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Research Article

Fuzzy $\overline{x}$ and s Control Charts: A Data-Adaptability and Human-Acceptance Approach

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For sequentially monitoring and controlling average and variability of an online manufacturing process, $\overline{x}$ and s control charts are widely utilized tools, whose constructions require the data to be real (precise) numbers. However, many quality characteristics in practice, such as surface roughness of optical lenses, have been long recorded as fuzzy data, in which the traditional $\overline{x}$ and s charts have manifested some inaccessibility. Therefore, for well accommodating this fuzzy-data domain, this paper integrates fuzzy set theories to establish the fuzzy charts under a general variable-sample-size condition. First, the resolution-identity principle is exerted to erect the sample-statistics' and control-limits' fuzzy numbers (SSFNs and CLFNs), where the sample fuzzy data are unified and aggregated through statistical and nonlinear-programming manipulations. Then, the fuzzy-number ranking approach based on left and right integral index is brought to differentiate magnitude of fuzzy numbers and compare SSFNs and CLFNs pairwise. Thirdly, the fuzzy-logic alike reasoning is enacted to categorize process conditions with intermittent classifications between in control and out of control. Finally, a realistic example to control surface roughness on the turning process in producing optical lenses is illustrated to demonstrate their data-adaptability and human-acceptance of those integrated methodologies under fuzzy-data environments.

1. Introduction

In nowadays fierce, competitive marketplaces, providing consistent and reliable quality products has been acknowledged as one of the most significant criteria for industrial manufacturers to persist their survival and sustainable growth. As such, establishing effective quality management systems and programs has become their prioritized strategy for lowering the percentage of nonconformities, slashing manufacturing costs, and fulfilling customer satisfaction [1]. Among them, the quality-control scheme has been widely advocated as a powerful control tool for achieving production effectiveness, as well as remaining quality-based competitive advantages [2].

Currently, for practically monitoring and controlling manufacturing processes, Shewhart-type control charts are extensively applied due to their notable capability on genuinely and early revealing process-abnormal conditions so as to ward off mistaken process intervention, prevent excessive product-defects, and eliminate costly scrap or rework of final products [1, 3]. Typically, the control chart constitutes three tracking lines, a center line (CL) and upper and lower control limits (UCL and LCL), whose construction is generally based on moderate numbers of subgroup sample data, each with either equal or unequal sample size, randomly being drawn from the process's key quality characteristic. For monitoring the process conditions, the control charts function under the situations that when all statistic points of collected sample data fall within the limits and do not exhibit any systematic pattern, the process is classified as in statistical control and no interference is needed; otherwise, the process is being suspected to be affected by some assignable causes that deserve to be comprehensively investigated through well-structured corrective actions [1].

Traditionally, Shewhart control charts are constructed based on random precise data collected from a key quality characteristic. However, in practice, crisp data may fail to describe the nature of several applications, such as the surface roughness of components, the transmission speed of certain lights through a material, and the coating thickness of industrial cutting tools, because they cannot be recorded or measured precisely [4, 5]. Besides, making decisions on whether those products are conforming or nonconforming or judging if a process is in control or out of control usually includes some extent of human subjectivity relating to decision-makers' intelligence and perceptions. These issues create the vagueness in the measurement system; therefore, the recorded data are considered as fuzzy data [6–9]. With the presence of fuzziness, the variance of normal observations tends to increase [10], and some intermediate decisions indispensably exist in-between the binary classification [6]. Thus, in order to adapt to these fuzzy environments, the traditional control charts with binary classifications are necessarily extended to *"fuzzy control charts"* [7, 11] which are considered as an inevitable and suitable choice in monitoring and controlling a manufacturing process with fuzzy data [7, 12–14]. As such, several fuzzy control charts have been proposed and constructed, for instance, fuzzy $\tilde{\bar{\bar{X}}}$-$\tilde{\bar{R}}$ and $\tilde{\bar{\bar{X}}}$-$\tilde{\bar{S}}$ control charts by Senturk and Erginel [7], fuzzy $\bar{x}$ and R control charts by Shu and Wu [11], fuzzy $\bar{x}$ and s control charts by Nguyen et al. [15], fuzzy MaxGWMA control chart by Shu et al. [4], and fuzzy $\bar{x}$ control chart for multiple objective decision-making problem [16].

Nevertheless, certain problems in the construction and evaluation of fuzzy control charts have been raised. Particularly, Wang and Raz [17] proposed multigrades linguistic terms such as perfect, good, medium, poor, and bad to express the key quality characteristic. However, the underlying probability distribution of the linguistic data was not considered [18]. Hence, Kanagawa et al. [18] suggested estimating the probability distribution existing behind the linguistic data before constructing the control charts. Then, Laviolette et al. [19] and Asai [20] pointed that the probability estimation cannot be easily determined. In addition, as the membership function of linguistic terms is obtained arbitrarily on a given scale regardless of the fuzziness in the judgments of experts [11, 21], the linguistic-based control charts are not firmly validated. Similarly, using defuzzification methods such as fuzzy midrange, fuzzy mode, fuzzy median, and fuzzy average in constructing fuzzy control charts proposed in several researches [7, 17, 22] has also raised a core controversial issue of losing the fuzziness information in the original data as well as misjudgement of the manufacturing process [11, 23, 24], although it allows the control charts to be constructed with binary classifications.

Thus, several scholars have put great effort into preserving the fuzziness of vague data in their approaches. For examples, Grzegorzewski and Hryniewicz [25] utilized the necessity index of strict dominance (NISD); however, Chien et al. [26] claimed that the NISD is content-dependent because the ranking results may change when a new fuzzy number is added. Also, Gülbay and Kahraman [13] came up with an acceptable percentage index called direct fuzzy approach (DFA) which was then found failing in obtaining the fuzzy sample means and variances with the simple using of α-cuts [11]. Shu and Wu [11] developed a fuzzy dominance approach (FDA) by extending Yuan's fuzzy-numbers ranking method [27]. Nevertheless, the FDA approach can only perform nicely at the dominance degree greater than 0.5. Nguyen et al. [28] proposed a detailed procedure to classify a process, but some of their rules were found indistinguishable by Nguyen et al. [15], who later proposed a remedy for a better performance. Though Shu et al. [4] established a thorough system to evaluate manufacturing processes, their classification rules seem quite complicated. Therefore, this paper aims at providing an easier procedure by simplifying Yu and Dat's ranking method [29]. In addition, our fuzzy $\bar{x}$ and s charts can generally deal with variable sample size which is the key advantage of our proposed control charts over those of Nguyen et al. [28].

This paper is organized as follows. Section 2 briefly provides key characteristics of traditional $\bar{x}$ and s control charts, playing as the foundation for our detailed procedure to construct fuzzy $\bar{x}$ and s control charts presented in Section 3. Also in Section 3, an empirical case in monitoring surface roughness of optical lenses in its turning process is conducted as a paradigm to illustrate the applicability of this new extended approach in building these fuzzy control charts. In order to effectively evaluate them, Section 4 apprises not only our advocated approach by simplifying Yu and Dat's ranking method [29] but also the elucidatory development of our novel classification mechanism which is then used in the case discussed in Section 3 to provide a thorough controlling and monitoring procedure for the application of our proposed fuzzy control charts in practice. Some concluding remarks make up the last section.

2. Review of Traditional $\bar{x}$ and s Control Charts

Literally, process variability must be fully controlled before process mean is monitored because larger variability in manufacturing process always results in higher percentage of nonconforming products, although the process mean is kept unchanged at its target value [1]. The process variability can be monitored with either R chart or s chart, while $\bar{x}$ chart is used to monitor process mean; thus, in practice, $\bar{x}$ chart usually goes with either R chart or s chart. Specifically, $\bar{x}$ chart and R chart are preferably used if sample size is small (no more than 10), whereas $\bar{x}$ chart and s chart are used when sample size is either larger than 10 or variable [2]. In practice, there are several applications where randomly choosing samples with variable sizes is economically preferred [1]. Moreover, many scholars claimed that control charts with variable sample size can detect process shifts markedly faster than the ones with equal sample size [30]. Therefore, this paper investigates samples with variable sizes; correspondingly, only $\overline{x}$ chart and s chart are taken into consideration.

Suppose a quality characteristic X has a normal distribution with a mean μ and a standard deviation σ, that is, $X \sim N(\mu, \sigma^2)$. Normally, μ and σ are not known in advance. They are usually estimated from initial samples taken from a process that is believed to be in control. Conventionally, 20 or 25 samples are investigated; their grand average is used as the best estimator of μ, and their average standard deviation can be used to obtain the estimated value of σ [1].

Let n_i be the size of the ith sample of the m samples investigated. Let x_{ij} denote the value of the quality characteristic in the sample ith at the observation jth ($i = \overline{1, m}$; $j = \overline{1, n_i}$). The average of the ith sample, denoted by $\overline{x}_i$, and the grand average of the m samples, denoted by $\overline{\overline{x}}$, are determined by

$$\begin{aligned} \overline{x}_i &= \frac{1}{n_i}\sum_{j=1}^{n_i} x_{ij} \\ \overline{\overline{x}} &= \frac{1}{\sum_{i=1}^{m} n_i}\sum_{i=1}^{m} n_i \overline{x}_i. \end{aligned} \tag{1}$$

Let s_i and $\overline{s}$ denote the standard deviation of the ith sample and the m samples, respectively. They are obtained by

$$\begin{aligned} s_i &= \sqrt{\frac{1}{n_i - 1}\sum_{j=1}^{n_i}\left(x_{ij} - \overline{x}_i\right)^2} \\ \overline{s} &= \sqrt{\frac{1}{\sum_{i=1}^{m} n_i - m}\sum_{i=1}^{m}\left(n_i - 1\right) s_i^2}. \end{aligned} \tag{2}$$

From the values of $\overline{\overline{x}}$ and $\hat{s}$, the centerline (CL), upper control limit (UCL), and lower control limit (LCL) of the $\overline{x}$ chart are constructed by

$$\begin{aligned} \mathrm{UCL}_{x_i} &= \overline{\overline{x}} + K\frac{\overline{s}}{c_{4_i}\sqrt{n_i}} \\ \mathrm{CL}_{x_i} &= \overline{\overline{x}} \\ \mathrm{LCL}_{x_i} &= \overline{\overline{x}} - K\frac{\overline{s}}{c_{4_i}\sqrt{n_i}}, \end{aligned} \tag{3}$$

where c_{4_i} is a constant determined by sample size n_i as shown in several textbooks [1, 2, 31]. And, K is the number of standard deviation units (usually called *Sigma*) that are allowed as tolerance; traditionally, $K = 3$ is usually used [1, 2].

And the control limits for s chart are determined by

$$\begin{aligned} \mathrm{UCL}_{s_i} &= \left(1 + \frac{K}{c_{4_i}}\sqrt{1 - c_{4_i}^2}\right)\overline{s} \\ \mathrm{CL}_{s_i} &= \overline{s} \\ \mathrm{LCL}_{s_i} &= \left(1 - \frac{K}{c_{4_i}}\sqrt{1 - c_{4_i}^2}\right)\overline{s}. \end{aligned} \tag{4}$$

The control limits of the control charts determined by (3) and (4) are fluctuated as shown in Figure 1 because each sample has its own value of c_4. By plotting all of $\overline{x}_i$ and s_i ($i = \overline{1, m}$) against the variable control limits, we can detect out-of-control signal (if any).

