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Research Article

FTC with Dynamic Virtual Actuators: Characterization via Dynamic Output Controllers and H_∞ Approach

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The paper presents new conditions, adequate in design of dynamic virtual actuators and utilizable in fault-tolerant control structures (FTC) for continuous-time linear systems, which are stabilizable by dynamic output controllers. Taking into account disturbance conditions and changes of variables in FTC after virtual actuator activation and applying the nominal control scheme relating to the biproper dynamic output controller of prescribed order, the design conditions are outlined in terms of the linear matrix inequalities within the enhanced bounded real lemma forms. Using a free tuning parameter in design, and with suitable choice of the controller order, the approach provides the way to obtain acceptable dynamics of the closed-loop system after activation of the dynamic virtual actuator.

1. Introduction

To increase the reliability of systems, FTC usually fix a system with faults so that it can continue its mission with certain limitations of functionality and quality. Considering this, the different approaches were studied in FTC design (see, e.g., [1, 2] and the references therein). The standard way of control reconfiguration discards the nominal controller from the control loop and replaces it with a new one so that its parameters are retuned in occurred fault conditions and, in dependency on the remaining set of sensors and actuators, to recover in a certain extent the performance given on the fault-free control system [3, 4]. Reconfiguration criterions for FTC are presented, for example, in [5, 6].

By contrast, instead of adapting the controller to the faulty plant, the virtual approach keeps the nominal controller in the reconfigured closed-loop system and virtually adapts the faulty plant to the nominal controller in such a way that the activated virtual reconfiguration block, together with the faulty plant, imitates the fault-free plant. The reconfiguration block is chosen so as to hide the fault from the controller point of view (the fault-hiding paradigm) and the approach tries to offer a way for the minimum invasive control reconfiguration. Since in healthy conditions the virtual reconfiguration blocks are not active, and the control action is realized by the nominal controller, the design of the virtual reconfiguration blocks is independent of the controller and can be aimed at preserving prescribed reduced closed-loop properties of the control in the presence of faults. Designated to sensor faults the reconfiguration block is termed virtual sensor (VS), while in the case of actuator faults is named virtual actuator (VA). In particular, an FTC strategy based on virtual actuator approach for linear piecewise affine systems with actuator faults is presented in [7], for nonlinear systems that can be approximated by linear parameter-varying (LPV) models in discrete-time or continuous-time description; this policy is proposed in [8–10] and [11], respectively, and applying to continuous-time Lipschitz nonlinear systems, this practice is introduced in [12]. Some practical case studies of using VA for linear systems are treated, for example, in [13, 14].

Until the first ideas of control reconfiguration by using VA, given for linear systems in [15, 16], could be summarized in the books [17, 18], several aspects have been used for VA design. Introducing the generalized virtual actuator [19], it was shown in [20] that reconfiguration after an actuator fault can be related to disturbance decoupling. Then, subsequently, H_∞-based virtual actuator synthesis for optimal trajectory recovery was presented in [21] and the dual principle was conveyed in the VA design methods [22]. In general, these conditions for VA design are formulated in terms of a finite set of linear matrix inequalities (LMI) for the static VAs. Some actual modifications for the static proportional-integral (PI) VAs can be found, for example, in [23–25].

Although the use of static VAs is not bound to the static output controllers (SOCs) [26, 27], in the vast majority of applications the control reconfiguration, exploiting static VA in faulty systems, is realized in conjunction with such type of regulators. This structure results in that the response of the subsystem for fault detection and isolation has to be fast, and, in addition to that, the peaks of the system output and control variables, immediately after the activation of a VA, are excessively high [28]. By adapting the dynamic output controller (DOC) design strategies given for LPV models in [29, 30], and considering characteristic conditions and changes of variables in FTC after VA activation, the synthesis of DOCs was stated in this context in [31, 32] and [33], respectively. Extending these results, the technique proposed in the paper is given by the above introduced virtual manner so that a single actuator fault in FTC structure is hidden for the DOC inputs by the dynamic VA (DVA). Based on the concept of quadratic stability, the design problems, respecting the H_∞ norm of the disturbance transfer matrix in DOC design, as well as the generalized disturbance transfer matrix in DVA design, are transferred into standard LMI optimization tasks, which includes enhanced bounded real lemma (BRL) formalism [34, 35]. To the best of the authors' knowledge, the paper presents a new formulation of the DVA design principle and newly defines the scheme relating to the order of DOC and DVA, respectively.

