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NONINCLUSION THEOREMS: SOME REMARKS ON A PAPER BY J. A. FRIDY

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Abstract. In 1997, J. A. Fridy gave conditions for noninclusion of ordinary and of absolute summability domains. In the present note, these conditions are interpreted in a natural topological context thus giving new proofs and also explaining why one of these conditions is too weak. Also an open question posed in Fridy's paper is answered.

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1. Noninclusion for ordinary summability. Recently, J. A. Fridy [2] stated a noninclusion theorem that can be formulated in the following way.

Theorem 1.1. *Let A and B be regular matrices such that c_A, the summability domain of A, is included in c_B, the summability domain of B. Then*

$$\lim_{n,k} a_{nk} = 0 \Longrightarrow \lim_{n,k} b_{nk} = 0. \tag{1.1}$$

Here $\lim_{n,k} a_{nk} = 0$ (and, similarly, $\lim_{n,k} b_{nk} = 0$) is taken in the Pringsheim sense, that is,

$$\forall \epsilon > 0\, \exists N > 0 : (n > N \text{ and } k > N) \Longrightarrow |a_{nk}| < \epsilon. \tag{1.2}$$

Of course, this is a noninclusion theorem, since if A has that limit property and B does not, then $c_A \not\subset c_B$. The reason for the above formulation is that it emphasizes an invariance property which is stated in an invariant form in the Lemma 1.2. Therein, e^k denotes the basic sequence $e^k = (0,\dots,0,1,0,\dots)$ with "1" in the kth position, and the summability domain

$$c_A = \left\{ x = (x_k) \,|\, Ax = \left(\sum_{k=1}^{\infty} a_{nk} x_k \right)_{n=1,2,\dots} \text{exists and converges} \right\} \tag{1.3}$$

is endowed with its FK-topology (see, e.g., [3, Ch. 22]) which is given by the seminorms

$$\begin{aligned} p_r(x) &:= |x_r| \quad (r = 1,2,\dots), \\ q_r(x) &:= \sup_m \left| \sum_{k=1}^{m} a_{rk} x_k \right| \quad (r = 1,2,\dots), \\ p_0(x) &:= \|Ax\|_\infty = \sup_n \left| \sum_{k=1}^{\infty} a_{nk} x_k \right|. \end{aligned} \tag{1.4}$$

Observe that all column limits of A exist if and only if $\varphi :=$ span $\{e^1, e^2, \ldots\} \subset c_A$.

Lemma 1.2. *Let A be a matrix with existing column limits. Then*

$$\Big(\lim_{k\to\infty} a_{nk} = 0 \text{ for } n = 1,2,\ldots \text{ and } \lim_{n,k} a_{nk} = 0\Big) \Longleftrightarrow \lim_{k\to\infty} e^k = 0 \quad \text{in } c_A. \tag{1.5}$$

Proof. Certainly, $p_r(e^k) \to 0$ as $k \to \infty$ for each r. Also, the condition $\lim_{k\to\infty} a_{nk} = 0$ for $n = 1,2,\ldots$ (all row limits of A are zero) is equivalent to $\lim_{k\to\infty} q_r(e^k) = 0$ for $r = 1,2,\ldots$. Now, let $\lim_{n,k} a_{nk} = 0$ in the Pringsheim sense. Then, given $\epsilon > 0$, there exists $N_1 > 0$ such that $|a_{nk}| < \epsilon$ for $n > N_1$ and $k > N_1$. If, in addition, $\lim_{k\to\infty} a_{rk} = 0$ for $r = 1,\ldots,N_1$, then there exists $N > N_1$ such that $|a_{nk}| < \epsilon$ for $1 \le r \le N_1$ and all $k > N$. Thus $p_0(e^k) = \sup_n |a_{nk}| \le \epsilon$ for all $k > N$. Hence $p_0(e^k) \to 0$ as $k \to \infty$, and $e^k \to 0$ in c_A follows.

Conversely, suppose $e^k \to 0$ in c_A. Then, in particular, $\lim_{k\to\infty} q_r(e^k) = 0$ for $r = 1,2,\ldots$ and $p_0(e^k) = \sup_n |a_{nk}| \to 0$ as $k \to \infty$; the former implies $\lim_{k\to\infty} a_{rk} = 0$, the latter $\lim_{n,k} a_{nk} = 0$. □

As a corollary we obtain Fridy's result.