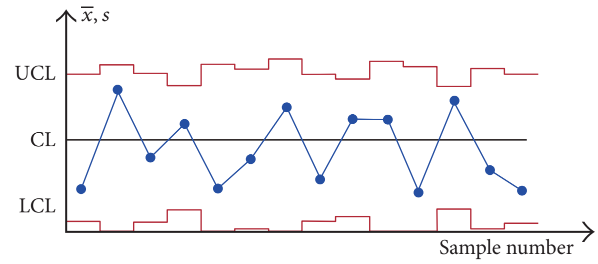

FIGURE 1: Typical $\overline{x}$ chart and s chart with variable sample size.

3. Construction of Fuzzy $\overline{x}$ Chart and s Chart

With sample fuzzy data obtained from a manufacturing process, this section shows how our proposed fuzzy $\overline{x}$ chart and s chart are constructed in order to monitor the underlying process average and variability, respectively.

Let $\widetilde{x}_{i1}, \widetilde{x}_{i2}, \ldots, \widetilde{x}_{in_i}$ ($i = \overline{1, m}$) be fuzzy observations (fuzzy data) which are assumed to be fuzzy real numbers discussed in [4, 28]. Upper and lower control limits are constructed based on these fuzzy data. For any given $\alpha \in [0, 1]$, we can obtain the corresponding real-valued data $(\widetilde{x}_{ij})_\alpha^L$ and $(\widetilde{x}_{ij})_\alpha^U$ for $i = \overline{1, m}$ and $j = \overline{1, n_i}$. In order to obtain the estimate of the fuzzy control limits for $(\widetilde{x}_{ij})_\alpha^L$ and $(\widetilde{x}_{ij})_\alpha^U$, the real-valued data $(\widetilde{x}_{i1})_\alpha^L, \ldots, (\widetilde{x}_{in_i})_\alpha^L$ and $(\widetilde{x}_{i1})_\alpha^U, \ldots, (\widetilde{x}_{in_i})_\alpha^U$ ($i = \overline{1, m}$) are used in (1)–(4), which results in the following:

$$\begin{aligned} \overline{x}_{i,\alpha}^U &= \frac{1}{n_i}\sum_{j=1}^{n_i}\left(\widetilde{x}_{ij}\right)_\alpha^U \\ \overline{\overline{x}}_\alpha^U &= \frac{1}{\sum_{i=1}^{m} n_i}\sum_{i=1}^{m} n_i \overline{x}_{i,\alpha}^U \\ \overline{x}_{i,\alpha}^L &= \frac{1}{n_i}\sum_{j=1}^{n_i}\left(\widetilde{x}_{ij}\right)_\alpha^L \\ \overline{\overline{x}}_\alpha^L &= \frac{1}{\sum_{i=1}^{m} n_i}\sum_{i=1}^{m} n_i \overline{x}_{i,\alpha}^L \\ s_{i,\alpha}^U &= \sqrt{\frac{1}{n_i - 1}\sum_{j=1}^{n_i}\left[\left(\widetilde{x}_{ij}\right)_{i,\alpha}^U - \overline{x}_{i,\alpha}^U\right]^2} \\ \overline{s}_\alpha^U &= \sqrt{\frac{1}{\sum_{i=1}^{m} n_i - m}\sum_{i=1}^{m}\left[\left(n_i - 1\right)\left(s_{i,\alpha}^U\right)^2\right]} \\ s_{i,\alpha}^L &= \sqrt{\frac{1}{n_i - 1}\sum_{j=1}^{n_i}\left[\left(\widetilde{x}_{ij}\right)_{i,\alpha}^L - \overline{x}_{i,\alpha}^L\right]^2} \\ \overline{s}_\alpha^L &= \sqrt{\frac{1}{\sum_{i=1}^{m} n_i - m}\sum_{i=1}^{m}\left[\left(n_i - 1\right)\left(s_{i,\alpha}^L\right)^2\right]} \end{aligned} \tag{5}$$

3.1. Fuzzy $\overline{x}$ Control Chart. Substituted with the fuzzy results in (5), the parameters for the fuzzy $\overline{x}$ chart in (3) are obtained by

$$\begin{aligned} u^U_{\overline{x}_i,\alpha} &\equiv \left(\mathrm{ucl}_{\overline{x}_i}\right)^U_\alpha = \overline{\overline{x}}^U_\alpha + K\frac{\overline{s}^U_\alpha}{c_{4_i}\sqrt{n_i}} \\ c^U_{\overline{x}_i,\alpha} &\equiv \left(\mathrm{cl}_{\overline{x}_i}\right)^U_\alpha = \overline{\overline{x}}^U_\alpha \\ l^U_{\overline{x}_i,\alpha} &\equiv \left(\mathrm{lcl}_{\overline{x}_i}\right)^U_\alpha = \overline{\overline{x}}^U_\alpha - K\frac{\overline{s}^U_\alpha}{c_{4_i}\sqrt{n_i}} \\ u^L_{\overline{x}_i,\alpha} &\equiv \left(\mathrm{ucl}_{\overline{x}_i}\right)^L_\alpha = \overline{\overline{x}}^L_\alpha + K\frac{\overline{s}^L_\alpha}{c_{4_i}\sqrt{n_i}} \\ c^L_{\overline{x}_i,\alpha} &\equiv \left(\mathrm{cl}_{\overline{x}_i}\right)^L_\alpha = \overline{\overline{x}}^L_\alpha \\ l^L_{\overline{x}_i,\alpha} &\equiv \left(\mathrm{lcl}_{\overline{x}_i}\right)^L_\alpha = \overline{\overline{x}}^L_\alpha - K\frac{\overline{s}^L_\alpha}{c_{4_i}\sqrt{n_i}}. \end{aligned} \tag{6}$$

(a) Construction of Fuzzy Upper Control Limit $\widetilde{u}_{\overline{x}}$. By using the above results, let us consider the closed interval $A_{i,\alpha}$ which is defined as follows:

$$\begin{aligned} A_{i,\alpha} &= \left[\min\left\{u^L_{\overline{x}_i,\alpha}, u^U_{\overline{x}_i,\alpha}\right\}, \max\left\{u^L_{\overline{x}_i,\alpha}, u^U_{\overline{x}_i,\alpha}\right\}\right] \\ &\equiv \left[l_{i,\alpha}, u_{i,\alpha}\right], \end{aligned} \tag{7}$$

where

$$\begin{aligned} l_{i,\alpha} &= \min\left\{u^L_{\overline{x}_i,\alpha}, u^U_{\overline{x}_i,\alpha}\right\} \\ u_{i,\alpha} &= \max\left\{u^L_{\overline{x}_i,\alpha}, u^U_{\overline{x}_i,\alpha}\right\}. \end{aligned} \tag{8}$$

Based on resolution identity [32], the control-limits' fuzzy numbers (CLFNs) for the membership function of the fuzzy upper control limit can be defined as

$$\xi_{\widetilde{u}_{\overline{x}_i}}(c) = \sup_{\alpha\in[0,1]} \alpha \cdot 1_{\widetilde{A}_{i,\alpha}}(c). \tag{9}$$

Since each $\widetilde{x}_{ij}$ is a fuzzy real number, $(\widetilde{x}_{ij})^L_\alpha$ and $(\widetilde{x}_{ij})^U_\alpha$ are continuous with respect to α on $[0,1]$, saying that $\overline{\overline{x}}^L_\alpha$, $\overline{\overline{x}}^U_\alpha$, $\overline{s}^L_\alpha$, and $\overline{s}^U_\alpha$ are continuous with respect to α on $[0,1]$. Under these facts, the α-level set $(\widetilde{u}_{\overline{x}_i})_\alpha$ of fuzzy upper control limit $\widetilde{u}_{\overline{x}_i}$ can be simply written as

$$\begin{aligned} \left(\widetilde{u}_{\overline{x}_i}\right)_\alpha &= \left\{c : \xi_{\widetilde{u}_{\overline{x}_i}}(c) \ge \alpha\right\} = \left[\min_{\alpha\le\beta\le1} l_{i,\alpha}, \max_{\alpha\le\beta\le1} u_{i,\alpha}\right] \\ &= \left[\left(\widetilde{u}_{\overline{x}_i}\right)^L_\alpha, \left(\widetilde{u}_{\overline{x}_i}\right)^U_\alpha\right], \end{aligned} \tag{10}$$

where $l_{i,\alpha}$ and $u_{i,\alpha}$ are shown in (8).

From (10), the relationship between $(\widetilde{u}_{\overline{x}_i})^L_\alpha$ and $u^L_{\overline{x}_i,\alpha}$, $u^U_{\overline{x}_i,\alpha}$ is found as

$$\left(\widetilde{u}_{\overline{x}_i}\right)^L_\alpha = \min_{\alpha\le\beta\le1} l(\beta) = \min_{\alpha\le\beta\le1} \min\left\{u^L_{\overline{x}_i,\beta}, u^U_{\overline{x}_i,\beta}\right\}. \tag{11}$$

Similarly, the relationship between $(\widetilde{u}_{\overline{x}_i})^U_\alpha$ and $u^L_{\overline{x}_i,\alpha}$, $u^U_{\overline{x}_i,\alpha}$ is found as

$$\left(\widetilde{u}_{\overline{x}_i}\right)^U_\alpha = \max_{\alpha\le\beta\le1} u(\beta) = \max_{\alpha\le\beta\le1} \max\left\{u^L_{\overline{x}_i,\beta}, u^U_{\overline{x}_i,\beta}\right\}. \tag{12}$$

(b) Construction of Fuzzy Lower Control Limit $\widetilde{l}_{\overline{x}}$. With the same procedure, the endpoints of the α-level closed interval of the fuzzy lower control limit $\widetilde{l}_{\overline{x}_i}$ are determined by

$$\begin{aligned} &\left(\widetilde{l}_{\overline{x}_i}\right)_\alpha \\ &\quad = \left[\min_{\alpha\le\beta\le1} \min\left\{l^L_{\overline{x}_i,\beta}, l^U_{\overline{x}_i,\beta}\right\}, \max_{\alpha\le\beta\le1} \max\left\{l^L_{\overline{x}_i,\beta}, l^U_{\overline{x}_i,\beta}\right\}\right]. \end{aligned} \tag{13}$$

3.2. Fuzzy s Control Chart. Substituted with the fuzzy results in (5), the parameters for the fuzzy s chart in (4) are obtained by

$$\begin{aligned} u^U_{s_i,\alpha} &\equiv \left(\mathrm{ucl}_{s_i}\right)^U_\alpha = \left(1 + \frac{K\sqrt{1-c^2_{4_i}}}{c_{4_i}}\right)\overline{s}^U_\alpha \\ c^U_{s_i,\alpha} &\equiv \left(\mathrm{cl}_{s_i}\right)^U_\alpha = \overline{s}^U_\alpha \\ l^U_{s_i,\alpha} &\equiv \left(\mathrm{lcl}_{s_i}\right)^U_\alpha = \left(1 - \frac{K\sqrt{1-c^2_{4_i}}}{c_{4_i}}\right)\overline{s}^U_\alpha \\ u^L_{s_i,\alpha} &\equiv \left(\mathrm{ucl}_{s_i}\right)^L_\alpha = \left(1 + \frac{K\sqrt{1-c^2_{4_i}}}{c_{4_i}}\right)\overline{s}^L_\alpha \\ c^L_{s_i,\alpha} &\equiv \left(\mathrm{cl}_{s_i}\right)^L_\alpha = \overline{s}^L_\alpha \\ l^L_{s_i,\alpha} &\equiv \left(\mathrm{lcl}_{s_i}\right)^L_\alpha = \left(1 - \frac{K\sqrt{1-c^2_{4_i}}}{c_{4_i}}\right)\overline{s}^L_\alpha. \end{aligned} \tag{14}$$