The paper is organized as follows. In Section 2, the H_∞ approach is presented with results on BRL and enhanced BRL for DOC design. Formulating the separation principle, and continuing with this formalism for DVA state-space description in Section 3, the equivalent BRL based design methods are outlined for DVAs. Finally, the example is given in Section 4 to illustrate the feasibility and properties of the proposed method and some concluding remarks are stated in Section 5.

Throughout the paper, the following notations are used: $\mathbf{x}^T$ and $\mathbf{X}^T$ denote the transpose of the vector $\mathbf{x}$ and the matrix $\mathbf{X}$, respectively, diag$[\{\mathbf{X}_i\}, i = 1, 2, \ldots, p]$ denotes a block diagonal matrix with p blocks, rank$(\cdot)$ remits the rank of a matrix, for a square matrix $\mathbf{X} < 0$ means that $\mathbf{X}$ is a symmetric negative definite matrix, the symbol $\mathbf{I}_n$ indicates the nth order unit matrix, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ and $\mathbb{R}^{n\times r}$ refer to the set of all n-dimensional real vectors and $n \times r$ real matrices, respectively, and $L_2\langle 0, +\infty)$ entails the space of square integrable functions over $\langle 0, +\infty)$.

2. Dynamic Output Controllers

In the paper, the continuous-time linear dynamic systems described in fault-free conditions as

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{u}_c(t) + \mathbf{V}\mathbf{v}(t), \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) \tag{2}$$

are taken into account, where $\mathbf{q}(t) \in \mathbb{R}^n$ stands for the system state, $\mathbf{u}_c(t) \in \mathbb{R}^r$ denotes the control input, $\mathbf{y}(t) \in \mathbb{R}^m$ is the measurable output, $\mathbf{v}(t) \in \mathbb{R}^{r_v}$ is the vector of unknown disturbance, the matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times r}$, $\mathbf{C} \in \mathbb{R}^{m\times n}$, and $\mathbf{V} \in \mathbb{R}^{n\times r_v}$ are finite valued. It is supposed that the exogenous disturbance is a nonanticipative process $\mathbf{v}(t) \in L_2(\langle 0, \infty); \mathbb{R}^{r_v})$.

It is assumed that the system is controlled by biproper DOC of the form

$$\dot{\mathbf{p}}(t) = \mathbf{J}\mathbf{p}(t) + \mathbf{L}\mathbf{y}(t), \tag{3}$$

$$\mathbf{u}_c(t) = \mathbf{M}\mathbf{p}(t) + \mathbf{N}\mathbf{y}(t) \tag{4}$$

and of an order p, where it can be accepted $1 \le p < n$ (reduced order), $p = n$ (full order), and $n < p \le p_m$ (upgraded order), while $\mathbf{p}(t) \in \mathbb{R}^p$ is the vector of the controller state variables. With respect to the real matrices $\mathbf{J} \in \mathbb{R}^{p\times p}$, $\mathbf{L} \in \mathbb{R}^{p\times m}$, $\mathbf{M} \in \mathbb{R}^{r\times p}$, and $\mathbf{N} \in \mathbb{R}^{r\times m}$, the controller parameter notation takes for $\mathbf{K}^\bullet \in \mathbb{R}^{(p+r)\times(p+m)}$ the following prescribed structure:

$$\mathbf{K}^\bullet = \begin{bmatrix} \mathbf{J} & \mathbf{L} \\ \mathbf{M} & \mathbf{N} \end{bmatrix}. \tag{5}$$

The objective is to design DOC to support the FTC structure with VAs so that the impact of the system disturbance $\mathbf{v}(t)$, expressed in terms of the H_∞ norm of the closed-loop disturbance transfer function matrix, is minimized in the mode where, after a single actuator fault, VA is used to mask the fault effects in the input of the controller.