Corollary 1.3. *Let A be a matrix with existing column limits and with row limits zero. If $c_A \subset c_B$, then*

$$\lim_{n,k} a_{nk} = 0 \Longrightarrow \lim_{n,k} b_{nk} = 0, \tag{1.6}$$

and then, in fact, B is a matrix with existing column limits and with row limits zero.

Proof. By the Lemma 1.2 we have $e^k \to 0$ in c_A. By $c_A \subset c_B$, the relative topology of c_B on c_A is weaker than the FK-topology of c_A (see [3, Ch. 17]; hence $e^k \to 0$ in c_B, and, by Lemma 1.2, this means $\lim_{n,k} b_{nk} = 0$, and the row limits of B are zero. □

Remark 1.4. In [2] it is already noticed that in Theorem 1.1 the supposition that A and B should be regular can be relaxed to the condition that both matrices have column and row limits zero. Corollary 1.3 is slightly more general; the existence of the column limits of A is needed in order that $e^k \in c_A$ for all k, and hence, by $c_A \subset c_B$, the column limits of B exist. It should also be remarked here that a K-space E containing φ is called a wedge space if $e^k \to 0$ in E, see G. Bennett [1, Thm. 27], asserting that c_A with $\varphi \subset c_A$ is a wedge space if and only if $\lim_{k\to\infty} \sup_n |a_{nk}| = 0$.

2. Noninclusion for absolute summability. In [2] noninclusion is also considered for absolute summability; here

$$\ell_A = \left\{ x = (x_k) \,\middle|\, Ax = \left(\sum_{k=1}^{\infty} a_{nk} x_k\right) \text{ exists and } Ax \in \ell \right\} \tag{2.1}$$

the absolute summability domain of A, is concerned, where

$$\ell = \left\{ x = (x_k) \mid \|x\|_1 := \sum_{k=1}^{\infty} |x_k| < \infty \right\}. \tag{2.2}$$

We state the result in the following form.

Theorem 2.1. *Let A be a matrix with its column sequences in ℓ (so that $e^k \in \ell_A$ for all k), and let B be a matrix with $\ell_A \subset \ell_B$. If there is an index sequence $(k(j))_{j=1,2,\dots}$ such that*

$$\lim_{j\to\infty}\sum_{n=1}^{\infty}\left|a_{n,k(j)}\right| = 0, \tag{2.3}$$

then

$$\lim_{j\to\infty}\sum_{n=1}^{\infty}\left|b_{n,k(j)}\right| = 0. \tag{2.4}$$

In [2], there is an extra condition $\ell \subset \ell_A$, but condition (2.3) is relaxed to

$$\lim_{j\to\infty}\sum_{n=\mu}^{\infty}\left|a_{n,k(j)}\right| = 0 \quad \text{for some integer } \mu, \tag{2.5}$$

and (2.4) is correspondingly weakened to

$$\lim_{j\to\infty}\sum_{n=\mu}^{\infty}\left|b_{n,k(j)}\right| = 0 \tag{2.6}$$

with the same μ as in (2.5). Unfortunately, this relaxed version fails for $\mu > 1$, even if $\ell \subset \ell_A$ and the μ in (2.6) is allowed to differ from that one in (2.5). This can be seen from the following example.

Example 2.2. For all $k = 1,2,\dots$, define

$$a_{nk} := \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \qquad \text{and} \qquad b_{nk} := \frac{1}{n^2} \quad \text{for } n = 1,2,\dots, \tag{2.7}$$

so that

$$(Ax)_n = \begin{cases} \sum_{k=1}^{\infty} x_k, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \qquad \text{and} \qquad (Bx)_n = \frac{1}{n^2}\sum_{k=1}^{\infty} x_k. \tag{2.8}$$

Then, clearly,

$$\begin{gathered} \ell \subset \ell_A = \ell_B = \left\{(x_k) \,\middle|\, \sum_{k=1}^{\infty} x_k \text{ converges}\right\}, \\ \lim_{j\to\infty}\sum_{n=2}^{\infty}\left|a_{n,k(j)}\right| = 0, \qquad \lim_{j\to\infty}\sum_{n=\mu}^{\infty}\left|b_{n,k(j)}\right| = \sum_{n=\mu}^{\infty}\frac{1}{n^2} > 0 \end{gathered} \tag{2.9}$$

for each integer μ and each index sequence $(k(j))$.