The construction of the fuzzy control limits for fuzzy s chart is done the same as that for fuzzy $\overline{x}$ chart. The results are summarized as follows:

(a) The endpoints of the α-level closed interval $(\widetilde{u}_{s_i})_\alpha = [(\widetilde{u}_{s_i})^L_\alpha, (\widetilde{u}_{s_i})^U_\alpha]$ of fuzzy upper control limit $\widetilde{u}_{s_i}$ are determined by

$$\begin{aligned} \left(\widetilde{u}_{s_i}\right)^L_\alpha &= \min_{\alpha\le\beta\le1} \min\left\{u^L_{s_i,\beta}, u^U_{s_i,\beta}\right\} \\ \left(\widetilde{u}_{s_i}\right)^U_\alpha &= \max_{\alpha\le\beta\le1} \max\left\{u^U_{s_i,\beta}, u^U_{s_i,\beta}\right\}. \end{aligned} \tag{15}$$

(b) The endpoints of the α-level closed interval $(\widetilde{l}_{s_i})_\alpha = [(\widetilde{l}_{s_i})^L_\alpha, (\widetilde{l}_{s_i})^U_\alpha]$ of fuzzy upper control limit $\widetilde{l}_{s_i}$ are determined by

$$\begin{aligned}\left(\widetilde{l}_{s_i}\right)^L_\alpha &= \min_{\alpha\le\beta\le1} \min\left\{l^L_{s_i,\beta}, l^U_{s_i,\beta}\right\}\\ \left(\widetilde{l}_{s_i}\right)^U_\alpha &= \max_{\alpha\le\beta\le1} \max\left\{l^U_{s_i,\beta}, l^U_{s_i,\beta}\right\}.\end{aligned} \tag{16}$$

3.3. Fuzzy Average $\widetilde{\overline{\overline{x}}}$ and Fuzzy Mean Standard Deviation $\widetilde{\overline{s}}$. In order to realize whether the fuzzy average $\widetilde{\overline{\overline{x}}}_i$ and fuzzy mean standard deviation $\widetilde{\overline{s}}_i$ are within the fuzzy control limits, we need to calculate their membership functions with the sample-statistics' fuzzy numbers (SSFNs) first. With a similar procedure for the CLFNs, the endpoints of the α-level closed interval $(\widetilde{\overline{\overline{x}}}_i)_\alpha = [(\widetilde{\overline{\overline{x}}}_i)^L_\alpha, (\widetilde{\overline{\overline{x}}}_i)^U_\alpha]$ of fuzzy average $\widetilde{\overline{\overline{x}}}_i$ are obtained by

$$\begin{aligned}\left(\widetilde{\overline{\overline{x}}}_i\right)^L_\alpha &= \min_{\alpha\le\beta\le1} \min\left\{\overline{x}^L_{i,\beta}, \overline{x}^U_{i,\beta}\right\}\\ \left(\widetilde{\overline{\overline{x}}}_i\right)^U_\alpha &= \max_{\alpha\le\beta\le1} \max\left\{\overline{x}^L_{i,\beta}, \overline{x}^U_{i,\beta}\right\}.\end{aligned} \tag{17}$$

And, the endpoints of the α-level closed interval $(\widetilde{\overline{s}}_i)_\alpha = [(\widetilde{\overline{s}}_i)^L_\alpha, (\widetilde{\overline{s}}_i)^U_\alpha]$ of fuzzy mean standard deviation $\widetilde{\overline{s}}_i$ are

$$\begin{aligned}\left(\widetilde{\overline{s}}_i\right)^L_\alpha &= \min_{\alpha\le\beta\le1} \min\left\{s^L_{i,\beta}, s^U_{i,\beta}\right\}\\ \left(\widetilde{\overline{s}}_i\right)^U_\alpha &= \max_{\alpha\le\beta\le1} \max\left\{s^L_{i,\beta}, s^U_{i,\beta}\right\}.\end{aligned} \tag{18}$$

In order to illustrate the practical applicability of the proposed fuzzy $\overline{x}$ and s control charts, a realistic example in monitoring and controlling surface roughness of optical lens on its turning process is presented in Section 3.4 as a paradigm for their data-adaptability.

3.4. Practical Application I. In recent years, optical lenses have become key components in many industrial products, such as digital cameras, microscope, and telescope [28]. Most of optical systems require accurate correspondence between the object and image, as well as high-quality image which can be achieved with high-quality and precisely centralized lens. A lens with better surface roughness significantly improves the optical resolution of the object image [33]. The quality of an optical lens depends not only on its design and its materials but also on the production process itself. The machining processes generate a wide variety of surface patterns, including lay, roughness, and waviness as illustrated in Figure 2. The effect of light scattering induced by the surface roughness has been well addressed in [34–37]. Literally, the less roughness on a lens surface the better [38, 39] because the roughness makes the light scattered as shown in Figure 3. Thus, surface roughness is one of the most important factors in evaluating the quality of a lens since quality lenses make the final optical products perform functionally as expected; that is, controlling the surface quality is a critical issue. However,

Table 1: The attained significance levels in Runs test.

Cut-points in Runs test	Median	Mean	Mode
Attained significance level	0.233	0.301	0.155

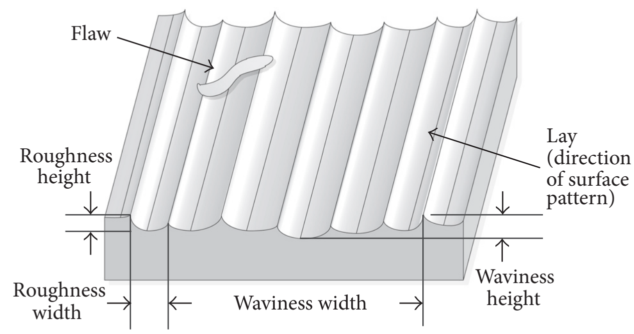

Figure 2: Roughness and waviness profiles.

as manufacturing condition is the key source of the surface texture irregularities which may form nucleation sites for cracks, imperfections, or quick corrosion [40], monitoring the manufacturing process becomes incredibly important in the optical lens industry.

The surface roughness is practically assessed by the height of the irregularities which can be measured with either contact methods using stylus profilometers [41] or noncontact methods including direct imaging using a commercial atomic force microscope in tapping mode [42], speckle-contrast method, light scattering method [43, 44], and so on. However, each method has certain disadvantages. A profilometer does not work accurately when the size of the features of the surface are quite close to the size of the stylus. Besides, it has difficulty in detecting flaws of the same size as the roughness of the surface. Under noncontact methods, some features that are less than some fraction of the frequency of their operating wavelength of light cannot be detected by the instruments that rely heavily on the optical inference [45]. Also, human eyes are capable of seeing the colors ranged from violet through red which lies in the wavelength range of 400–700 nanometers (nm), respectively [46]. Due to these problems, the surface roughness cannot be measured precisely; thus, the recorded value is interpreted as a triangular fuzzy number as depicted in Figure 4.

In this study, twenty-five samples have been randomly taken from a current turning process. Due to the random sampling, the sizes of the samples are variable. The collected data are shown in Table 6. As mentioned above, the uncertainty existing in this measurement could be affected by randomness and the phenomenon of fuzziness. Therefore, both randomness and fuzziness should be taken into consideration [4]. Runs test is considered as one of the good methods to test for the randomness of fuzzy numbers [47]. The attained significance levels for different cut-points (median, mean, and mode) under this test are shown in Table 1. As all of the levels

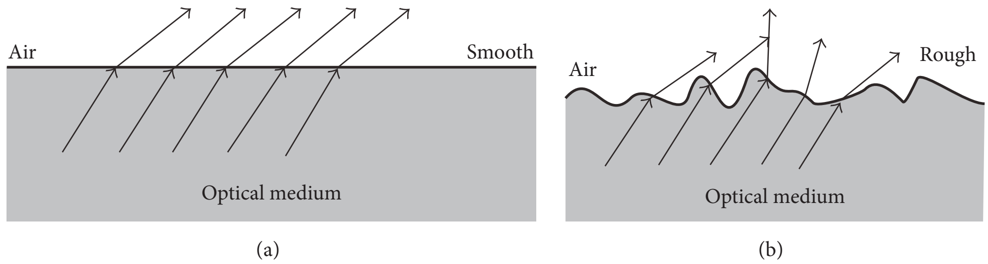

Figure 3: (a) Schema of lights propagation at smooth and (b) rough optical surface.

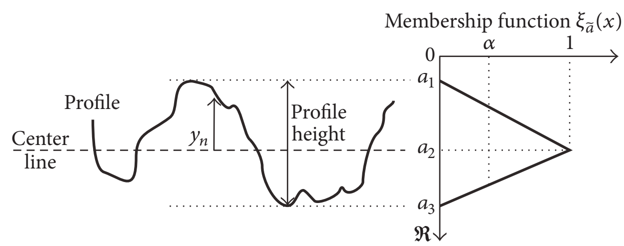

Figure 4: Schema of surface profile as produced by a stylus device.

are greater than 0.155, at a given significance level of 0.01 was used in this study, the surface roughness can obviously be considered as a random variable [2].

Moreover, it is critical to test if these data are normally distributed because normal distribution is the basic assumption to construct the control charts. The normal distribution is tested under Kolmogorov-Smirnov (K-S) statistics [31]. With our data, the attained significance level of 0.05 in this test indicates that the surface roughness is normally distributed. As a result, the data can be used to calculate the fuzzy control limits and construct fuzzy control charts. The fuzzy s chart and $\overline{x}$ chart with $K = 3$ for the twenty-five investigated samples are plotted in Figures 5 and 6, respectively.

In evaluating a fuzzy control chart, we need to compare SSFNs and CLFNs. Thus, a method for ranking fuzzy numbers is mandatory. Among several existing ranking approaches, a recent method proposed Yu and Dat is considered in this paper due to its claimed ability in not only providing consistent ranking results and easy applicability but also effectively ranking a mix of various types of fuzzy numbers [29]. The next section, Section 4, elucidates the development of our thorough classification mechanism based on a simplified approach of the chosen method.

