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ON THE POSITIVE SOLUTIONS OF A HIGHER ORDER FUNCTIONAL DIFFERENTIAL EQUATION WITH A DISCONTINUITY

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ABSTRACT. The n-th order nonlinear functional differential equation

$$[r(t)x^{(n-\nu)}(t)]^{(\nu)} = f(t,x(g(t)))$$

is considered; necessary and sufficient conditions are given for this equation to have: (i) a positive bounded solution $x(t) \to B > 0$ as $t \to \infty$; and (ii) all positive bounded solutions converging to 0 as $t \to \infty$. Other results on the asymptotic behavior of solutions are also given. The conditions imposed are such that the equation with a discontinuity

$$[r(t)x^{(n-\nu)}(t)]^{(\nu)} = q(t)x^{-\lambda}, \quad \lambda > 0$$

is included as a special case.

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1. INTRODUCTION.

Kitamura and Kusano [1] have recently studied the problems of the existence and asymptotic behavior of positive solutions of the equation

$$[r(t)x']' = q(t)x^{-\lambda} \tag{1.1}$$

where r and q are positive and continuous on $[0,\infty)$ and λ is a positive constant. Other authors, including Taliaferro [2] and [3], have studied the behavior of the solutions of (1.1) with either $q(t) < 0$ or $\lambda < 0$. However, as pointed out in [1], there seems to be no literature concerning (1.1) with $q(t) > 0$ and $\lambda > 0$ other than [1] and a superficial treatment of some of its special cases by Kamke[4].

Here we are concerned with the existence and the asymptotic properties of the positive continuable solutions of the functional differential equation

$$[r(t)x^{(n-\nu)}(t)]^{(\nu)} = f(t,x(g(t))) \tag{1.2}$$

where $1 \le \nu \le n-1$, $r,g\colon [t_0,\infty) \to R$ are continuous, $r(t) > 0$, $\int^{\infty}[1/r(s)]ds = \infty$, $g(t) \to \infty$ as $t \to \infty$ and $f(t,y)$ is positive, continuous, and nonincreasing with respect to y on $[t_0,\infty) \times (0,\infty)$. Clearly (1.1) is a special case of (1.2) and the results here extend some of those obtained in [1].

2. EXISTENCE AND ASYMPTOTIC PROPERTIES.

We first state two lemmas which will be used in some of our proofs.

LEMMA 1. ([5,6; Lemma 1]). Let u be a positive $(n-\nu)$-times continuously differentiable function on the interval $[a,\infty)$ and let μ be a positive continuous function on $[a,\infty)$ such that

$$\int^{\infty}[1/\mu(t)]dt = \infty,$$

and the function $w \equiv \mu u^{(n-\nu)}$ is ν-times continuously differentiable on $[a,\infty)$. Moreover, let

$$\omega_k = \begin{cases} u^{(k)}, & \text{if } 0 \le k \le n-\nu-1 \\ w^{(k-n+\nu)}, & \text{if } n-\nu \le k \le n. \end{cases}$$

If $\omega_n(t) \equiv w^{(\nu)}(t)$ is of constant sign and not identically zero for all large t, then there exists $t_u \ge a$ and an integer ℓ, $0 \le \ell \le n$, with $n+\ell$ even for ω_n non-negative or $n+\ell$ odd for ω_n non positive, and such that for every $t \ge t_u$

$$\ell > 0 \text{ implies } \omega_k(t) > 0 \ (k = 0,1,\ldots, \ell-1)$$

and

$$\ell \le n-1 \text{ implies } (-1)^{\ell+k}\omega_k(t) > 0 \ (k = \ell, \ell+1,\ldots, n-1).$$

LEMMA 2. ([5,6; Lemma 2]). If the functions u,μ,w and ω_k are as in Lemma 1 and for some $k = 0,1,\ldots,n-2$ $\omega_k(t) \to c$ as $t \to \infty$, then $\omega_{k+1}(t) \to 0$ as $t \to \infty$.

