

**ON COEFFICIENT BOUNDS OF A CERTAIN CLASS  
 p-VALENT  $\lambda$ -SPIRAL FUNCTIONS OF ORDER  $\alpha$**

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ABSTRACT. Let  $S^\lambda(A, B, p, \alpha)$  ( $|\lambda| < \frac{\pi}{2}$ ,  $-1 \leq A < B \leq 1$  and  $0 \leq \alpha < p$ ), denote the class of functions  $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$  analytic in  $U = \{z: |z| < 1\}$ , which satisfy for  $z = re^{i\theta} \in U$

$$e^{i\lambda} \sec \lambda \frac{zf'(z)}{f(z)} - ip \tan \lambda = \frac{p + [pB + (A-B)(p-\alpha)]w(z)}{1 + Bw(z)},$$

$w(z)$  is analytic in  $U$  with  $w(0) = 0$  and  $|w(z)| \leq |z|$  for  $z \in U$ . In this paper we obtain the bounds of  $a_n$  and we maximize  $|a_{p+2} - \mu a_{p+1}^2|$  over the class  $S^\lambda(A, B, p, \alpha)$  for complex values of  $\mu$ .

KEY WORDS AND PHRASES. *p*-Valent, analytic, bounds,  $\lambda$ -spirallike functions of order  $\alpha$ .  
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1. INTRODUCTION.

Let  $A_p$  ( $p$  a fixed integer greater than zero) denote the class of functions  $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$  which are analytic in  $U = \{z: |z| < 1\}$ . We use  $\Omega$  to denote the class of bounded analytic functions  $w(z)$  in  $U$  satisfies the conditions  $w(0) = 0$  and  $|w(z)| \leq |z|$  for  $z \in U$ . Also let  $P(p, \alpha)$  (with  $p$  a positive integer) denote the class of functions with positive real part of order  $\alpha$  that have the form

$$P(z) = p + \sum_{k=1}^{\infty} c_k z^k \tag{1.1}$$

which are analytic in  $U$  and satisfy the conditions  $P(0) = p$  and  $\text{Re}\{P(z)\} > \alpha$  ( $0 \leq \alpha < p$ ) in  $U$ . The class  $P(p, \alpha)$  was introduced by Patil and Thakare [1]. It was shown in [1] that the function  $P \in P(p, \alpha)$  if and only if

$$P(z) = \frac{p-(p-2\alpha)w(z)}{1+w(z)}, \quad w \in \Omega. \tag{1.2}$$

For  $|\lambda| < \frac{\pi}{2}$  and  $p$  a fixed integer greater than zero, let  $S^\lambda(p, \alpha)$  denote the class of functions  $f(z) \in A_p$  which satisfy

$$\operatorname{Re} \left\{ e^{i\lambda} \frac{zf'(z)}{f(z)} \right\} > \alpha \cos \lambda \tag{1.3}$$

for  $z \in U$  and  $0 \leq \alpha < p$ . We say the functions in  $S^\lambda(p, \alpha)$  are  $p$ -valent  $\lambda$ -spiral-like of order  $\alpha$ . The class  $S^\lambda(p, \alpha)$  was introduced by Patil and Thakare [1]. It was shown in [1] that  $f \in S^\lambda(p, \alpha)$  if and only if there exists a function  $P \in P(p, \alpha)$  such that

$$e^{i\lambda} \frac{zf'(z)}{f(z)} = \cos \lambda \cdot P(z) + ip \sin \lambda. \tag{1.4}$$

Let  $P(A, B) (-1 \leq A < B \leq 1)$  denote the class of functions  $P_1(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$  analytic in  $U$  and such that  $P_1(z) \in P(A, B)$  if and only if

$$P_1(z) = \frac{1+Aw(z)}{1+Bw(z)}, \quad w \in \Omega, \quad z \in U. \tag{1.5}$$

The class  $P(A, B)$  was introduced by Janowski [2].