4. Proposed Classification Mechanism

With fuzzy control charts shown in Figures 5 and 6, as each fuzzy observation needs compared with its fuzzy control limits only, for better efficiency, Yu and Dat's approach [29] can be efficiently simplified by adding an extra normal triangular fuzzy number $\tilde{0} = (0, 0, 0)$ which is used as a radical number to support our comparison. Specifically, consider a generalized normal fuzzy numbers $\tilde{a}_i = (b_i, c_i, d_i, e_i)$ whose membership functions are defined by

$$\xi_{\tilde{a}_i}(x) = \begin{cases} f^L_{\tilde{a}_i}(x) & \text{if } x \in [b_i, c_i] \\ 1 & \text{if } x \in [c_i, d_i] \\ f^R_{\tilde{a}_i}(x) & \text{if } x \in [d_i, e_i] \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Let $g^L_{\tilde{a}_i}(y)$ and $g^R_{\tilde{a}_i}(y)$ be the inverse functions of the $f^L_{\tilde{a}_i}(x)$ and $f^R_{\tilde{a}_i}(x)$, respectively. Then, the left and right integral values of $\tilde{a}_i$ to $\tilde{0}$, respectively, denoted by LV_i and RV_i, are calculated by

$$\begin{aligned} \mathrm{LV}_i &= \int_0^1 g^L_{\tilde{a}_i}(y)\, dy \\ \mathrm{RV}_i &= \int_0^1 g^R_{\tilde{a}_i}(y)\, dy \end{aligned} \tag{20}$$

By incorporating optimism level β, an index presenting subjective attitude of decision-maker ($\beta \in [0, 1]$), with LV_i and RV_i, our proposed ranking index of $\tilde{a}_i$, denoted by SV^β_i, is defined by

$$\mathrm{SV}^\beta_i = \beta \mathrm{RV}_i + (1 - \beta)\, \mathrm{LV}_i. \tag{21}$$

4.1. Monitoring Process Variability. In order to monitor process variability, we need to compare the collected fuzzy data $\tilde{s}_i$ with its fuzzy control limits $\tilde{u}_{s_i}$ and $\tilde{l}_{s_i}$. Hence, based on the ranking index in (21), the following procedure is suggested:

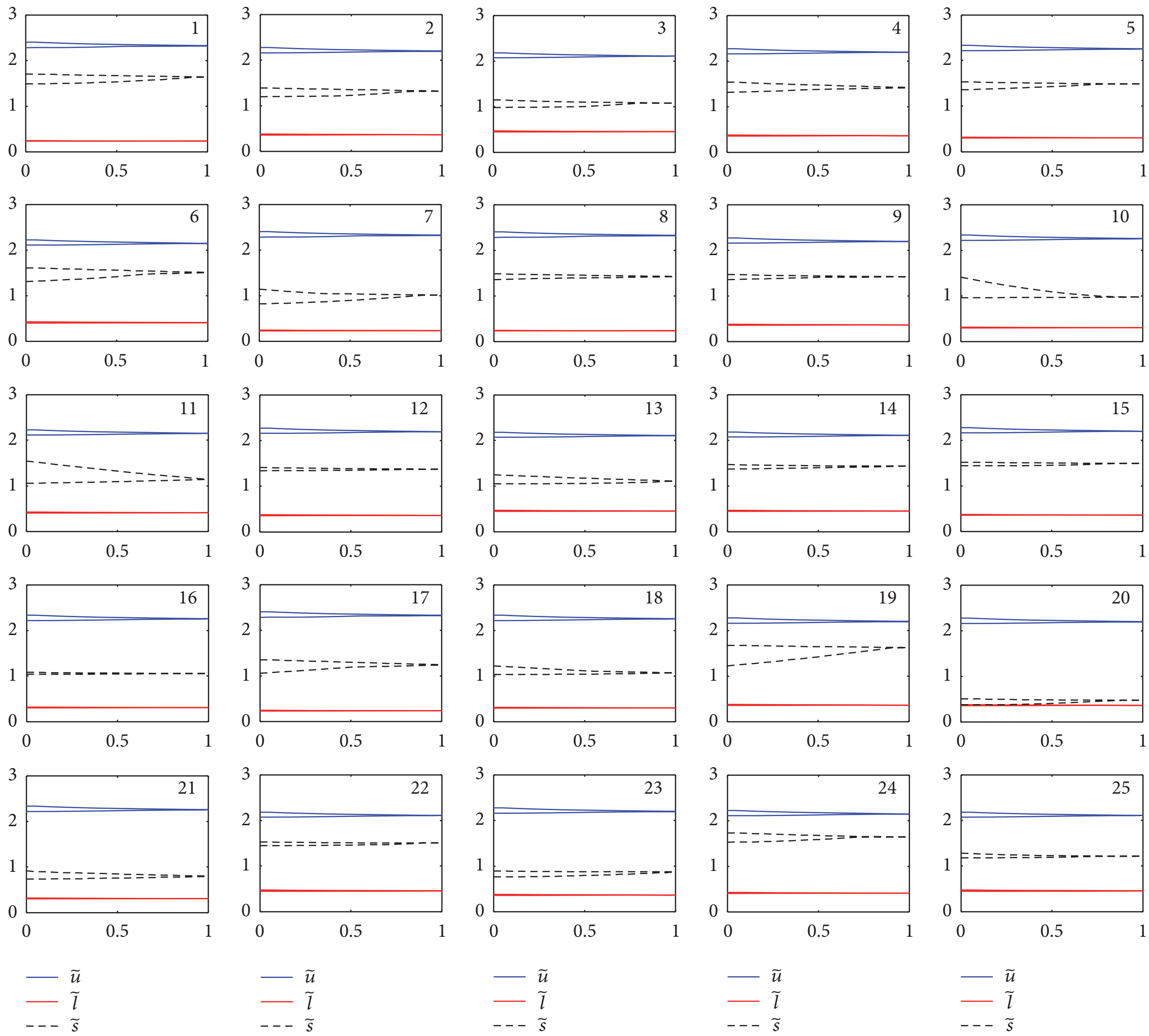

Figure 5: Fuzzy s chart for roughness height of optical lens.

(1) Calculate $\mathrm{SV}^{\beta}_{\tilde{u}_{s_i}}$, $\mathrm{SV}^{\beta}_{\tilde{s}_i}$, and $\mathrm{SV}^{\beta}_{\tilde{l}_{s_i}}$ for $\tilde{u}_{s_i}$, $\tilde{s}_i$, and $\tilde{l}_{s_i}$, respectively.

(2) For $\tilde{s}_i$, we calculate its standard deviation, denoted by Sd^{β}_{s} across the m samples by the following formulas:

$$\overline{\mathrm{SV}}^{\beta}_{\tilde{s}} = \frac{1}{m}\sum_{i=1}^{m}\mathrm{SV}^{\beta}_{\tilde{s}_i}$$
$$\mathrm{Sd}^{\beta}_{s} = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}\left(\mathrm{SV}^{\beta}_{\tilde{s}_i} - \overline{\mathrm{SV}}^{\beta}_{\tilde{s}}\right)^2}. \tag{22}$$

(3) For $\tilde{u}_{s_i}$, from the obtained $\mathrm{SV}^{\beta}_{\tilde{u}_{s_i}}$ and Sd^{β}_{s}, we establish two relevant control limits for $\tilde{u}_{s_i}$ as

$$\mathrm{SV}^{U,\beta}_{\tilde{u}_{s_i}} = \mathrm{SV}^{\beta}_{\tilde{u}_{s_i}} + \mathrm{Sd}^{\beta}_{s}$$
$$\mathrm{SV}^{L,\beta}_{\tilde{u}_{s_i}} = \mathrm{SV}^{\beta}_{\tilde{u}_{s_i}} - \mathrm{Sd}^{\beta}_{s}. \tag{23}$$

(4) For $\tilde{l}_{s_i}$, from the obtained $\mathrm{SV}^{\beta}_{\tilde{l}_{s_i}}$ and Sd^{β}_{s}, we establish two relevant control limits for $\tilde{l}_{s_i}$ as follows:

$$\mathrm{SV}^{U,\beta}_{\tilde{l}_{s_i}} = \mathrm{SV}^{\beta}_{\tilde{l}_{s_i}} + \mathrm{Sd}^{\beta}_{s}$$
$$\mathrm{SV}^{L,\beta}_{\tilde{l}_{s_i}} = \mathrm{SV}^{\beta}_{\tilde{l}_{s_i}} - \mathrm{Sd}^{\beta}_{s}. \tag{24}$$

(5) Before proposing the classification of the manufacturing process, we first reorder the values of $\mathrm{SV}^{U,\beta}_{\tilde{u}_{s_i}}$, $\mathrm{SV}^{\beta}_{\tilde{u}_{s_i}}$, $\mathrm{SV}^{L,\beta}_{\tilde{u}_{s_i}}$, $\mathrm{SV}^{U,\beta}_{\tilde{l}_{s_i}}$, $\mathrm{SV}^{\beta}_{\tilde{l}_{s_i}}$, and $\mathrm{SV}^{L,\beta}_{\tilde{l}_{s_i}}$ in a descending

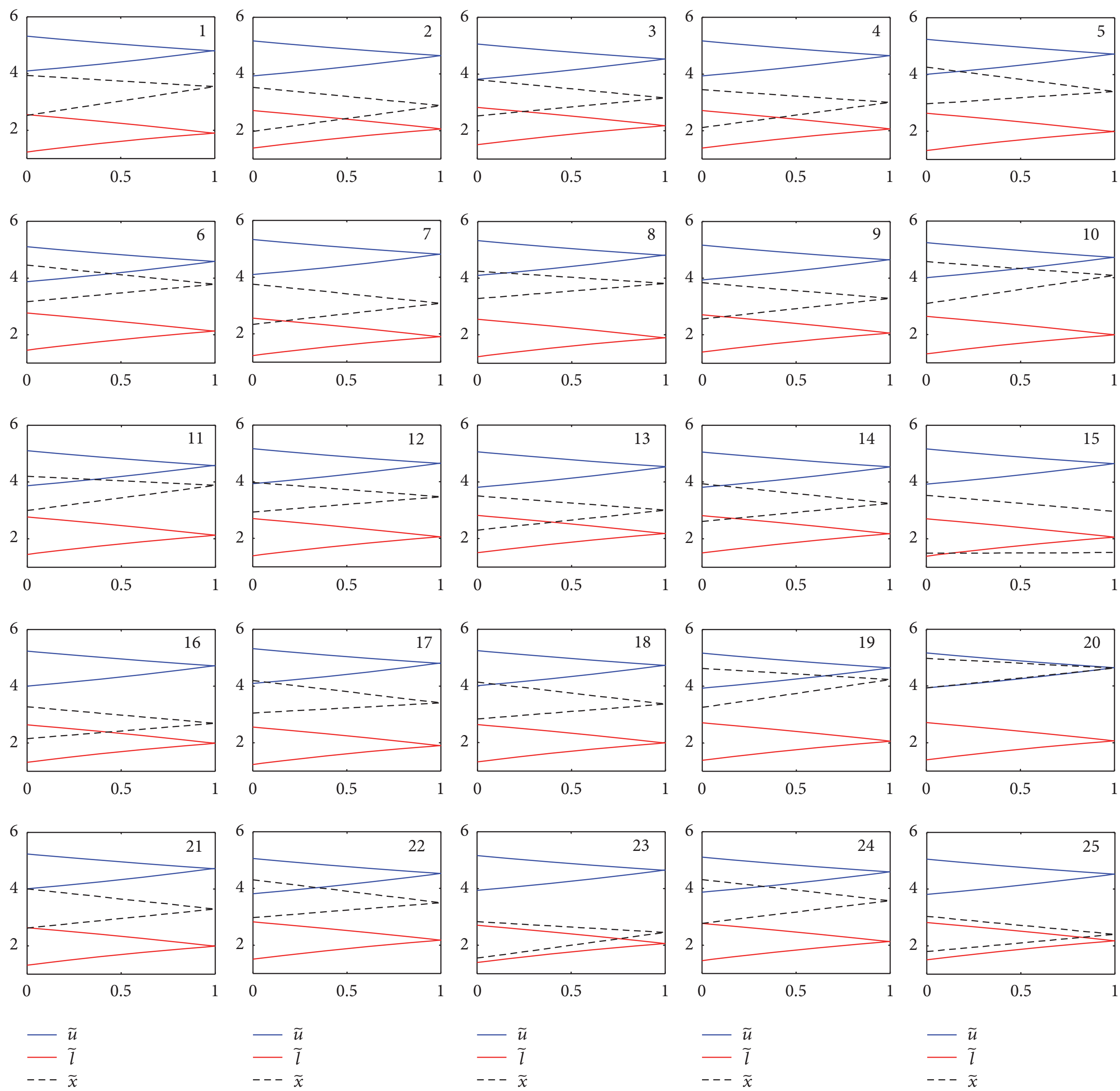

FIGURE 6: Fuzzy $\overline{x}$ chart for roughness height of optical lens.

order represented by six critical values S_i^{β} $(i = \overline{1,6})$ where $S_1^{\beta} > S_2^{\beta} > S_3^{\beta} > S_4^{\beta} > S_5^{\beta} > S_6^{\beta}$.