To prove Theorem 2.1 in a topological way—similar to the proof of Corollary 1.3 (and Theorem 1.1)—we need the following lemma.

Lemma 2.3. *Let A be a matrix with its column sequences in ℓ, and let $(k(j))_{j=1,2,\dots}$ be an index sequence. Then*

$$\lim_{j\to\infty}\sum_{n=1}^{\infty}\left|a_{n,k(j)}\right| = 0 \Longleftrightarrow e^{k(j)} \longrightarrow 0 \quad \textit{in } \ell_A \textit{ as } j \longrightarrow \infty. \tag{2.10}$$

Proof. The FK-topology of the FK-space ℓ_A is given by the seminorms p_r, q_r (see above) and

$$p_0^{\ell}(x) := \|Ax\|_1 = \sum_{n=1}^{\infty}\left|\sum_{k=1}^{\infty} a_{nk}x_k\right|. \tag{2.11}$$

Thus $e^{k(j)} \to 0$ in ℓ_A is equivalent to $p_r(e^{k(j)}) \to 0, q_r(e^{k(j)}) \to 0$ for each fixed $r = 1,2,\dots$ and $\|Ae^{k(j)}\|_1 = \sum_{n=1}^{\infty} |a_{n,k(j)}| \to 0$. These conditions are equivalent to the single condition $\|Ae^{k(j)}\|_1 \to 0$, since $q_r(e^{k(j)}) \le \|Ae^{k(j)}\|_1$ and $p_rH(e^{k(j)}) = 0$ for $k(j) > r$. The lemma follows. □

Theorem 2.1 is now a simple corollary of Lemma 2.3. By $\ell_A \subset \ell_B$ the FK-topology of ℓ_A is stronger than the relative FK-topology of ℓ_B on ℓ_A. Hence $e^{k(j)} \to 0$ in ℓ_A implies $e^{k(j)} \to 0$ in ℓ_B. Lemma 2.3 now yields the assertion of Theorem 2.1.

In [2] it is asked whether in Theorem 2.1 conditions (2.3) and (2.4) can be replaced by

$$\lim_{j\to\infty}\left|\sum_{n=1}^{\infty} a_{n,k(j)}\right| = 0 \qquad \text{and} \qquad \lim_{j\to\infty}\left|\sum_{n=1}^{\infty} b_{n,k(j)}\right| = 0, \tag{2.12}$$

respectively. The answer is negative as can be seen by the following example.

Example 2.4. Define $A = (a_{nk})$ and $B = (b_{nk})$ by

$$a_{nk} := \begin{cases} 1, & \text{if } n = 1, \\ -1, & \text{if } n = 2, \\ 0, & \text{if } n > 2, \end{cases} \quad \text{for } k = 1,2,\dots \tag{2.13}$$

and

$$b_{nk} := \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \quad \text{for } k = 1,2,\dots, \tag{2.14}$$

so that $Ax = \left(\sum_{k=1}^{\infty} x_k, -\sum_{k=1}^{\infty} x_k, 0, 0, \dots\right)$ and $Bx = \left(\sum_{k=1}^{\infty} x_k, 0, 0, \dots\right)$.

Then, clearly, $(\ell \subset)\ell_A = \ell_B = \{x = (x_k) \mid \sum_{k=1}^{\infty} x_k \text{ converges}\}$ and

$$\lim_{j\to\infty}\left|\sum_{n=1}^{\infty} a_{n,k(j)}\right| = 0, \qquad \lim_{j\to\infty}\left|\sum_{n=1}^{\infty} b_{n,k(j)}\right| = 1 \tag{2.15}$$

for any index sequence $(k(j))_{j=1,2,\dots}$.

References

[1] G. Bennett, *A new class of sequence spaces with applications in summability theory*, J. Reine Angew. Math. **266** (1974), 49–75. MR 49 9585. Zbl 277.46012.

[2] J. A. Fridy, *Noninclusion theorems for summability matrices*, Internat. J. Math. Math. Sci. **20** (1997), no. 3, 511–516. MR 98f:40003. Zbl 886.40005.

[3] K. Zeller and W. Beekmann, *Theorie der Limitierungsverfahren*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 15, Springer-Verlag, Berlin, New York, 1970. MR 41#8863. Zbl 199.11301.

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