It will be convenient to make use of the following notation in the remainder of this paper. For any $T \ge t_0$ and all $t \ge T$ we let

$$z(t) = r(t)x^{(n-\nu)}(t),$$

$$\omega_k(t) = \begin{cases} x^{(k)}(t), & 0 \le k \le n-\nu-1 \\ z^{(k-n+\nu)}(t), & n-\nu \le k \le n, \end{cases}$$

$$J(T,t) = \int_T^t [(t-s)^{n-\nu-1}s^{\nu-1}/r(s)]ds/(n-\nu-1)!(\nu-1)!,$$

and

$$S(T,t) = \int_T^t [(t-s)^{\nu-1}s^{n-\nu-1}/r(s)]ds/(n-\nu-1)!(\nu-1)!.$$

From our assumptions regarding the functions in (1.2), it is easy to see that if $x(t)$ is a positive continuable solution of (1.2), then there exists $t \ge t_0$ such that $x(t)$ belongs to one of the two classes:

$$\text{(I)} \quad \omega_{n-1}(t) = z^{(\nu-1)}(t) > 0 \text{ for } t \ge t_1$$

or

$$\text{(II)} \quad \omega_{n-1}(t) = z^{(\nu-1)}(t) < 0 \text{ for } t \ge t_1.$$

As is indicated in the discussion following its proof, our first theorem is very near being a necessary and sufficient result.

THEOREM 1. Let $x(t)$ be a positive solution of type (I).

(i) If for every constant $c > 0$

$$\int^\infty f(s, cJ(t_0, g(s)))ds < \infty, \tag{2.1}$$

then there exists a positive constant A such that

$$[x(t)/J(T,t)] \to A \text{ as } t \to \infty \tag{2.2}$$

for all sufficiently large T.

(ii) If (2.2) holds then (2.1) holds for some constant $c > 0$.

PROOF. let $x(t)$ be a positive solution of (1.2) of type (I), then there exists $t_2 > \max\{t_1, 0\}$ such that $x(t)$, $x(g(t))$, and $z^{(\nu-1)}(t)$ are all positive on $[t_2,\infty)$. Since $x(t) > 0$ implies that $z^{(\nu)}(t) > 0$, we see that $z^{(\nu-1)}(t)$ is increasing on $[t_2,\infty)$. It then follows from Lemma 2 that $\omega_k(t) \to \infty$ as $t \to \infty$ for $k = 0,1,\ldots,n-2$. Then, by repeated application of L'Hôpital's rule, we obtain

$$\lim_{t\to\infty} [x(t)/J(t_2,t)] = \lim_{t\to\infty} z^{(\nu-1)}(t)/(\nu-1)! > K \tag{2.3}$$

for some positive constant K. Thus there exists $T \geq t_2$ so that

$$x(g(t)) \geq KJ(T,g(t)) \tag{2.4}$$

for $t \geq T$. To show that (2.2) holds it suffices (in view of (2.3)) to show that $x(t)/J(T,t)$ is bounded above. For this purpose we integrate (1.2) n-times over $[T,t]$ obtaining

$$x(t) = Q_{n-\nu-1}(t) + \int_T^t [(t-s)^{n-\nu-1}/r(s)]P_{\nu-1}(s)ds/(n-\nu-1)!$$

$$+ \int_T^t [(t-s)^{n-\nu-1}/r(s)] \int_T^s (s-u)^{\nu-1} f(u,x(g(u)))duds/(n-\nu-1)!(\nu-1)!, \tag{2.5}$$

where $Q_{n-\nu-1}(t)$ and $P_{\nu-1}(t)$ are polynomials of degree at most $n-\nu-1$ and $\nu-1$ respectively. By (2.4), we see that

$$\int_T^s (s-u)^{\nu-1} f(u,x(g(u))du \leq s^{\nu-1} \int_T^t f(u,KJ(T,g(u)))du,$$

and therefore we have

$$x(t)/J(T,t) \leq \int_T^\infty f(x,KJ(T,g(s)))ds + H$$

for some positive constant H. (2.1) implies that $x(t)/J(T,t)$ is bounded above.

Now suppose that (2.2) holds; then there exists positive constants A_1 and $T_1 \geq T$ such that $x(t) \leq A_1J(T_1,t)$ and $x(g(t)) \leq A_1J(T_1,g(t))$ for $t \geq T_1$. But (2.5) holds with T replaced by T_1 so we have

$$A_1 \geq x(t)/J(T_1,t)$$

$$\geq [1/J(T_1,t)]\{Q_{n-\nu-1}(t) + \int_{T_1}^t [(t-s)^{n-\nu-1}/r(s)]P_{\nu-1}(s)ds/(n-\nu-1)!$$

$$+ \int_{T_1}^t [(t-s)^{n-\nu-1}/r(s)] \int_{T_1}^s (s-u)^{\nu-1} f(u,A_1J(T_1,g(u)))duds/(n-\nu-1)!(\nu-1)!\}.$$

Therefore

$$A_1 \geq \lim_{t\to\infty} [x(t)/J(T_1,t)] \geq \lim_{t\to\infty} \int_{T_1}^t f(s,A_1J(T_1,g(s)))ds$$

which implies that

$$\int_{T_1}^\infty f(s,A_1J(T_1,g(s)))ds < \infty,$$

and it is not difficult to see that this implies that (2.1) holds for some positive constant c.