(6) Based on the six critical values, a manufacturing process can be classified based on the following rules:

(a) Process is in control at the optimism level β if the following condition happens:

$$S_4^{\beta} < \mathrm{SV}_{\tilde{s}_i}^{\beta} < S_3^{\beta}. \tag{25}$$

(b) Process is rather in control at the optimism level β if one of the following conditions happens:

(b1) $S_3^{\beta} \le \mathrm{SV}_{\tilde{s}_i}^{\beta} \le S_2^{\beta}$.

(b2) $S_5^{\beta} \le \mathrm{SV}_{\tilde{s}_i}^{\beta} \le S_4^{\beta}$.

(c) Process is rather out of control at the optimism level β if one of the following conditions happens:

(c1) $S_2^{\beta} < \mathrm{SV}_{\tilde{s}_i}^{\beta} < S_1^{\beta}$.

(c2) $S_6^{\beta} < \mathrm{SV}_{\tilde{s}_i}^{\beta} < S_5^{\beta}$.

(d) Process is out of control at the optimism level β if one of the following conditions happens:

(d1) $SV^{\beta}_{\widetilde{s}_i} \geq S^{\beta}_1$.

(d2) $SV^{\beta}_{\widetilde{s}_i} \leq S^{\beta}_6$.

4.2. Monitoring Process Average. In monitoring process average, we need to compare the collected fuzzy data with its fuzzy control limits $\widetilde{u}_{\overline{x}_i}$ and $\widetilde{l}_{\overline{x}_i}$. Similarly, we suggest the following procedure:

(1) Calculate $SV^{\beta}_{\widetilde{u}_{\overline{x}_i}}$, $SV^{\beta}_{\widetilde{\overline{x}}_i}$, and $SV^{\beta}_{\widetilde{l}_{\overline{x}_i}}$ for $\widetilde{u}_{\overline{x}_i}$, $\widetilde{\overline{x}}_i$, and $\widetilde{l}_{\overline{x}_i}$, respectively.

(2) For $\widetilde{\overline{x}}_i$, we calculate its standard deviation, denoted by $Sd^{\beta}_{\overline{x}}$ across the m samples by the following formulas:

$$\overline{SV}^{\beta}_{\widetilde{\overline{x}}} = \frac{1}{m}\sum_{i=1}^{m} SV^{\beta}_{\widetilde{\overline{x}}_i}$$
$$Sd^{\beta}_{\overline{x}} = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}\left(SV^{\beta}_{\widetilde{\overline{x}}_i} - \overline{SV}^{\beta}_{\widetilde{\overline{x}}}\right)^2}. \tag{26}$$

(3) For $\widetilde{u}_{\overline{x}_i}$, from the obtained $SV^{\beta}_{\widetilde{u}_{\overline{x}_i}}$ and $Sd^{\beta}_{\overline{x}}$, we establish two relevant control limits for $\widetilde{u}_{\overline{x}_i}$ as follows:

$$SV^{U,\beta}_{\widetilde{u}_{\overline{x}_i}} = SV^{\beta}_{\widetilde{u}_{\overline{x}_i}} + Sd^{\beta}_{\overline{x}}$$
$$SV^{L,\beta}_{\widetilde{u}_{\overline{x}_i}} = SV^{\beta}_{\widetilde{u}_{\overline{x}_i}} - Sd^{\beta}_{\overline{x}}. \tag{27}$$

(4) For $\widetilde{l}_{\overline{x}_i}$, from the obtained $SV^{\beta}_{\widetilde{l}_{\overline{x}_i}}$ and $Sd^{\beta}_{\overline{x}}$, we establish two relevant control limits for $\widetilde{l}_{\overline{x}_i}$ as follows:

$$SV^{U,\beta}_{\widetilde{l}_{\overline{x}_i}} = SV^{\beta}_{\widetilde{l}_{\overline{x}_i}} + Sd^{\beta}_{\overline{x}}$$
$$SV^{L,\beta}_{\widetilde{l}_{\overline{x}_i}} = SV^{\beta}_{\widetilde{l}_{\overline{x}_i}} - Sd^{\beta}_{\overline{x}}. \tag{28}$$

(5) Before proposing the classification of the manufacturing process, we first reorder the values of $SV^{U,\beta}_{\widetilde{u}_{\overline{x}_i}}$, $SV^{\beta}_{\widetilde{u}_{\overline{x}_i}}$, $SV^{L,\beta}_{\widetilde{u}_{\overline{x}_i}}$, $SV^{U,\beta}_{\widetilde{l}_{\overline{x}_i}}$, $SV^{\beta}_{\widetilde{l}_{\overline{x}_i}}$, and $SV^{L,\beta}_{\widetilde{l}_{\overline{x}_i}}$ in a descending order represented by six critical values S^{β}_i $(i = \overline{1,6})$ where $S^{\beta}_1 > S^{\beta}_2 > S^{\beta}_3 > S^{\beta}_4 > S^{\beta}_5 > S^{\beta}_6$.

(6) Then the manufacturing process can be classified based on the following rules:

(a) Process is in control at the optimism level β if the following condition happens:

$S^{\beta}_4 < SV^{\beta}_{\widetilde{\overline{x}}_i} < S^{\beta}_3$.

(b) Process is rather in control at the optimism level β if one of the following conditions happen:

(b1) $S^{\beta}_3 \leq SV^{\beta}_{\widetilde{\overline{x}}_i} \leq S^{\beta}_2$.

(b2) $S^{\beta}_5 \leq SV^{\beta}_{\widetilde{\overline{x}}_i} \leq S^{\beta}_4$.

(c) Process is rather out of control at the optimism level β if one of the following conditions happens:

(c1) $S^{\beta}_2 < SV^{\beta}_{\widetilde{\overline{x}}_i} < S^{\beta}_1$.

(c2) $S^{\beta}_6 < SV^{\beta}_{\widetilde{\overline{x}}_i} < S^{\beta}_5$.

(d) Process is out of control at the optimism level β if one of the following conditions happens:

(d1) $SV^{\beta}_{\widetilde{\overline{x}}_i} \geq S^{\beta}_1$.

(d2) $SV^{\beta}_{\widetilde{\overline{x}}_i} \leq S^{\beta}_6$.

4.3. Practical Application II. The above classification mechanism is now employed in evaluating the fuzzy control charts shown in Figures 5 and 6. Particularly, with the simplified left and right integral value approach presented in Section 4, the six critical values in the 5th step for the process variability and process average across the different optimism levels are, respectively, displayed in Tables 4 and 5. Moreover, with the suggested classification rules for the process variability and process average, the current manufacturing process can be classified as in Tables 2 and 3 where we briefly present the numerical results at five optimism levels 0.1, 0.3, 0.5, 0.7, and 0.9 for brevity (full results are supplemented on request). The numerical results in Table 2 indicate that the current process variability is in control although the process variability of the 20th sample is considered acceptable (rather in control) only. The in-control process variability is well considered as the first and important criterion to further analyze the process under $\overline{x}$ chart because its control limits depend on the process variability. Moreover, if assignable causes of the variability are detected and eliminated, no systematic pattern is found on $\overline{x}$ chart [1].

Based on Figure 6 and the numerical results the twenty-five samples investigated in Table 3, five alarming signals, including 15th, 19th, 20th, 23rd, and 25th samples, are detected. Among them, 19th, 23rd, and 25th samples are considered *rather in-control (R-In)*, while a pessimistic decision-maker (at optimism level 0.1) believes that the 15th and 20th samples are *rather out-of-control (R-Out)* and if he/she is more optimistic, these two samples can be still claimed to be *rather in-control (R-In)*; that is, the incorporation of the human subjectivity, practitioner's experience, and optimism level λ in our proposed ranking index provides certain flexibility in the process evaluation. Ultimately, with such outliers, a careful investigation is needed to look for the assignable causes occurred in the production of these mentioned samples. Once detected, the causes should be completely eliminated from the current process; or at least, their impacts escorting the process variability must be significantly reduced with proper corrective actions undertaken in order to improve the surface quality of the manufactured optical lenses.

From managerial perspective, the classification with four intermediate levels plays important role in avoiding unnecessary adjustments to the current process, thus avoiding vain expenditures and operational risks. Specifically, for the

Table 2: Process variability classification.

Number	$SV^{\beta}_{\tilde{\tilde{s}}_i}$					Status				
	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
(S.1)	1.640	1.666	1.691	1.717	1.743	In	In	In	In	In
(S.2)	1.321	1.344	1.366	1.389	1.412	In	In	In	In	In
(S.3)	1.083	1.101	1.118	1.136	1.153	In	In	In	In	In
(S.4)	1.451	1.473	1.496	1.518	1.540	In	In	In	In	In
(S.5)	1.508	1.524	1.540	1.556	1.572	In	In	In	In	In
(S.6)	1.503	1.533	1.563	1.593	1.624	In	In	In	In	In
(S.7)	0.970	1.001	1.031	1.061	1.091	In	In	In	In	In
(S.8)	1.470	1.483	1.495	1.507	1.520	In	In	In	In	In
(S.9)	1.472	1.482	1.491	1.500	1.509	In	In	In	In	In
(S.10)	1.034	1.068	1.102	1.135	1.169	In	In	In	In	In
(S.11)	1.175	1.225	1.274	1.324	1.373	In	In	In	In	In
(S.12)	1.426	1.433	1.441	1.448	1.456	In	In	In	In	In
(S.13)	1.132	1.155	1.178	1.201	1.224	In	In	In	In	In
(S.14)	1.477	1.487	1.496	1.505	1.515	In	In	In	In	In
(S.15)	1.538	1.547	1.556	1.565	1.575	In	In	In	In	In
(S.16)	1.101	1.105	1.108	1.112	1.116	In	In	In	In	In
(S.17)	1.242	1.269	1.296	1.324	1.351	In	In	In	In	In
(S.18)	1.108	1.126	1.144	1.162	1.180	In	In	In	In	In
(S.19)	1.524	1.570	1.616	1.661	1.707	In	In	In	In	In
(S.20)	0.445	0.460	0.474	0.489	0.503	R-In	R-In	R-In	R-In	R-In
(S.21)	0.806	0.825	0.844	0.863	0.882	In	In	In	In	In
(S.22)	1.545	1.554	1.564	1.573	1.583	In	In	In	In	In
(S.23)	0.849	0.866	0.882	0.898	0.915	In	In	In	In	In
(S.24)	1.680	1.699	1.718	1.737	1.757	In	In	In	In	In
(S.25)	1.258	1.267	1.276	1.284	1.293	In	In	In	In	In

Notes. In: *in-control*; R-In: *rather in-control*.