For  $-1 \leq A < B \leq 1$  and  $0 \leq \alpha < p$ , denote by  $P(A, B, p, \alpha)$  the class of functions  $P_2(z)$  of form (1.1) which satisfy that  $P_2(z) \in P(A, B, p, \alpha)$  if and only if

$$P_2(z) = (p-\alpha)P_1(z) + \alpha, \quad P_1(z) \in P(A, B) \tag{1.6}$$

Using (1.5) in (1.6), one can show that  $P_2(z) \in P(A, B, p, \alpha)$  if and only if

$$P_2(z) = \frac{p + [pB+(A-B)(p-\alpha)]w(z)}{1 + Bw(z)}, \quad w \in \Omega. \tag{1.7}$$

Also let  $S^\lambda(A, B, p, \alpha) (|\lambda| < \frac{\pi}{2}, -1 \leq A < B \leq 1 \text{ and } 0 \leq \alpha < p)$  denote the class of functions  $f(z) \in A_p$  which satisfy

$$e^{i\lambda} \frac{zf'(z)}{f(z)} = \cos \lambda P_2(z) + ip \sin \lambda, \quad P_2(z) \in P(A, B, p, \alpha). \tag{1.8}$$

Using (1.7) in (1.8) one can easily show that:

$f(z) \in S^\lambda(A, B, p, \alpha)$  if and only if

$$(i) \quad e^{i\lambda} \sec \lambda \frac{zf'(z)}{f(z)} - ip \tan \lambda = \frac{p + [pB+(A-B)(p-\alpha)]w(z)}{1 + Bw(z)}, \quad w \in \Omega. \tag{1.9}$$

$$(ii) \quad \frac{zf'(z)}{f(z)} = \frac{p + [pB+(A-B)(p-\alpha)\cos \lambda e^{-j\lambda}]w(z)}{1 + Bw(z)}, \quad w \in \Omega. \tag{1.10}$$

We shall need the following lemma in our investigation:

LEMMA 1 [3]. Let  $w(z) = \sum_{k=1}^{\infty} b_k z^k \in \Omega$ , if  $v$  is any complex number, then

$$|b_2 - v b_1^2| \leq \max \{1, |v|\}. \tag{1.11}$$

Equality is attained for  $w(z) = z^2$  and  $w(z) = z$ .

2. COEFFICIENT ESTIMATES FOR THE CLASS  $S^\lambda(A, B, p, \alpha)$ .

LEMMA 2. If integers  $p$  and  $m$  are greater than zero;  $0 \leq \alpha < p$ ,  $|\lambda| < \frac{\pi}{2}$  and  $-1 \leq A < B \leq 1$ , then

$$\prod_{j=0}^{m-1} \frac{|(B-A)(p-\alpha)\cos \lambda e^{-i\lambda} + B_j|^2}{(j+1)^2} =$$

$$\frac{\cos^2 \lambda}{m^2} \left\{ (B-A)^2 (p-\alpha)^2 + \sum_{k=1}^{m-1} [k^2 (B^2 - 1) \sec^2 \lambda] \right.$$

$$\left. + (B-A)^2 (p-\alpha)^2 + 2kB(B-A)(p-\alpha) \prod_{j=0}^{k-1} \frac{|(B-A)(p-\alpha)\cos \lambda e^{-i\lambda} + B_j|^2}{(j+1)^2} \right\}. \quad (2.1)$$

PROOF. We prove the lemma by induction on  $m$ . For  $m=1$ , the lemma is obvious. Next suppose that the result is true for  $m=q-1$ . We have

$$\frac{\cos^2 \lambda}{q^2} \left\{ (B-A)^2 (p-\alpha)^2 + \sum_{k=1}^{q-1} [k^2 (B^2 - 1) \sec^2 \lambda] \right.$$

$$\left. + (B-A)^2 (p-\alpha)^2 + 2kB(B-A)(p-\alpha) \prod_{j=0}^{k-1} \frac{|(B-A)(p-\alpha)\cos \lambda e^{-i\lambda} + B_j|^2}{(j+1)^2} \right\} =$$

$$\frac{\cos^2 \lambda}{q^2} \left\{ (B-A)^2 (p-\alpha)^2 + \sum_{k=1}^{q-2} [k^2 (B^2 - 1) \sec^2 \lambda] \right.$$

$$\left. + (B-A)^2 (p-\alpha)^2 + 2kB(B-A)(p-\alpha) \prod_{j=0}^{k-1} \frac{|(B-A)(p-\alpha)\cos \lambda e^{-i\lambda} + B_j|^2}{(j+1)^2} \right\} +$$

$$[(q-1)^2 (B^2-1)\sec^2 \lambda + (B-A)^2 (p-\alpha)^2 + 2(q-1)B(B-A)(p-\alpha)] \times$$

