} \mu_j^0 \partial g_j(x^0),$$
$$\langle w_i^0, x^0 \rangle = s(x^0|D_i),\ i = 1, \cdots, p,$$
$$\mu_j^0 g_j(x^0) = 0,\ j = 1, \cdots, q,$$
$$(\lambda_1^0, \cdots, \lambda_p^0) \neq (0, \cdots, 0).$$

Thus $(x^0, w^0, \lambda^0, \mu^0)$ is a feasible for (MOD) and $(x^0)^T w_i^0 = s(x^0|D_i)$, $i = 1, \dots, p$. By Theorem 4.1, we obtain that the following cannot hold: □

$$f_i(x^0) + (x^0)^T w_i^0 = f_i(x^0) + s(x^0|D_i) < f_i(u) + u^T w_i,\ i = 1, \cdots, p,$$

where (u, w, λ, μ) is any feasible solution of (MOD). Since $c_i \in int\ \mathbb{R}^p$ such that for all x^0, $u \in S$

$$f_i(x^0) + (x^0)^T w_i^0 + c_i||u - x^0||^m \not< f_i(u) + u^T w_i,\ i = 1, \cdots, p.$$

Thus $(x^0, w^0, \lambda^0, \mu^0)$ is a strictly minimizer of order m for (MOD). Hence, the result holds.

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Authors' contributions
DSK presented necessary and sufficient optimality conditions, formulated Mond-Weir type dual problem and established weak and strong duality theorems for a strict minimizer of order m. KDB carried out the optimality and duality studies, participated in the sequence alignment and drafted the manuscript. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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