Table 3: Process average classification.

Number	$SV^{\beta}_{\tilde{\tilde{x}}_i}$					Process status				
	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
(S.1)	3.257	3.405	3.552	3.700	3.848	In	In	In	In	In
(S.2)	2.631	2.794	2.957	3.120	3.284	In	In	In	In	In
(S.3)	3.046	3.180	3.314	3.448	3.582	In	In	In	In	In
(S.4)	2.750	2.890	3.031	3.171	3.311	In	In	In	In	In
(S.5)	3.407	3.543	3.679	3.815	3.951	In	In	In	In	In
(S.6)	3.714	3.849	3.984	4.120	4.255	In	In	In	In	In
(S.7)	2.913	3.062	3.211	3.360	3.509	In	In	In	In	In
(S.8)	3.770	3.872	3.973	4.075	4.176	In	In	In	In	In
(S.9)	3.133	3.267	3.401	3.535	3.669	In	In	In	In	In
(S.10)	3.841	3.996	4.151	4.306	4.460	In	In	In	In	In
(S.11)	3.675	3.801	3.927	4.054	4.180	In	In	In	In	In
(S.12)	3.415	3.525	3.634	3.744	3.853	In	In	In	In	In
(S.13)	2.847	2.973	3.099	3.225	3.352	In	In	In	In	In
(S.14)	3.143	3.283	3.422	3.562	3.701	In	In	In	In	In
(S.15)	1.760	2.127	2.494	2.861	3.228	R-Out	R-In	R-In	R-In	In
(S.16)	2.599	2.717	2.835	2.953	3.071	In	In	In	In	In
(S.17)	3.447	3.567	3.688	3.808	3.929	In	In	In	In	In
(S.18)	3.319	3.455	3.591	3.728	3.864	In	In	In	In	In
(S.19)	4.001	4.146	4.290	4.434	4.579	R-In	R-In	R-In	R-In	R-In
(S.20)	4.546	4.655	4.765	4.875	4.985	R-Out	R-In	R-In	R-In	R-In
(S.21)	3.179	3.323	3.468	3.613	3.758	In	In	In	In	In
(S.22)	3.463	3.603	3.743	3.883	4.024	In	In	In	In	In
(S.23)	2.163	2.298	2.433	2.568	2.703	R-In	R-In	R-In	R-In	R-In
(S.24)	3.407	3.568	3.730	3.892	4.054	In	In	In	In	In
(S.25)	2.269	2.399	2.529	2.660	2.790	R-In	R-In	R-In	R-In	R-In

Notes. R-Out: *rather out-of-control*; R-In: *rather in-control*.

Table 4: Critical values to evaluate fuzzy s-chart.

	λ	S_1^λ	S_2^λ	S_3^λ	S_4^λ	S_5^λ	S_6^λ
	0.1	2.724	2.425	2.126	0.546	0.247	−0.052
	0.3	2.735	2.437	2.138	0.547	0.248	−0.051
(S.1)	0.5	2.747	2.448	2.149	0.548	0.249	−0.050
	0.7	2.760	2.460	2.160	0.550	0.250	−0.050
	0.9	2.772	2.471	2.170	0.553	0.251	−0.050
	0.1	2.591	2.292	1.993	0.679	0.380	0.081
	0.3	2.602	2.303	2.004	0.681	0.382	0.083
(S.2)	0.5	2.613	2.314	2.015	0.683	0.383	0.084
	0.7	2.625	2.325	2.025	0.685	0.385	0.085
	0.9	2.637	2.336	2.035	0.688	0.387	0.086
	0.1	2.498	2.199	1.899	0.772	0.473	0.174
	0.3	2.508	2.209	1.910	0.774	0.475	0.176
(S.3)	0.5	2.519	2.220	1.920	0.777	0.478	0.178
	0.7	2.530	2.230	1.930	0.780	0.480	0.180
	0.9	2.542	2.241	1.940	0.783	0.482	0.181
	0.1	2.591	2.292	1.993	0.679	0.380	0.081
	0.3	2.602	2.303	2.004	0.681	0.382	0.083
(S.4)	0.5	2.613	2.314	2.015	0.683	0.383	0.084
	0.7	2.625	2.325	2.025	0.685	0.385	0.085
	0.9	2.637	2.336	2.035	0.688	0.387	0.086
	0.1	2.652	2.352	2.053	0.618	0.319	0.020
	0.3	2.663	2.364	2.065	0.620	0.321	0.022
(S.5)	0.5	2.674	2.375	2.076	0.622	0.322	0.023
	0.7	2.686	2.386	2.086	0.624	0.324	0.024
	0.9	2.698	2.397	2.096	0.626	0.325	0.024
	0.1	2.541	2.242	1.942	0.729	0.430	0.131
	0.3	2.551	2.252	1.953	0.731	0.432	0.133
(S.6)	0.5	2.562	2.263	1.964	0.733	0.434	0.135
	0.7	2.574	2.274	1.974	0.736	0.436	0.136
	0.9	2.585	2.284	1.983	0.739	0.438	0.137
	0.1	2.724	2.425	2.126	0.546	0.247	−0.052
	0.3	2.735	2.437	2.138	0.547	0.248	−0.051
(S.7)	0.5	2.747	2.448	2.149	0.548	0.249	−0.050
	0.7	2.760	2.460	2.160	0.550	0.250	−0.050
	0.9	2.772	2.471	2.170	0.553	0.251	−0.050
	0.1	2.724	2.425	2.126	0.546	0.247	−0.052
	0.3	2.735	2.437	2.138	0.547	0.248	−0.051
(S.8)	0.5	2.747	2.448	2.149	0.548	0.249	−0.050
	0.7	2.760	2.460	2.160	0.550	0.250	−0.050
	0.9	2.772	2.471	2.170	0.553	0.251	−0.050
	0.1	2.591	2.292	1.993	0.679	0.380	0.081
	0.3	2.602	2.303	2.004	0.681	0.382	0.083
(S.9)	0.5	2.613	2.314	2.015	0.683	0.383	0.084
	0.7	2.625	2.325	2.025	0.685	0.385	0.085
	0.9	2.637	2.336	2.035	0.688	0.387	0.086
	0.1	2.652	2.352	2.053	0.618	0.319	0.020
	0.3	2.663	2.364	2.065	0.620	0.321	0.022
(S.10)	0.5	2.674	2.375	2.076	0.622	0.322	0.023
	0.7	2.686	2.386	2.086	0.624	0.324	0.024
	0.9	2.698	2.397	2.096	0.626	0.325	0.024
	0.1	2.541	2.242	1.942	0.729	0.430	0.131
	0.3	2.551	2.252	1.953	0.731	0.432	0.133
(S.11)	0.5	2.562	2.263	1.964	0.733	0.434	0.135
	0.7	2.574	2.274	1.974	0.736	0.436	0.136
	0.9	2.585	2.284	1.983	0.739	0.438	0.137

Table 4: Continued.

	λ	S_1^λ	S_2^λ	S_3^λ	S_4^λ	S_5^λ	S_6^λ
	0.1	2.591	2.292	1.993	0.679	0.380	0.081
	0.3	2.602	2.303	2.004	0.681	0.382	0.083
(S.12)	0.5	2.613	2.314	2.015	0.683	0.383	0.084
	0.7	2.625	2.325	2.025	0.685	0.385	0.085
	0.9	2.637	2.336	2.035	0.688	0.387	0.086
	0.1	2.498	2.199	1.899	0.772	0.473	0.174
	0.3	2.508	2.209	1.910	0.774	0.475	0.176
(S.13)	0.5	2.519	2.220	1.920	0.777	0.478	0.178
	0.7	2.530	2.230	1.930	0.780	0.480	0.180
	0.9	2.542	2.241	1.940	0.783	0.482	0.181
	0.1	2.498	2.199	1.899	0.772	0.473	0.174
	0.3	2.508	2.209	1.910	0.774	0.475	0.176
(S.14)	0.5	2.519	2.220	1.920	0.777	0.478	0.178
	0.7	2.530	2.230	1.930	0.780	0.480	0.180
	0.9	2.542	2.241	1.940	0.783	0.482	0.181
	0.1	2.591	2.292	1.993	0.679	0.380	0.081
	0.3	2.602	2.303	2.004	0.681	0.382	0.083
(S.15)	0.5	2.613	2.314	2.015	0.683	0.383	0.084
	0.7	2.625	2.325	2.025	0.685	0.385	0.085
	0.9	2.637	2.336	2.035	0.688	0.387	0.086
	0.1	2.652	2.352	2.053	0.618	0.319	0.020
	0.3	2.663	2.364	2.065	0.620	0.321	0.022
(S.16)	0.5	2.674	2.375	2.076	0.622	0.322	0.023
	0.7	2.686	2.386	2.086	0.624	0.324	0.024
	0.9	2.698	2.397	2.096	0.626	0.325	0.024
	0.1	2.724	2.425	2.126	0.546	0.247	−0.052
	0.3	2.735	2.437	2.138	0.547	0.248	−0.051
(S.17)	0.5	2.747	2.448	2.149	0.548	0.249	−0.050
	0.7	2.760	2.460	2.160	0.550	0.250	−0.050
	0.9	2.772	2.471	2.170	0.553	0.251	−0.050
	0.1	2.652	2.352	2.053	0.618	0.319	0.020
	0.3	2.663	2.364	2.065	0.620	0.321	0.022
(S.18)	0.5	2.674	2.375	2.076	0.622	0.322	0.023
	0.7	2.686	2.386	2.086	0.624	0.324	0.024
	0.9	2.698	2.397	2.096	0.626	0.325	0.024
	0.1	2.591	2.292	1.993	0.679	0.380	0.081
	0.3	2.602	2.303	2.004	0.681	0.382	0.083
(S.19)	0.5	2.613	2.314	2.015	0.683	0.383	0.084
	0.7	2.625	2.325	2.025	0.685	0.385	0.085
	0.9	2.637	2.336	2.035	0.688	0.387	0.086
	0.1	2.591	2.292	1.993	0.679	0.380	0.081
	0.3	2.602	2.303	2.004	0.681	0.382	0.083
(S.20)	0.5	2.613	2.314	2.015	0.683	0.383	0.084
	0.7	2.625	2.325	2.025	0.685	0.385	0.085
	0.9	2.637	2.336	2.035	0.688	0.387	0.086
	0.1	2.652	2.352	2.053	0.618	0.319	0.020
	0.3	2.663	2.364	2.065	0.620	0.321	0.022
(S.21)	0.5	2.674	2.375	2.076	0.622	0.322	0.023
	0.7	2.686	2.386	2.086	0.624	0.324	0.024
	0.9	2.698	2.397	2.096	0.626	0.325	0.024
	0.1	2.498	2.199	1.899	0.772	0.473	0.174
	0.3	2.508	2.209	1.910	0.774	0.475	0.176
(S.22)	0.5	2.519	2.220	1.920	0.777	0.478	0.178
	0.7	2.530	2.230	1.930	0.780	0.480	0.180
	0.9	2.542	2.241	1.940	0.783	0.482	0.181

Table 4: Continued.