Notice that if the function f were such that condition (2.1) holding for some $c > 0$ implies that (2.1) holds for all positive constants, then condition (2.1) would be necessary and sufficient for every positive solution of (1.2) of type (I) to satisfy (2.2). This would be the case if $f(t,y)$ were homogeneous of some degree α in y, i.e. $f(t,sy) = s^{\alpha}f(t,y)$. Since $f(t,y) = q(t)y^{-\lambda}$, $\lambda > 0$, is homogeneous of degree $s = -\lambda$, we see that our Theorem 1 includes Theorem 1 and part (ii) of Theorem 4 in [1]. A simple example to which Theorem 1 applies is the equation

$$[x'(t)/t]'' = 6(\ln t + t^3)^2/t^4x^2(t), \quad t > 1$$

which has the positive solution $x(t) = \ln t + t^3$. Notice that here $J(T,t) = (t^3 - T^3)/3$ so that

$$A = \lim_{t\to\infty} [x(t)/J(T,t)] = \lim_{t\to\infty} 3(\ln t + t^3)/(t^3 - T^3) = 3.$$

THEOREM 2. A necessary and sufficient condition for (1.2) to have a bounded positive solution $x(t)$ satisfying $x(t) \to B > 0$ as $t \to \infty$ is that $\int^{\infty} S(T,s)f(s,c)ds < \infty$ for some constant $c > 0$ and all sufficiently large T.

PROOF. To prove necessity let $x(t)$ be a positive bounded solution of (1.2) and let $c > 0$ and $T > \max\{t_0,0\}$ be such that

$$0 < x(g(t)) \le c, \quad t \ge T. \tag{2.6}$$

Notice that from (1.2) we have $z^{(\nu)}(t) > 0$ for $t \ge T$ so that the hypotheses of Lemma 1 are satisfied. Also, it is easy to see that the boundedness of $x(t)$ implies that the integer ℓ assigned to $x(t)$ by Lemma 1 must satisfy $\ell < 2$. Moreover, since $z^{(\nu)}(t) > 0$ implies that $n + \ell$ is even, we see that $\ell = 0$ for n even and $\ell = 1$ for n odd. Consequently, by Lemma 1, we have for n even that

$$(-1)^i\omega_i(t) > 0, \quad i = 1,2,\dots,n-1 \tag{2.7}$$

and for n odd that

$$(-1)^{i+1}\omega_i(t) > 0, \quad i = 1,2,\dots,n-1 \tag{2.8}$$

for $t \ge T$. Next we multiply (1.2) by $S(T,t)$ and integrate to obtain

$$\int_T^t S(T,s)f(s,x(g(s)))ds = \int_T^t S(T,s)z^{(\nu)}(s)ds. \tag{2.9}$$

But successive integration by parts yields

$$\int_T^t S(T,s)z^{(\nu)}(s)ds = S(T,t)z^{(\nu-1)}(t) - S'(T,t)z^{(\nu-2)}(t) + \dots + (-1)^{\nu-1}S^{(\nu-1)}(t)z(t)$$
$$+ (-1)^{\nu}t^{n-\nu-1}x^{(n-\nu-1)}(t)/(n-\nu-1)! + \dots + (-1)^{n-1}x(t) + L$$

where L is a constant. Therefore, in view of (2.7) and (2.8), it follows from (2.9) that $\int_T^\infty S(T,s)f(s,x(g(s)))ds < \infty$ and by (2.6) we see that

$$\int_T^\infty S(T,s)f(s,c)ds < \infty.$$

To prove sufficiency, let $T_0 > \max\{t_0, 0\}$ and $c > 0$ be such that

$$\int_T^\infty S(T_0,s)f(s,c)ds < c \tag{2.10}$$

and consider the integral equation

$$x(t) = 2c + (-1)^n \int_t^\infty \left(\int_t^s [(s-u)^{\nu-1}(u-t)^{n-\nu-1}/r(u)]du \right) f(s,x(g(s)))ds/(\nu-1)!(n-\nu-1)!. \tag{2.11}$$

It is not difficult to verify by differentiation that a solution of (2.11) is also a solution of (1.2). We will show that (1.2) has a solution $x(t) \to B > 0$ as $t \to \infty$ by using the following special case of Tychonov's fixed point theorem:

THEOREM. Let F be a Fréchet space and X be a closed convex subset of F. If $G : X \to X$ is continuous and the closure $\overline{G(X)}$ is a compact subset of X, then there exists at least one fixed point x in X.