To analyze the stability of the closed-loop system structure with DOC ((3) and (4)), the following form of the system description can be formulated:

$$\begin{gathered} \begin{bmatrix} \dot{\mathbf{q}}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} + \mathbf{BNC} & \mathbf{BM} \\ \mathbf{LC} & \mathbf{J} \end{bmatrix} \begin{bmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{V} \\ \mathbf{0} \end{bmatrix} \mathbf{v}(t), \\ \mathbf{y}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{I}_p \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{bmatrix}. \end{gathered} \tag{6}$$

Introducing the notations

$$\mathbf{q}^{\bullet T}(t) = \begin{bmatrix} \mathbf{q}^T(t) & \mathbf{p}^T(t) \end{bmatrix}, \tag{7}$$

$$\begin{gathered} \mathbf{A}^\bullet = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \\ \mathbf{B}^\bullet = \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{I}_p & \mathbf{0} \end{bmatrix}, \\ \mathbf{C}^\bullet = \begin{bmatrix} \mathbf{0} & \mathbf{I}_p \\ \mathbf{C} & \mathbf{0} \end{bmatrix}, \end{gathered} \tag{8}$$

$$\begin{gathered} \mathbf{V}^{\bullet T} = \begin{bmatrix} \mathbf{V}^T & \mathbf{0} \end{bmatrix}, \\ \mathbf{I}^\bullet = \begin{bmatrix} \mathbf{0} & \mathbf{I}_m \end{bmatrix}, \end{gathered} \tag{9}$$

where $\mathbf{A}^{\bullet} \in \mathbb{R}^{(n+p)\times(n+p)}$, $\mathbf{B}^{\bullet} \in \mathbb{R}^{(n+p)\times(p+r)}$, $\mathbf{C}^{\bullet} \in \mathbb{R}^{(p+m)\times(n+p)}$, $\mathbf{V}^{\bullet} \in \mathbb{R}^{(n+p)\times r_v}$, and $\mathbf{I}^{\bullet} \in \mathbb{R}^{m\times(p+m)}$, the closed-loop state-space equations take the form

$$\dot{\mathbf{q}}^{\bullet}(t) = \mathbf{A}_c^{\bullet}\mathbf{q}^{\bullet}(t) + \mathbf{V}^{\bullet}\mathbf{v}(t), \quad (10)$$

$$\mathbf{y}^{\bullet}(t) = \mathbf{I}^{\bullet}\mathbf{C}^{\bullet}\mathbf{q}^{\bullet}(t), \quad (11)$$

where, with $\mathbf{K}^{\bullet}$ given in (5),

$$\mathbf{A}_c^{\bullet} = \mathbf{A}^{\bullet} + \mathbf{B}^{\bullet}\mathbf{K}^{\bullet}\mathbf{C}^{\bullet}. \quad (12)$$

In the sequel, it is supposed that $(\mathbf{A}^{\bullet}, \mathbf{B}^{\bullet})$ is stabilizable and $(\mathbf{A}^{\bullet}, \mathbf{C}^{\bullet})$ is detectable [36].

Lemma 1 (see [33] (bounded real lemma)). *The closed-loop system, consisting of the plant (1), (2), and DOC (6), is stable with the quadratic performance $\gamma^{\bullet}$ if there exist a symmetric positive definite matrix $\mathbf{Q}^{\bullet} \in \mathbb{R}^{(n+p)\times(n+p)}$, a regular matrix $\mathbf{H}^{\bullet} \in \mathbb{R}^{(p+m)\times(p+m)}$, a matrix $\mathbf{Y}^{\bullet} \in \mathbb{R}^{(p+r)\times(p+m)}$, and a positive scalar $\gamma^{\bullet} \in \mathbb{R}$ such that*