	λ	S_1^λ	S_2^λ	S_3^λ	S_4^λ	S_5^λ	S_6^λ
(S.23)	0.1	2.591	2.292	1.993	0.679	0.380	0.081
	0.3	2.602	2.303	2.004	0.681	0.382	0.083
	0.5	2.613	2.314	2.015	0.683	0.383	0.084
	0.7	2.625	2.325	2.025	0.685	0.385	0.085
	0.9	2.637	2.336	2.035	0.688	0.387	0.086
(S.24)	0.1	2.541	2.242	1.942	0.729	0.430	0.131
	0.3	2.551	2.252	1.953	0.731	0.432	0.133
	0.5	2.562	2.263	1.964	0.733	0.434	0.135
	0.7	2.574	2.274	1.974	0.736	0.436	0.136
	0.9	2.585	2.284	1.983	0.739	0.438	0.137
(S.25)	0.1	2.498	2.199	1.899	0.772	0.473	0.174
	0.3	2.508	2.209	1.910	0.774	0.475	0.176
	0.5	2.519	2.220	1.920	0.777	0.478	0.178
	0.7	2.530	2.230	1.930	0.780	0.480	0.180
	0.9	2.542	2.241	1.940	0.783	0.482	0.181

"rather in-control" process, no intervention to the process is suggested unless its cost is bearable. On the other hand, for "rather out-of-control" process, mediating the manufacturing process is strongly required if either the setup cost is small or tolerable or the gains of quality products outweigh the large expenditure for possible intervention.

5. Conclusion

In monitoring a manufacturing process, the conventional control charts are only applicable for real-valued data and categorizing the process as either in control or out of control; however, due to some certain problems in measuring, the data are not precisely collected, which is said to be fuzzy data. Therefore, in the fuzzy environment, the traditional control charts turn out to be inappropriate. Hence, in this paper, we proposed fuzzy $\overline{x}$ and s control charts whose control limits are obtained based on the results of the resolution identity in the well-known fuzzy set theory. Moreover, in order to monitor the process based on these fuzzy control charts, by simplifying a recently proposed ranking method based on the left and right integral value, we developed thorough evaluation rules which can classify a manufacturing process in the fuzzy environment into four different linguistic statuses, including *in-control, out-of-control, rather out-of-control*, and *rather in-control*. Basically, since the incorporation of optimism level into the ranking index provides critical flexibility in decision-making procedure, decision-makers can take their own advantages and experiences in implementing proper actions in order to fully control the quality of manufactured products. In the empirical case study of monitoring the surface roughness of optical lens with fuzzy data in manufacturing processes, both randomness and fuzziness are

Table 5: Critical values to evaluate fuzzy $\overline{x}$-chart.

	λ	S_1^λ	S_2^λ	S_3^λ	S_4^λ	S_5^λ	S_6^λ
(S.1)	0.1	5.326	4.710	4.094	2.352	1.736	1.121
	0.3	5.434	4.842	4.250	2.464	1.873	1.281
	0.5	5.545	4.973	4.402	2.580	2.009	1.438
	0.7	5.659	5.105	4.551	2.699	2.145	1.591
	0.9	5.778	5.237	4.696	2.822	2.282	1.741
(S.2)	0.1	5.158	4.543	3.927	2.519	1.904	1.288
	0.3	5.266	4.674	4.083	2.632	2.040	1.448
	0.5	5.378	4.806	4.235	2.747	2.176	1.605
	0.7	5.492	4.938	4.384	2.866	2.312	1.758
	0.9	5.611	5.070	4.530	2.989	2.448	1.907
(S.3)	0.1	5.037	4.422	3.806	2.640	2.025	1.409
	0.3	5.145	4.554	3.962	2.752	2.161	1.569
	0.5	5.257	4.686	4.115	2.868	2.297	1.725
	0.7	5.372	4.818	4.264	2.986	2.432	1.878
	0.9	5.491	4.950	4.409	3.109	2.568	2.028
(S.4)	0.1	5.158	4.543	3.927	2.519	1.904	1.288
	0.3	5.266	4.674	4.083	2.632	2.040	1.448
	0.5	5.378	4.806	4.235	2.747	2.176	1.605
	0.7	5.492	4.938	4.384	2.866	2.312	1.758
	0.9	5.611	5.070	4.530	2.989	2.448	1.907
(S.5)	0.1	5.234	4.619	4.003	2.443	1.827	1.212
	0.3	5.342	4.751	4.159	2.555	1.964	1.372
	0.5	5.454	4.882	4.311	2.671	2.100	1.529
	0.7	5.568	5.014	4.460	2.790	2.236	1.682
	0.9	5.687	5.146	4.605	2.913	2.372	1.832
(S.6)	0.1	5.093	4.478	3.862	2.584	1.969	1.353
	0.3	5.201	4.610	4.018	2.696	2.105	1.513
	0.5	5.313	4.742	4.171	2.812	2.241	1.669
	0.7	5.428	4.874	4.320	2.931	2.377	1.823
	0.9	5.547	5.006	4.465	3.053	2.513	1.972
(S.7)	0.1	5.326	4.710	4.094	2.352	1.736	1.121
	0.3	5.434	4.842	4.250	2.464	1.873	1.281
	0.5	5.545	4.973	4.402	2.580	2.009	1.438
	0.7	5.659	5.105	4.551	2.699	2.145	1.591
	0.9	5.778	5.237	4.696	2.822	2.282	1.741
(S.8)	0.1	5.326	4.710	4.094	2.352	1.736	1.121
	0.3	5.434	4.842	4.250	2.464	1.873	1.281
	0.5	5.545	4.973	4.402	2.580	2.009	1.438
	0.7	5.659	5.105	4.551	2.699	2.145	1.591
	0.9	5.778	5.237	4.696	2.822	2.282	1.741
(S.9)	0.1	5.158	4.543	3.927	2.519	1.904	1.288
	0.3	5.266	4.674	4.083	2.632	2.040	1.448
	0.5	5.378	4.806	4.235	2.747	2.176	1.605
	0.7	5.492	4.938	4.384	2.866	2.312	1.758
	0.9	5.611	5.070	4.530	2.989	2.448	1.907
(S.10)	0.1	5.234	4.619	4.003	2.443	1.827	1.212
	0.3	5.342	4.751	4.159	2.555	1.964	1.372
	0.5	5.454	4.882	4.311	2.671	2.100	1.529
	0.7	5.568	5.014	4.460	2.790	2.236	1.682
	0.9	5.687	5.146	4.605	2.913	2.372	1.832

Table 5: Continued.

	λ	S_1^λ	S_2^λ	S_3^λ	S_4^λ	S_5^λ	S_6^λ
	0.1	5.093	4.478	3.862	2.584	1.969	1.353
	0.3	5.201	4.610	4.018	2.696	2.105	1.513
(S.11)	0.5	5.313	4.742	4.171	2.812	2.241	1.669
	0.7	5.428	4.874	4.320	2.931	2.377	1.823
	0.9	5.547	5.006	4.465	3.053	2.513	1.972
	0.1	5.158	4.543	3.927	2.519	1.904	1.288
	0.3	5.266	4.674	4.083	2.632	2.040	1.448
(S.12)	0.5	5.378	4.806	4.235	2.747	2.176	1.605
	0.7	5.492	4.938	4.384	2.866	2.312	1.758
	0.9	5.611	5.070	4.530	2.989	2.448	1.907
	0.1	5.037	4.422	3.806	2.640	2.025	1.409
	0.3	5.145	4.554	3.962	2.752	2.161	1.569
(S.13)	0.5	5.257	4.686	4.115	2.868	2.297	1.725
	0.7	5.372	4.818	4.264	2.986	2.432	1.878
	0.9	5.491	4.950	4.409	3.109	2.568	2.028
	0.1	5.037	4.422	3.806	2.640	2.025	1.409
	0.3	5.145	4.554	3.962	2.752	2.161	1.569
(S.14)	0.5	5.257	4.686	4.115	2.868	2.297	1.725
	0.7	5.372	4.818	4.264	2.986	2.432	1.878
	0.9	5.491	4.950	4.409	3.109	2.568	2.028
	0.1	5.158	4.543	3.927	2.519	1.904	1.288
	0.3	5.266	4.674	4.083	2.632	2.040	1.448
(S.15)	0.5	5.378	4.806	4.235	2.747	2.176	1.605
	0.7	5.492	4.938	4.384	2.866	2.312	1.758
	0.9	5.611	5.070	4.530	2.989	2.448	1.907
	0.1	5.234	4.619	4.003	2.443	1.827	1.212
	0.3	5.342	4.751	4.159	2.555	1.964	1.372
(S.16)	0.5	5.454	4.882	4.311	2.671	2.100	1.529
	0.7	5.568	5.014	4.460	2.790	2.236	1.682
	0.9	5.687	5.146	4.605	2.913	2.372	1.832
	0.1	5.326	4.710	4.094	2.352	1.736	1.121
	0.3	5.434	4.842	4.250	2.464	1.873	1.281
(S.17)	0.5	5.545	4.973	4.402	2.580	2.009	1.438
	0.7	5.659	5.105	4.551	2.699	2.145	1.591
	0.9	5.778	5.237	4.696	2.822	2.282	1.741
	0.1	5.234	4.619	4.003	2.443	1.827	1.212
	0.3	5.342	4.751	4.159	2.555	1.964	1.372
(S.18)	0.5	5.454	4.882	4.311	2.671	2.100	1.529
	0.7	5.568	5.014	4.460	2.790	2.236	1.682
	0.9	5.687	5.146	4.605	2.913	2.372	1.832
	0.1	5.158	4.543	3.927	2.519	1.904	1.288
	0.3	5.266	4.674	4.083	2.632	2.040	1.448
(S.19)	0.5	5.378	4.806	4.235	2.747	2.176	1.605
	0.7	5.492	4.938	4.384	2.866	2.312	1.758
	0.9	5.611	5.070	4.530	2.989	2.448	1.907
	0.1	5.158	4.543	3.927	2.519	1.904	1.288
	0.3	5.266	4.674	4.083	2.632	2.040	1.448
(S.20)	0.5	5.378	4.806	4.235	2.747	2.176	1.605
	0.7	5.492	4.938	4.384	2.866	2.312	1.758
	0.9	5.611	5.070	4.530	2.989	2.448	1.907

Table 5: Continued.