In order to utilize this theorem, let $u_0 = \min\{T_0, \min_{t \ge t_0} g(t)\}$ and let F be the Fréchet space of all continuous functions $x: [u_0,\infty) \to R$ with the topology of uniform convergence on compact subintervals of $[u_0,\infty)$. Let the closed convex subset X of F defined by

$$X = \{x \in F : c \le x(g(t)) \le 3c, \quad t \ge u_0\},$$

and define the operator G on X by

$$(Gx)(t) = \begin{cases} 2c + (-1)^n Q(t), & \text{if } t \ge T_0 \\ 2c + (-1)^n Q(T_0), & \text{if } u_0 \le t \le T_0 \end{cases}$$

where

$$Q(m) = \int_m^\infty \left(\int_m^s [(s-u)^{\nu-1}(u-m)^{n-\nu-1}/r(u)]du \right) f(s,x(g(s)))ds/(n-\nu-1)!(\nu-1)!.$$

To complete the proof we show that G satisfies all the hypotheses of the fixed point theorem stated above. First observe that for any $x \in X$

$$|(Gx)(t) - 2c| \le Q(T_0)$$

for $t \ge u_0$, and that

$$Q(T_0) \le \int_{T_0}^{\infty} S(T_0,s)f(s,c)ds < c.$$

Thus we see that G maps X into X.

To show that G is continuous let $\{x_\lambda\}$, $\lambda = 1,2,\ldots$ be any sequence of functions in X converging uniformly to $x \in X$ on every compact subinterval of $[u_0,\infty)$. Let $t \ge u_0$ and $T_2 > \max\{t,T_0\}$, then $f(t,x_\lambda(g(t))) \to f(t,x(g(t)))$ uniformly on $[u_0,T_2]$. However,

$$|(Gx_\lambda)(t) - (Gx)(t)| \le \int_{T_0}^{T_2} S(T_0,s)|f(s,x_\lambda(g(s))) - f(s,x(g(s)))|ds,$$

and we see from (2.10) that Gx_λ converges uniformly to Gx on any compact subinterval of $[u_0,\infty)$. Hence, we conclude that G is continuous.

Finally, in order to show that $\overline{GX}$ is a compact subset of X, it is sufficient to show that GX is relatively compact since $GX \subset X$ and X is closed. Furthermore, since X is bounded, it suffices to show that GX is equicontinuous. For this purpose, we distinguish two cases. If $n - \nu \ne 1$, then from the definitions of GX, S, and X, we have that there exists a constant L_1 such that

$$|(Gx)'(t)| \le \int_T^{\infty}\left(\int_t^s [(s-u)^{\nu-1}(u-t)^{n-\nu-2}/r(u)]du\right) f(s,x(g(s)))ds/(\nu-1)!(n-\nu-2)!$$
$$\le L_1 \int_T^{\infty} S(T_0,s)f(s,c)ds$$

for $t \ge u_0$. Hence it follows from (2.10) that there is constant L_2 such that $|(Gx)'(t)| \ge L_2$ where L_2 is independent of both x and t. It then follows that GX is equicontinuous on $[u_0,\infty)$.

If $n - \nu = 1$, we have for each $t \ge T_0$ that

$$|(Gx)'(t)| \le L_3 \int_t^{\infty}(s-t)^{\nu-1}f(s,x(g(s)))ds/r(t)$$

for some positive constant L_3. Since Gx is constant on $[u_0,T_0]$, then

$$|(Gx)'(t)| \le L_3 \int_{T_0}^{\infty} (s-T_0)^{\nu-1} f(s,c)ds/r(t)$$

for all $t \ge u_0$. Noticing that $[(s-T_0)^{\nu-1}/S(T_0,s)] \to 0$ as $s \to \infty$, we see that

$$|(Gx)'(t)| \le [L_4/r(t)] \int_{T_0}^{\infty} S(T_0,s)f(s,c)ds$$

for some constant L_4. Therefore, for any given closed subinterval $[u_0,T_1]$, of $[u_0,\infty)$, with $T_1 > T_0$, there exists a constant $L(T_1)$ such that