$$\mathbf{Q}^{\bullet} = \mathbf{Q}^{\bullet T} > 0,$$
$$\gamma^{\bullet} > 0,$$
$$\begin{bmatrix} \mathbf{A}^{\bullet}\mathbf{Q}^{\bullet} + \mathbf{Q}^{\bullet}\mathbf{A}^{\bullet T} + \mathbf{B}^{\bullet}\mathbf{Y}^{\bullet}\mathbf{C}^{\bullet} + \mathbf{C}^{\bullet T}\mathbf{Y}^{\bullet T}\mathbf{B}^{\bullet T} & * & * \\ \mathbf{V}^{\bullet T} & -\gamma^{\bullet}\mathbf{I}_{r_v} & * \\ \mathbf{I}^{\bullet}\mathbf{C}^{\bullet}\mathbf{Q}^{\bullet} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0, \quad (13)$$
$$\mathbf{C}^{\bullet}\mathbf{Q}^{\bullet} = \mathbf{H}^{\bullet}\mathbf{C}^{\bullet},$$

where the generalized system matrices are defined in (8) and (9).

When the above conditions hold,

$$\mathbf{K}^{\bullet} = \mathbf{Y}^{\bullet}\left(\mathbf{H}^{\bullet}\right)^{-1}. \quad (14)$$

Here and hereafter, $*$ denotes the symmetric item in a symmetric matrix.

In order to adjust fault detection and isolation time to the dynamics of the closed-loop system, the selection of the order p of the DOC is provided with a free tuning parameter in control design. One serviceable method is based on incorporation of a slack matrix into LMI design conditions. This augmentation is proposed in the following theorem.

Theorem 2 (enhanced bounded real lemma). *The closed-loop system, consisting of the plant (1), (2), and the DOC (6), is stable with the quadratic performance $\gamma^{\bullet}$ if for the given positive scalar $\delta^{\bullet} \in \mathbb{R}$ there exist symmetric positive definite matrices $\mathbf{R}^{\bullet}$, $\mathbf{U}^{\bullet} \in \mathbb{R}^{(n+p)\times(n+p)}$, a regular matrix $\mathbf{H}^{\bullet} \in \mathbb{R}^{(p+m)\times(p+m)}$, a matrix $\mathbf{Y}^{\bullet} \in \mathbb{R}^{(p+r)\times(p+m)}$, and a positive scalar $\gamma^{\bullet} \in \mathbb{R}$ such that*

$$\mathbf{R}^{\bullet} = \mathbf{R}^{\bullet T} > 0,$$
$$\mathbf{U}^{\bullet} = \mathbf{U}^{\bullet T} > 0, \quad (15)$$
$$\gamma^{\bullet} > 0,$$
$$\begin{bmatrix} \mathbf{A}^{\bullet}\mathbf{R}^{\bullet} + \mathbf{R}^{\bullet}\mathbf{A}^{\bullet T} + \mathbf{B}^{\bullet}\mathbf{Y}^{\bullet}\mathbf{C}^{\bullet} + \mathbf{C}^{\bullet T}\mathbf{Y}^{\bullet T}\mathbf{B}^{\bullet T} & * & * & * \\ \mathbf{U}^{\bullet} - \mathbf{R}^{\bullet} + \delta\mathbf{A}^{\bullet}\mathbf{R}^{\bullet} + \delta\mathbf{B}^{\bullet}\mathbf{Y}^{\bullet}\mathbf{C}^{\bullet} & -2\delta^{\bullet}\mathbf{R}^{\bullet} & * & * \\ \mathbf{V}^{\bullet T} & \delta^{\bullet}\mathbf{V}^{\bullet T} & -\gamma^{\bullet}\mathbf{I}_{r_v} & * \\ \mathbf{I}^{\bullet}\mathbf{C}^{\bullet}\mathbf{R}^{\bullet} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0, \quad (16)$$
$$\mathbf{C}^{\bullet}\mathbf{R}^{\bullet} = \mathbf{H}^{\bullet}\mathbf{C}^{\bullet}, \quad (17)$$

where the generalized system matrices are defined in (8) and (9) and the positive $\delta^{\bullet} \in \mathbb{R}$ is the tuning parameter.