$$\prod_{j=0}^{q-2} \frac{|(B-A)(p-\alpha)\cos \lambda e^{-i\lambda} + B_j|^2}{(j+1)^2}$$

$$= \prod_{j=0}^{q-2} \frac{|(B-A)(p-\alpha)\cos \lambda e^{-i\lambda} + B_j|^2}{(j+1)^2} \times$$

$$\left\{ \frac{(q-1)^2 B^2 + (B-A)^2 (p-\alpha)^2 \cos^2 \lambda + 2(q-1) \cdot B(B-A)(p-\alpha) \cos^2 \lambda}{q^2} \right\}$$

$$= \prod_{j=0}^{q-1} \frac{|(B-A)(p-\alpha)\cos \lambda e^{-i\lambda} + B_j|^2}{(j+1)^2}.$$

Showing that the result is valid for  $m=q$ . This proves the lemma.

THEOREM 1. If  $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \in S^\lambda(A, B, p, \alpha)$ , then

$$|a_n| \leq \prod_{k=0}^{n-(p+1)} \frac{|(B-A)(p-\alpha)\cos \lambda e^{-i\lambda} + B_k|}{k+1} \quad (2.2)$$

for  $n \geq p+1$  and these bounds are sharp for all admissible  $A, B, \lambda$  and  $\alpha$  and for each  $n$ .

PROOF. As  $f \in S^\lambda(A, B, p, \alpha)$ , from (1.9), we have

$$e^{i\lambda} \sec \lambda \cdot \frac{zf'(z)}{f(z)} - ip \tan \lambda = \frac{p+[pB+(A-B)(p-\alpha)]w(z)}{1+Bw(z)}, \quad w \in \Omega.$$

This may be written as

$$\begin{aligned} & \{Be^{i\lambda} \sec \lambda sf'(z) + [-pB+(B-A)(p-\alpha)-ipB \tan \lambda]f(z)\} w(z) \\ & = (p+ip \tan \lambda)f(z) - e^{i\lambda} \sec \lambda zf'(z). \end{aligned}$$

Hence

$$\begin{aligned} & \left[ Be^{i\lambda} \sec \lambda \{pz^p + \sum_{k=1}^{\infty} (p+k)a_{p+k}z^{p+k}\} + \right. \\ & \left. [-pB+(B-A)(p-\alpha)-ipB \tan \lambda] \{z^p + \sum_{k=1}^{\infty} a_{p+k}z^{p+k}\} \right] w(z) \\ & = (p+ip \tan \lambda) \{z^p + \sum_{k=1}^{\infty} (p+k)a_{p+k}z^{p+k}\} - \\ & e^{i\lambda} \sec \lambda \{pz^p + \sum_{k=1}^{\infty} (p+k)a_{p+k}z^{p+k}\} \end{aligned}$$

or

$$\begin{aligned} & \left[ pBe^{i\lambda} \sec \lambda + [-pB+(B-A)(p-\alpha) - ip B \tan \lambda] + \right. \\ & \left. \sum_{k=1}^{\infty} \{(p+k)Be^{i\lambda} \sec \lambda + [-pB+(B-A)(p-\alpha) - ip B \tan \lambda]\} a_{p+k}z^k \right] w(z) \\ & = (p+ip \tan \lambda - p e^{i\lambda} \sec \lambda) + \sum_{k=1}^{\infty} \{p+ip \tan \lambda - (p+k)e^{i\lambda} \sec \lambda\} a_{p+k}z^k \end{aligned}$$

which may be written as

$$\begin{aligned} & \sum_{k=0}^{\infty} \left[ \{(p+k)Be^{i\lambda} \sec \lambda + [-pB+(B-A)(p-\alpha) - ip B \tan \lambda]\} a_{p+k}z^k \right] w(z) \\ & = \sum_{k=0}^{\infty} [p+ip \tan \lambda - (p+k)e^{i\lambda} \sec \lambda] a_{p+k}z^k \end{aligned} \quad (2.3)$$

where  $a_p=1$  and  $w(z) = \sum_{k=0}^{\infty} b_{k+1} z^{k+1}$ .