$$|(Gx)'(t)| \le L(T_1)$$

where $L(T_1)$ is independent of both $x \in X$ and t in $[u_0,T_1]$. Also, if $t_2 > t_1 \ge T_1$, then

$$\begin{aligned}|(Gx)(t_2) - (Gx)(t_1)| &\le \int_{t_2}^{\infty} S(T_1,s)f(x,c)ds + \int_{t_1}^{\infty} S(T_1,s)f(s,c)ds \\ &\le 2\int_{T_1}^{\infty} S(T_1,s)f(s,c)ds ,\end{aligned}$$

where the last integral tends to zero as $T_1 \to \infty$ independent of $x \in X$ and t_1, t_2 in $[T_1,\infty)$. If is now easy to see that GX is equicontinuous on $[u_0,\infty)$ for $n - \nu = 1$.

We now have all the hypotheses of the fixed point theorem satisfied and therefore we have the existence of $x \in X$ such that $Gx = x$, i.e. x is a solution of both (1.2) and (2.11) and satisfies $c \le x(t) \le 3c$. By differentiating both sides of (2.11) we see that $x'(t)$ has fixed sign and hence $x(t) \to B$ as $t \to \infty$ for some B in $[c,3c]$.

REMARK. Theorem 2 reduces to Theorem 2 in [1] when $r(t) \equiv 1$, $f(t,y) = q(t)y^{-\lambda}$, $\lambda > 0$, and $n = 2$. It also includes part (iii) of Theorem 4 in [1]. The equations

$$x^{(2k)}(t) = ce^{-4t/3}(e^t+c)^{1/3}x^{-1/3}(t), \quad t > 0, \tag{2.12}$$

where c is any positive constant and $k = 1,2,\ldots$ are examples of equations satisfying the hypotheses of Theorem 2. Notice that for each k, $x(t) = e^{-t}(e^t + c)$ is a solution of (2.12) satisfying $x(t) \to 1$ as $t \to \infty$.

THEOREM 3. A necessary and sufficient condition for every positive bounded solution $x(t) = \omega_0(t)$ of (1.2) to be such that $\omega_k(t) \to 0$ monotonically as $t \to \infty$ for $k = 0,1,\ldots,n-1$ is that

$$\int^{\infty} S(T,s)f(s,c)ds = \infty \tag{2.13}$$

for every $c > 0$ and all sufficiently large T.

PROOF. It follows from Theorem 2 that every positive bounded solution $x(t)$ of (1.2) satisfies $\omega_0(t) = x(t) \to 0$ as $t \to \infty$ if and only if (2.13) holds. Since $\omega_0(t) \to 0$ as $t \to \infty$, then $\omega_k(t) \to 0$ as $t \to \infty$ for $k = 1,2,\dots,n-1$ by Lemma 2.

In contrast to Theorem 2 and 3, the next two theorems give sufficient conditions for the positive solutions of (1.2) to be unbounded. In their proofs we utilize the function

$$J_1(c,T,t) = \int_T^t [c(t-s)^{n-\nu-1}h(s)/r(s)]ds$$

where c is a positive constant and $h(t) = t^{\nu-2}$ for $\nu - 2 \geq 0$ and $h(t) = 1$ for $\nu = 1$.

THEOREM 4. If for all positive constants c and L and all sufficiently large T

$$\limsup_{t \to \infty} \left\{ \int_T^t [(t-s)^{n-\nu-1}/r(s)] \int_T^s (s-u)^{\nu-1} f(u,J_1(c,T,g(u)))duds - LJ(T,t) \right\} = \infty, \tag{2.14}$$

then every positive solution of (1.2) of type (II) is unbounded.

PROOF. Let $x(t)$ be a positive solution of (1.2) of type (II). Since $z^{(\nu-1)}(t)$ is eventually negative, it is easy to see from Taylor's formula that there exists $T_1 > \max\{0,t_0\}$ and a positive constant c_1 so that $r(t)x^{(n-\nu)}(t) = z(t) \leq c_1h(t)$, $t \geq T_1$. Thus $x^{(n-\nu)}(t) \leq c_1h(t)/r(t)$ for $t \geq T_1$. Integrating each member of the last inequality $(n-\nu)$ times we obtain $x(t) \leq J_1(c_2,T_1,t)$ for some constant $c_2 > 0$. Hence there exists $T \geq T_1$ such that