When the above conditions hold,

$$\mathbf{K}^{\bullet} = \mathbf{Y}^{\bullet}\left(\mathbf{H}^{\bullet}\right)^{-1}. \quad (18)$$

Proof. Since the differential equation (10) can be rewritten as

$$\mathbf{A}_c^{\bullet}\mathbf{q}^{\bullet}(t) + \mathbf{V}^{\bullet}\mathbf{v}(t) - \dot{\mathbf{q}}^{\bullet}(t) = \mathbf{0}, \quad (19)$$

then with an arbitrary symmetric positive definite matrix $\mathbf{S}^{\bullet} \in \mathbb{R}^{(n+p)\times(n+p)}$ and a positive scalar $\delta^{\bullet} \in \mathbb{R}$ it yields [37, 38]

$$\left(\mathbf{q}^{\bullet T}(t)\mathbf{S}^{\bullet} + \delta^{\bullet}\dot{\mathbf{q}}^{\bullet T}(t)\mathbf{S}^{\bullet}\right)\left(\mathbf{A}_c^{\bullet}\mathbf{q}^{\bullet}(t) + \mathbf{V}^{\bullet}\mathbf{v}(t) - \dot{\mathbf{q}}^{\bullet}(t)\right) = 0. \quad (20)$$

Defining the Lyapunov function candidate as follows:

$$v\left(\mathbf{q}^{\bullet}(t)\right) = \mathbf{q}^{\bullet T}(t)\mathbf{P}^{\bullet}\mathbf{q}^{\bullet}(t) + \int_0^t \left(\mathbf{y}^T(\tau)\mathbf{y}(\tau) - \gamma^{\bullet}\mathbf{v}^T(\tau)\mathbf{v}(\tau)\right)d\tau > 0, \quad (21)$$

where $\mathbf{P}^{\bullet} > 0$ is symmetric positive definite and $\sqrt{\gamma^{\bullet}} > 0$ is H_{∞} norm of the closed-loop transfer matrix between the disturbance input and the system output, then

$$\dot{v}\left(\mathbf{q}^{\bullet}(t)\right) = \dot{\mathbf{q}}^{\bullet T}(t)\mathbf{P}^{\bullet}\mathbf{q}^{\bullet}(t) + \mathbf{q}^{\bullet T}(t)\mathbf{P}^{\bullet}\dot{\mathbf{q}}^{\bullet}(t) + \mathbf{y}^T(t)\mathbf{y}(t) - \gamma^{\bullet}\mathbf{v}^T(t)\mathbf{v}(t) < 0. \quad (22)$$

Therefore, adding (20) and its transposition to (22) gives

$$\begin{aligned} \dot{v}\left(\mathbf{q}^{\bullet}(t)\right) &= \dot{\mathbf{q}}^{\bullet T}(t)\mathbf{P}^{\bullet}\mathbf{q}^{\bullet}(t) + \mathbf{q}^{\bullet T}(t)\mathbf{P}^{\bullet}\dot{\mathbf{q}}^{\bullet}(t) + \mathbf{q}^{\bullet T}(t) \\ &\cdot \mathbf{C}^{\bullet T}\mathbf{I}^{\bullet T}\mathbf{I}^{\bullet}\mathbf{C}^{\bullet}\mathbf{q}^{\bullet}(t) - \gamma^{\bullet}\mathbf{v}^T(t)\mathbf{v}(t) \\ &+ \left(\mathbf{q}^{\bullet T}(t)\mathbf{S}^{\bullet} + \delta^{\bullet}\dot{\mathbf{q}}^{\bullet T}(t)\mathbf{S}^{\bullet}\right) \\ &\cdot \left(\mathbf{A}_c^{\bullet}\mathbf{q}^{\bullet}(t) + \mathbf{V}^{\bullet}\mathbf{v}(t) - \dot{\mathbf{q}}^{\bullet}(t)\right) \\ &+ \left(\mathbf{q}^{\bullet T}(t)\mathbf{A}_c^{\bullet T} + \mathbf{v}^T(t)\mathbf{V}^{\bullet T} - \dot{\mathbf{q}}^{\bullet T}(t)\right) \\ &\cdot \left(\mathbf{S}^{\bullet}\mathbf{q}^{\bullet}(t) + \delta^{\bullet}\mathbf{S}^{\bullet}\dot{\mathbf{q}}^{\bullet}(t)\right) < 0. \end{aligned} \quad (23)$$