	λ	S_1^λ	S_2^λ	S_3^λ	S_4^λ	S_5^λ	S_6^λ
	0.1	5.234	4.619	4.003	2.443	1.827	1.212
	0.3	5.342	4.751	4.159	2.555	1.964	1.372
(S.21)	0.5	5.454	4.882	4.311	2.671	2.100	1.529
	0.7	5.568	5.014	4.460	2.790	2.236	1.682
	0.9	5.687	5.146	4.605	2.913	2.372	1.832
	0.1	5.037	4.422	3.806	2.640	2.025	1.409
	0.3	5.145	4.554	3.962	2.752	2.161	1.569
(S.22)	0.5	5.257	4.686	4.115	2.868	2.297	1.725
	0.7	5.372	4.818	4.264	2.986	2.432	1.878
	0.9	5.491	4.950	4.409	3.109	2.568	2.028
	0.1	5.158	4.543	3.927	2.519	1.904	1.288
	0.3	5.266	4.674	4.083	2.632	2.040	1.448
(S.23)	0.5	5.378	4.806	4.235	2.747	2.176	1.605
	0.7	5.492	4.938	4.384	2.866	2.312	1.758
	0.9	5.611	5.070	4.530	2.989	2.448	1.907
	0.1	5.093	4.478	3.862	2.584	1.969	1.353
	0.3	5.201	4.610	4.018	2.696	2.105	1.513
(S.24)	0.5	5.313	4.742	4.171	2.812	2.241	1.669
	0.7	5.428	4.874	4.320	2.931	2.377	1.823
	0.9	5.547	5.006	4.465	3.053	2.513	1.972
	0.1	5.037	4.422	3.806	2.640	2.025	1.409
	0.3	5.145	4.554	3.962	2.752	2.161	1.569
(S.25)	0.5	5.257	4.686	4.115	2.868	2.297	1.725
	0.7	5.372	4.818	4.264	2.986	2.432	1.878
	0.9	5.491	4.950	4.409	3.109	2.568	2.028

taken into consideration to avoid potential bias and loss of efficiency. By comparing the fuzzy averages and variances to their respective fuzzy control limits, our proposed control charts have effectively detected key alarming signals whose underlying roots need to be carefully investigated so that corrective actions can be undertaken in order to either preclude them from the process or reduce their effect on the variation in the processes to improve the surface quality of the optical lenses. In addition, the intermediate classification provided by our proposed control charts practically signifies the trade-off between the intervention cost and the expected quality gain; thus, our approach can obviously avoid not only unnecessary adjustment to the current process but also the potential loss in their business. As such, our proposed fuzzy control charts can fulfill the current literature of the conventional $\overline{x}$ chart and s chart in terms of effectiveness when fuzzy data inevitably present in the manufacturing processes.

Appendix

See Tables 4, 5, and 6.

Table 6: Roughness height (μm) of optical lens.

Obser. number		(S.1)			(S.2)			(S.3)			(S.4)	
(1)	0.76	1.03	1.34	0.79	1.16	2.12	3.46	3.68	3.86	3.04	4.79	4.84
(2)	4.87	5.31	5.59	2.26	2.79	3.49	2.68	2.72	2.85	4.38	5.11	5.29
(3)	3.58	4.55	5.14	3.08	3.22	4.16	2.15	4.85	5.50	0.84	1.63	1.75
(4)	2.76	3.22	3.85	1.02	1.62	1.88	1.85	2.18	2.62	0.80	0.96	1.35
(5)	2.46	3.28	3.67	0.57	3.24	4.02	1.14	2.00	2.29	2.04	2.96	3.69
(6)	0.96	1.32	1.45	3.92	4.51	5.00	3.82	4.11	4.60	0.89	3.26	3.52
(7)	4.27	4.64	5.34	0.76	1.10	1.79	4.05	4.61	4.95	3.22	4.01	4.43
(8)	0.53	4.96	5.07	1.45	2.04	2.95	2.99	3.53	3.79	3.19	4.16	4.65
(9)				3.31	4.17	4.34	3.55	4.08	4.39	2.10	2.24	2.27
(10)				2.56	4.98	5.52	1.16	2.10	2.78	0.57	0.86	2.63
(11)							2.64	2.73	3.72			
(12)							0.74	1.26	4.21			
Obser. number		(S.5)			(S.6)			(S.7)			(S.8)	
(1)	2.04	2.27	4.20	4.98	5.23	5.39	3.49	4.05	5.00	3.20	3.65	3.95
(2)	1.31	1.58	2.30	3.05	3.87	5.42	0.81	1.25	3.01	3.19	3.62	3.99
(3)	3.34	4.27	4.78	1.01	1.45	3.98	0.54	3.07	3.95	1.34	1.61	2.35
(4)	1.17	1.93	2.77	2.46	3.07	3.27	2.20	2.52	2.66	1.26	1.72	2.29
(5)	4.78	4.83	5.48	2.78	3.29	4.18	2.29	2.35	2.99	3.72	4.51	4.52
(6)	5.21	5.96	6.31	1.78	1.95	2.74	3.59	4.33	4.60	4.95	5.50	5.63
(7)	2.93	3.75	4.65	4.48	5.19	5.54	2.59	3.00	3.71	4.40	5.28	5.93
(8)	4.02	4.03	4.99	1.89	2.41	2.77	3.14	4.04	4.08	4.15	4.58	5.28
(9)	1.86	1.96	2.83	4.95	5.87	5.89						
(10)	2.25	3.11	3.43	5.17	6.15	6.37						
Obser. number		(S.9)			(S.10)			(S.11)			(S.12)	
(1)	1.63	2.54	3.39	0.79	2.14	3.02	3.45	4.69	4.94	2.67	3.50	4.13
(2)	4.33	4.73	5.50	0.88	4.51	4.96	4.98	5.11	5.21	4.43	4.66	4.86
(3)	0.96	3.54	4.02	4.92	5.52	6.19	4.53	5.01	5.12	4.94	5.52	5.69
(4)	2.75	3.02	3.59	3.04	3.32	3.34	4.92	5.22	5.41	2.71	3.01	3.56
(5)	2.92	3.34	3.68	3.40	3.84	4.26	3.26	4.10	4.28	4.64	5.47	6.39
(6)	1.33	1.66	2.26	3.25	3.98	4.21	1.27	3.17	3.83	0.99	1.34	1.40
(7)	4.73	5.59	5.79	4.01	4.25	4.68	2.29	3.11	3.22	1.62	1.98	2.93
(8)	1.19	1.52	2.43	3.28	4.12	5.20	3.03	3.67	3.73	3.06	3.22	3.58
(9)	1.49	1.78	2.23	4.25	5.02	5.23	3.62	4.33	5.15	2.04	2.90	3.51
(10)	4.24	5.10	5.45				1.06	1.67	2.29	2.21	3.10	3.69
(11)							0.55	2.63	3.01			
Obser. number		(S.13)			(S.14)			(S.15)			(S.16)	
(1)	1.90	2.36	3.14	1.61	1.86	2.32	4.87	5.54	6.29	2.49	3.17	4.07
(2)	3.08	4.07	4.75	4.36	5.16	6.12	0.59	0.80	1.36	3.96	4.30	4.68
(3)	4.11	4.80	5.09	1.23	2.02	2.05	1.63	2.52	2.61	1.95	2.80	3.69
(4)	2.05	2.41	3.12	3.65	4.26	4.99	1.27	1.71	1.88	0.64	2.02	2.67
(5)	1.33	1.76	1.96	3.89	4.48	5.30	2.08	2.91	3.12	2.45	2.67	3.07
(6)	1.91	2.44	3.24	1.53	2.03	2.50	3.42	3.71	3.77	2.41	2.50	3.49
(7)	1.92	2.75	3.41	3.06	3.84	4.30	4.53	4.61	4.84	0.58	0.86	1.27
(8)	4.04	4.87	5.85	3.62	4.51	4.92	2.36	2.99	3.97	1.83	1.92	2.22
(9)	1.20	1.32	1.38	1.99	2.83	3.77	3.38	3.82	4.34	3.02	3.97	4.28
(10)	0.65	3.16	3.52	0.46	0.93	2.31	0.89	1.05	3.14			
(11)	2.43	2.72	2.93	1.47	2.11	2.96						
(12)	2.98	3.36	3.64	4.42	4.93	5.69						
Obser. number		(S.17)			(S.18)			(S.19)				
(1)	3.29	3.99	4.48	4.04	4.24	5.20	1.87	1.93	2.37			

Table 6: Continued.

(2)	1.72	1.88	2.75	4.15	4.34	5.00	3.48	4.03	4.44
(3)	4.63	4.98	5.20	4.03	4.55	4.84	0.45	0.59	1.18
(4)	3.48	4.42	5.24	1.51	1.67	2.16	4.78	5.40	6.34
(5)	2.35	2.46	3.16	2.30	2.94	3.88	3.80	4.30	4.59
(6)	4.28	4.39	5.27	1.95	2.47	2.59	3.52	5.10	5.46
(7)	3.41	3.85	4.41	3.78	4.17	5.12	3.68	4.85	5.15
(8)	1.21	1.26	3.05	2.82	3.48	4.31	3.60	5.05	5.47
(9)				0.92	2.36	4.09	3.24	5.46	5.62
(10)							4.10	5.61	5.65
Obser. number		(S.20)			(S.21)			(S.22)	
(1)	3.83	4.35	5.41	2.43	3.25	3.87	0.93	0.99	1.95
(2)	3.40	4.42	4.60	2.52	2.89	3.64	3.90	4.61	5.04
(3)	4.64	5.21	5.35	2.69	2.74	3.82	2.92	3.60	4.47
(4)	4.20	5.51	5.64	2.86	4.14	4.85	3.99	4.42	5.11
(5)	4.18	4.40	4.85	1.63	1.80	1.93	0.76	1.92	4.96
(6)	3.62	4.62	4.80	3.29	3.32	3.92	4.69	4.95	5.74
(7)	3.86	4.62	4.94	0.76	3.21	4.12	2.81	3.22	3.60
(8)	4.05	5.09	5.34	3.47	3.73	4.64	4.57	5.40	6.23
(9)	4.04	4.26	4.80	3.98	4.54	5.25	0.83	0.92	1.30
(10)	3.44	3.80	3.98				2.35	3.28	3.97
(11)	3.05	3.60	3.98						
(12)	4.83	5.01	5.30						
Obser. number		(S.23)			(S.24)			(S.25)	
(1)	1.61	2.57	2.87	0.95	1.03	1.35	2.28	2.69	3.05
(2)	1.07	1.56	1.80	5.26	6.10	6.41	1.20	1.86	3.57
(3)	1.56	1.71	1.82	3.96	4.19	4.43	0.68	1.25	3.61
(4)	2.17	3.16	3.62	2.02	2.42	2.48	2.19	2.9	3.25
(5)	0.86	3.26	3.84	2.52	5.31	5.41	3.18	3.75	3.94
(6)	0.56	2.84	3.14	3.60	4.37	4.88	3.35	3.91	4.27
(7)	0.46	1.11	1.55	2.56	3.20	4.16	0.47	0.76	1.33
(8)	1.48	1.57	2.38	3.66	4.11	4.75	0.72	1.01	1.08
(9)	2.70	3.43	3.96	4.84	5.36	6.29	0.68	2.96	3.04
(10)	2.96	3.28	3.31	0.61	1.86	3.24	1.83	2.26	3.16
(11)				0.49	1.26	4.02	0.65	1.13	1.18
(12)							4.31	4.34	4.97

Notes. Obser. number: observation number; S.: sample number.

Abbreviations

SSFNs: Sample-statistics' fuzzy numbers
CLFNs: Control-limits' fuzzy numbers
CL: Center line
UCL: Upper control limit
LCL: Lower control limit
NISD: Necessity index of strict dominance
DFA: Direct fuzzy approach
FDA: Fuzzy dominance approach
LV: Left integral values
RV: Right integral values
R-In: Rather in-control
R-Out: Rather out-of-control.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

The authors all reviewed the current literature relating to these two control charts in Section 1 and worked on the classification mechanism. Dinh-Chien Dang and Bi-Min Hsu focused more on Section 2, while the fuzzy control charts in Section 3 were developed by Ming-Hung Shu and Thanh-Lam Nguyen. Ngoc-Son Phan developed the Matlab codes for the construction and classification of the control charts. Dinh-Chien Dang and Bi-Min Hsu collected the data which were then analyzed by Ming-Hung Shu and Thanh-Lam Nguyen. Ming-Hung Shu, Dinh-Chien Dang, Thanh-Lam Nguyen, and Ngoc-Son Phan wrote the paper.

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