$$x(g(t)) \leq J_1(c_2,T,g(t)), \quad t \geq T. \tag{2.15}$$

Next we integrate (1.2) n-times obtaining

$$x(t) = Q_{n-\nu-1}(t) + \int_T^t [(t-s)^{n-\nu-1}/r(s)]P_{\nu-1}(s)ds/(n-\nu-1)!$$

$$+ \int_T^t [(t-s)^{n-\nu-1}/r(s)] \int_T^s (s-u)^{\nu-1} f(u,x(g(u))duds/(n-\nu-1)!(\nu-1)!$$

where $Q_{n-\nu-1}(t)$ and $P_{\nu-1}(t)$ are polynomials of degree at most $n-\nu-1$ and $\nu-1$ respectively. But the last equation, together with (2.15), implies that there exists constants $c_3 > 0$ and $L > 0$ such that

$$c_3x(t) \geq -LJ(T,t) + \int_T^t [(t-s)^{n-\nu-1}/r(s)] \int_T^s (s-u)^{\nu-1} f(u,J_1(c_2,T,g(u)))duds,$$

and the conclusion of the theorem follows form (2.14).

REMARK. It is easy to see from the proof of Theorem 4 that, if (2.14) were replaced by

$$\liminf_{t \to \infty} \{ \int_T^t [(t-s)^{n-\nu-1}/r(s)] \int_T^s (s-u)^{\nu-1} f(u,J_1(c,T,g(u)))duds - LJ(T,t)\} > 0,$$

in the hypotheses of Theorem 4, then we could conclude that every positive solution of (1.2) of type (II) is bounded away from zero.

THEOREM 5. Suppose that

$$\int^\infty f(s,J_1(c,T,g(s))ds = \infty \tag{2.16}$$

for every constant $c > 0$ and all sufficiently large T. If $x(t)$ is a positive solution of (1.2), then $x(t)$ is of type (I) and there exists positive constants c, t_2 and $T_1 \geq t_2$ such that $x(t) \geq cJ(t_2,t)$ for $t \geq T_1$.

PROOF. Let $x(t)$ be a positive solution of (1.2) then there exists $t_2 > \max\{t_0,0\}$ so that $x(t)$, $x(g(t))$, $z^{(\nu)}(t)$ and $|z^{(\nu-1)}(t)|$ are all positive on $[t_2,\infty)$. First suppose that $x(t)$ is of type (II), i.e. $z^{(\nu-1)}(t) < 0$ for $t \geq t_2$. Then, by Taylor's formula, there exists a constant $c_1 > 0$ and $T \geq t_2$ such that (2.15) holds for $t \geq T$. From (2.15) and an integration of (1.2) we have

$$z^{(\nu-1)}(t) = z^{(\nu-1)}(T) + \int_T^t f(s,x(g(s)))ds \geq z^{(\nu-1)}(T) + \int_T^t f(x,J_1(c,T,g(s)))ds.$$

But then (2.16) implies that $z^{(\nu-1)}(t) \to \infty$ as $t \to \infty$ contradicting the assumption that $x(t)$ is of type (II). Therefore, we conclude that $x(t)$ is of type (I).

Now from L'Hôpital's rule, we obtain

$$\lim_{t\to\infty} [x(t)/J(t_2,t)] = \lim_{t\to\infty} z^{(\nu-1)}(t)/(\nu-1)! \tag{2.17}$$

as in the proof of Theorem 1. Since $x(t)$ being of type (I) implies that $z^{(\nu-1)}(t)$ is positive and increasing, from (2.17) there exists constants $c > 0$ and $T_1 \geq t_2$ such that $x(t) \geq cJ(t_2,t)$ for $t \geq T_1$.

REMARK. If $\int^\infty f(s,cJ(t_0,g(s)))ds = \infty$ for every constant $c > 0$, then if follows from Theorem 1 and condition (2.17) that $[x(t)/J(t_2,t)] \to \infty$ as $t \to \infty$. Notice also that $\nu=1$, then $J_1(c,T,t) = cJ(T,t)$. If, in addition, $f(t,y)$ is homogeneous of some degree α in y, then $\int^{\infty} f(s,cJ(t_0,g(s)))ds = \infty$ is equivalent to (2.16). Thus Theorem 5 includes the corollary in [1]. The equation

$$(tx'(t))' = (2 + \ln t)(t\ln t)^2/x^2(t), \quad t > 1, \tag{2.18}$$

satisfies all the hypotheses of Theorem 5 and has the solution $x(t) = t\ln t$.

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