Using the following notation:

$$\mathbf{q}_c^{\bullet T}(t) = \left[\mathbf{q}^{\bullet T}(t)\ \ \dot{\mathbf{q}}^{\bullet T}(t)\ \ \mathbf{v}^T(t)\right], \tag{24}$$

the derivative of the Lyapunov function (23) can be written as

$$\dot{v}\left(\mathbf{q}^{\bullet}(t)\right) = \mathbf{q}_c^{\bullet T}(t)\,\mathbf{P}_c^{\bullet}\mathbf{q}_c^{\bullet}(t) < 0, \tag{25}$$

where

$$\mathbf{P}_c^{\bullet} = \begin{bmatrix} \mathbf{S}^{\bullet}\mathbf{A}_c^{\bullet} + \mathbf{A}_c^{\bullet T}\mathbf{S}^{\bullet} + \mathbf{C}^{\bullet T}\mathbf{I}^{\bullet T}\mathbf{I}^{\bullet}\mathbf{C}^{\bullet} & * & * \\ \mathbf{P}^{\bullet} - \mathbf{S}^{\bullet} + \delta^{\bullet}\mathbf{S}^{\bullet}\mathbf{A}_c^{\bullet} & -2\delta^{\bullet}\mathbf{S}^{\bullet} & * \\ \mathbf{V}^{\bullet T}\mathbf{S}^{\bullet} & \delta^{\bullet}\mathbf{V}^{\bullet T}\mathbf{S}^{\bullet} & -\gamma^{\bullet}\mathbf{I}_{r_v} \end{bmatrix} < 0. \tag{26}$$

Thus, using Schur complement property, (26) implies

$$\begin{bmatrix} \mathbf{S}^{\bullet}\mathbf{A}_c^{\bullet} + \mathbf{A}_c^{\bullet T}\mathbf{S}^{\bullet} & * & * & * \\ \mathbf{P}^{\bullet} - \mathbf{S}^{\bullet} + \delta^{\bullet}\mathbf{S}^{\bullet}\mathbf{A}_c^{\bullet} & -2\delta^{\bullet}\mathbf{S}^{\bullet} & * & * \\ \mathbf{V}^{\bullet T}\mathbf{S}^{\bullet} & \delta^{\bullet}\mathbf{V}^{\bullet T}\mathbf{S}^{\bullet} & -\gamma^{\bullet}\mathbf{I}_{r_v} & * \\ \mathbf{I}^{\bullet}\mathbf{C}^{\bullet} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0. \tag{27}$$

Defining the transform matrix

$$\mathbf{T}^{\bullet} = \operatorname{diag}\left[\mathbf{R}^{\bullet}\ \ \mathbf{R}^{\bullet}\ \ \mathbf{I}_{r_v}\ \ \mathbf{I}_m\right], \quad \mathbf{R}^{\bullet} = \left(\mathbf{S}^{\bullet}\right)^{-1}, \tag{28}$$

premultiplying the left-hand side and postmultiplying the right-hand side of (27) by (28), results in

$$\begin{bmatrix} \mathbf{A}_c^{\bullet}\mathbf{R}^{\bullet} + \mathbf{R}^{\bullet}\mathbf{A}_c^{\bullet T} & * & * & * \\ \mathbf{R}^{\bullet}\mathbf{P}^{\bullet}\mathbf{R}^{\bullet} - \mathbf{R}^{\bullet} + \delta^{\bullet}\mathbf{A}_c^{\bullet}\mathbf{R}^{\bullet} & -2\delta^{\bullet}\mathbf{R}^{\bullet} & * & * \\ \mathbf{V}^{\bullet T} & \delta^{\bullet}\mathbf{V}^{\bullet T} & -\gamma^{\bullet}\mathbf{I}_{r_v} & * \\ \mathbf{I}^{\bullet}\mathbf{C}^{\bullet}\mathbf{R}^{\bullet} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0. \tag{29}$$