Equating coefficients of  $z^m$  on both sides of (2.3), we obtain

$$\begin{aligned} & \sum_{k=0}^{m-1} \{(p+k)Be^{i\lambda} \sec \lambda + [-pB + (B-A)(p-\alpha) - ip B \tan \lambda]\} a_{p+k} b_{m-k} \\ & = \{p+ip \tan \lambda - (p+m)e^{i\lambda} \sec \lambda\} a_{p+m}; \end{aligned}$$

which shows that  $a_{p+m}$  on right-hand side depends only on

$$a_p, a_{p+1}, \dots, a_{p+(m-1)}$$

of left-hand side. Hence we can write

$$\begin{aligned} & \sum_{k=0}^{m-1} \left[ \{(p+k)B e^{i\lambda} \sec \lambda + [-pB+(B-A)(p-\alpha) - ip B \tan \lambda]\} a_{p+k} z^k \right] w(z) = \\ & = \sum_{k=0}^m [p+ip \tan \lambda - (p+k)e^{i\lambda} \sec \lambda] a_{p+k} z^k + \sum_{k=m+1}^{\infty} A_k z^k \end{aligned}$$

for  $m=1,2,3,\dots$  and a proper choice of  $A_k (k \geq 0)$ .

Let  $z = re^{i\theta}$ ,  $0 < r < 1$ ,  $0 \leq \theta \leq 2\pi$ , then

$$\begin{aligned} & \sum_{k=0}^{m-1} |(p+k)B e^{i\lambda} \sec \lambda + [-pB + (B-A)(p-\alpha) - ip B \tan \lambda]|^2 |a_{p+k}|^2 r^{2k} \\ & = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^{m-1} \{(p+k)B e^{i\lambda} \sec \lambda + [-pB+(B-A)(p-\alpha) - ip B \tan \lambda]\} a_{p+k} r^k e^{i\theta k} \right|^2 d\theta \\ & \geq \frac{1}{2\pi} \cdot \int_0^{2\pi} \left| \sum_{k=0}^{m-1} \{(p+k)B e^{i\lambda} \sec \lambda + [-pB + (B-A)(p-\alpha) - ip B \tan \lambda]\} a_{p+k} r^k e^{i\theta k} \right|^2 |w(re^{i\theta})|^2 \cdot d\theta \\ & \geq \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^m \{p+ip \tan \lambda - (p+k)e^{i\lambda} \sec \lambda\} a_{p+k} r^k e^{i\theta k} + \sum_{k=m+1}^{\infty} A_k r^k e^{i\theta k} \right|^2 \cdot d\theta \\ & \geq \sum_{k=0}^m |p+ip \tan \lambda - (p+k)e^{i\lambda} \sec \lambda|^2 |a_{p+k}|^2 r^{2k} + \sum_{k=m+1}^{\infty} |A_k|^2 r^{2k} \\ & \geq \sum_{k=0}^m |p+ip \tan \lambda - (p+k)e^{i\lambda} \sec \lambda|^2 |a_{p+k}|^2 r^{2k}. \end{aligned} \tag{2.4}$$

Setting  $r \rightarrow 1$  in (2.4), the inequality (2.4) may be written as

$$\begin{aligned} & \sum_{k=0}^{m-1} \left\{ |(p+k)B e^{i\lambda} \sec \lambda + [-pB+(B-A)(p-\alpha) - ip B \tan \lambda]|^2 - \right. \\ & \left. - |p+ip \tan \lambda - (p+k)e^{i\lambda} \sec \lambda|^2 \right\} |a_{p+k}|^2 \\ & \geq |p + ip \tan \lambda - (p+m)e^{i\lambda} \sec \lambda|^2 |a_{p+m}|^2. \end{aligned} \tag{2.5}$$

Simplification of (2.5) leads to

$$|a_{p+m}|^2 \leq \frac{\cos^2 \lambda}{m^2} \cdot \sum_{k=0}^{m-1} \{k^2(B^2-1)\sec^2 \lambda + (B-A)(p-\alpha)[(B-A)(p-\alpha)+2kB]\} |a_{p+k}|^2 \tag{2.6}$$

Replacing  $p+m$  by  $n$  in (2.6), we are led to

$$|a_n|^2 \leq \frac{\cos^2 \lambda}{(n-p)^2} \cdot \sum_{k=0}^{n-(p+1)} \{k^2(B^2-1)\sec^2 \lambda + (B-A)(p-\alpha)[(B-A)(p-\alpha)+2kB]\} |a_{p+k}|^2, \tag{2.7}$$

where  $n \geq p+1$ .

For  $n=p+1$ , (2.7) reduces to

$$|a_{p+1}|^2 \leq (B-A)^2(p-\alpha)^2 \cos^2 \lambda$$

or  $|a_{p+1}|^2 \leq (B-A)(p-\alpha) \cos \lambda$  (2.8)

which is equivalent to (2.2).

To establish (2.2) for  $n > p+1$ , we will apply induction argument.

Fix  $n$ ,  $n \geq p+2$ , and suppose (2.2) holds for  $k = 1, 2, \dots, n-(p+1)$ . Then

$$|a_n|^2 \leq \frac{\cos^2 \lambda}{(n-p)^2} \{ (B-A)^2(p-\alpha)^2 +$$

$$\sum_{k=1}^{n-(p+1)} \{ k^2(B^2-1) \sec^2 \lambda + (B-A)(p-\alpha) [(B-A)(p-\alpha) + 2kB] \} \\ \times \prod_{j=0}^{k-1} \frac{|(B-A)(p-\alpha) \cos \lambda e^{-i\lambda} + B_j|^2}{(j+1)^2} \} .$$
 (2.9)

Thus from (2.7), (2.9) and lemma 2 with  $m=n-p$ , we obtain

$$|a_n|^2 \leq \prod_{j=0}^{n-(p+1)} \frac{|(B-A)(p-\alpha) \cos \lambda e^{-i\lambda} + B_j|^2}{(j+1)^2} .$$

This completes the proof of (2.2). This proof is based on a technique found in Clunie [4].

For sharpness of (2.2) consider

$$f(z) = \frac{z^p}{(1-B\delta z) \left( \frac{B-A}{B} \right)^{p-\alpha} \cos \lambda e^{-i\lambda}} , \quad |\delta| = 1, B \neq 0$$

Remarks on Theorem 1:

- (1) Setting  $B=1$  and  $A=-1$  in Theorem 1, we get the result of Patil and Thakare [1].
- (2) Setting  $B=1, A=-1$  and  $p=1$  in Theorem 1, we get the result of Libera [5].
- (3) Setting  $B=1, A=-1, p=1$  and  $\alpha=0$  in Theorem 1, we get the result of Zamorski [6].
- (4) Setting  $B=1, A=-1, p=1$  and  $\lambda=0$  in Theorem 1, we get the result of Robertson [7] and Schild [8].

THEOREM 2. If  $f(z)=z^p + \sum_{k=p+1}^{\infty} a_k z^k \in S^\lambda(A,B,p,\alpha)$  and  $\mu$  is any complex number, then

$$|a_{p+2}^{-\mu} a_{p+1}^2| \leq \frac{(B-A)(p-\alpha)}{2} \cos \lambda \max \{ 1, |(B-A)(p-\alpha)(2\mu-1) \cos \lambda - e^{i\lambda}| \}$$
 (2.10)

This inequality is sharp for each  $\mu$ .

PROOF. As  $f \in S^\lambda(A,B,p,\alpha)$ , from (1.9) we have

$$e^{i\lambda} \sec \lambda \frac{zf''(z)}{f(z)} - ip \tan \lambda = \frac{p+[pB+(A-B)(p-\alpha)]w(z)}{1+Bw(z)}$$
 (2.11)

where  $w(z) = \sum_{k=1}^{\infty} b_k z^k \in \Omega$ .

Rewriting the form (2.11) as

$$\begin{aligned} w(z) &= \frac{p-e^{i\lambda} \sec \lambda \frac{zf'(z)}{f(z)} + ip \tan \lambda}{Be^{i\lambda} \sec \lambda \frac{zf'(z)}{f(z)} + [-pB+(B-A)(p-\alpha)-ip \tan \lambda B]} \\ &= \frac{e^{i\lambda} \sec \lambda [pf(z)-zf'(z)]}{Be^{i\lambda} \sec \lambda \cdot (zf'(z)) + [-Bp e^{i\lambda} \sec \lambda + (B-A)(p-\alpha)]f(z)} \\ &= \frac{-e^{i\lambda} \sec \lambda \sum_{k=1}^{\infty} k a_{p+k} z^k}{(B-A)(p-\alpha) [1 + \sum_{k=1}^{\infty} a_{p+k} z^k] + Be^{i\lambda} \sec \lambda \sum_{k=1}^{\infty} k a_{p+k} z^k} \\ &= \frac{-e^{i\lambda} \sec \lambda \sum_{k=1}^{\infty} k a_{p+k} z^k}{(B-A)(p-\alpha) + \sum_{k=1}^{\infty} [ \{ (B-A)(p-\alpha) + \kappa Be^{i\lambda} \sec \lambda \} a_{p+k} ] z^k} \\ &= -e^{i\lambda} \sec \lambda \left[ \frac{a_{p+1}}{(B-A)(p-\alpha)} z + \frac{1}{(B-A)(p-\alpha)} \times \right. \\ &\quad \left. \times \{ 2 a_{p+2} - \left( \frac{(B-A)(p-\alpha) + Be^{i\lambda} \sec \lambda}{(B-A)(p-\alpha)} \right) a_{p+1}^2 \} z^2 + \dots \right] \end{aligned}$$

and then comparing coefficients of  $z$  and  $z^2$  on both sides, we have

$$\begin{aligned} b_1 &= - \frac{e^{i\lambda} \sec \lambda}{(B-A)(p-\alpha)} a_{p+1} \\ b_2 &= - \frac{e^{i\lambda} \sec \lambda}{(B-A)^2 (p-\alpha)^2} [2(B-A)(p-\alpha) a_{p+2} - \{ (B-A)(p-\alpha) + e^{i\lambda} \sec \lambda \} a_{p+1}^2]. \end{aligned}$$

Thus

$$a_{p+1} = - \frac{(B-A)(p-\alpha)}{e^{i\lambda} \sec \lambda} b_1$$

and

$$a_{p+2} = - \frac{(B-A)(p-\alpha)}{2 e^{i\lambda} \sec \lambda} b_2 + \frac{(B-A)(p-\alpha) + e^{i\lambda} \sec \lambda}{2(B-A)(p-\alpha)} a_{p+1}^2$$

Hence

$$\begin{aligned} a_{p+2} - \mu a_{p+1}^2 &= \\ &= - \frac{(B-A)(p-\alpha)}{2e^{i\lambda} \sec \lambda} b_2 + \left[ \frac{(B-A)(p-\alpha) + e^{i\lambda} \sec \lambda}{2(B-A)(p-\alpha)} - \mu \right] a_{p+1}^2 \\ &= - \frac{(B-A)(p-\alpha)}{2e^{i\lambda} \sec \lambda} b_2 + \left[ \frac{(B-A)(p-\alpha) + e^{i\lambda} \sec \lambda}{2(B-A)(p-\alpha)} - \mu \right] \frac{(B-A)^2 (p-\alpha)^2}{e^{2i\lambda} \sec^2 \lambda} b_1^2. \end{aligned} \tag{2.12}$$

Thus taking modulus of both sides of (2.12), we are led to

$$\begin{aligned}
 & |a_{p+2} - \mu a_{p+1}^2| = \\
 & \frac{(B-A)(p-\alpha)}{2} \cos \lambda \left| b_2 - \left\{ \frac{(B-A)(p-\alpha)e^{i\lambda} \sec \lambda}{2(B-A)(p-\alpha)} - \mu \right\} \frac{2(B-A)(p-\alpha)}{e^{i\lambda} \sec \lambda} b_1^2 \right| \\
 & = \frac{(B-A)(p-\alpha)}{2} \cos \lambda \left| b_2 - \left\{ \frac{e^{i\lambda} \sec \lambda - (B-A)(p-\alpha)(2\mu-1)}{e^{i\lambda} \sec \lambda} \right\} b_1^2 \right|. \quad (2.13)
 \end{aligned}$$

Using lemma 1 in (2.13), we get

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(B-A)(p-\alpha)}{2} \cos \lambda \max \{1, |(B-A)(p-\alpha)(2\mu-1) \cos \lambda - e^{i\lambda}|\}$$

and since (1.11) is sharp, then (2.10) is also sharp.

Remark on Theorem 2. Setting (i)  $B=1$  and  $A=-1$ , (ii)  $B=1$ ,  $A=-1$  and  $p=1$ , (iii)  $B=1$ ,  $A=-1$ ,  $p=1$  and  $\alpha=0$ , (iv)  $B=1$ ,  $A=-1$  and  $\lambda=0$ , in Theorem 2, we get the results of Patil and Thakare [1].

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