Substituting (12) and analyzing the matrix element at the upper left corner of (29), that is,

$$\begin{aligned} \mathbf{A}_c^{\bullet}\mathbf{R}^{\bullet} + \mathbf{R}^{\bullet}\mathbf{A}_c^{\bullet T} &= \left(\mathbf{A}^{\bullet} + \mathbf{B}^{\bullet}\mathbf{K}^{\bullet}\mathbf{C}^{\bullet}\right)\mathbf{R}^{\bullet} \\ &\quad + \mathbf{R}^{\bullet}\left(\mathbf{A}^{\bullet} + \mathbf{B}^{\bullet}\mathbf{K}^{\bullet}\mathbf{C}^{\bullet}\right)^T, \end{aligned} \tag{30}$$

it can be set that

$$\mathbf{B}^{\bullet}\mathbf{K}^{\bullet}\mathbf{H}^{\bullet}\left(\mathbf{H}^{\bullet}\right)^{-1}\mathbf{C}^{\bullet}\mathbf{R}^{\bullet} = \mathbf{B}^{\bullet}\mathbf{Y}^{\bullet}\mathbf{C}^{\bullet}, \tag{31}$$

where

$$\begin{aligned} \left(\mathbf{H}^{\bullet}\right)^{-1}\mathbf{C}^{\bullet} &= \mathbf{C}^{\bullet}\left(\mathbf{R}^{\bullet}\right)^{-1}, \\ \mathbf{Y}^{\bullet} &= \mathbf{K}^{\bullet}\mathbf{H}^{\bullet}. \end{aligned} \tag{32}$$

Thus, with (31), and with the notation

$$\mathbf{U}^{\bullet} = \mathbf{R}^{\bullet}\mathbf{P}^{\bullet}\mathbf{R}^{\bullet}, \tag{33}$$

(29) implies (16), and (32) specifies ((17) and (18)). This concludes the proof. □

Consider the case $r = m$ (square plants), where with each output signal is associated with a reference signal. Such regime is called the forced regime and for DOC it is defined as follows.

Definition 3. The forced regime for (1) and (2) with DOC ((3) and (4)) is given by the control policy

$$\begin{aligned} \dot{\mathbf{p}}(t) &= \mathbf{J}\mathbf{p}(t) + \mathbf{L}\mathbf{y}(t), \\ \mathbf{u}(t) &= \mathbf{M}\mathbf{p}(t) + \mathbf{N}\mathbf{y}(t) + \mathbf{W}\mathbf{w}(t), \end{aligned} \tag{34}$$

where $\mathbf{w}(t) \in \mathbb{R}^m$ is desired output signal vector and $\mathbf{W} \in \mathbb{R}^{m\times m}$ is the signal gain matrix.

Theorem 4. *If square systems (1) and (2) is stabilizable by the control policy (34), and [39]*

$$\operatorname{rank}\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} = n + m, \tag{35}$$

then the matrix **W** *takes the form*

$$\mathbf{W} = -\left(\mathbf{C}\left(\mathbf{A} - \mathbf{B}\mathbf{M}\mathbf{J}^{-1}\mathbf{L}\mathbf{C} + \mathbf{B}\mathbf{N}\mathbf{C}\right)^{-1}\mathbf{B}\right)^{-1}. \tag{36}$$

Proof. In a steady state, the disturbance-free equations (1) and (2) and the control law (34) imply

$$\begin{aligned} \mathbf{0} &= \mathbf{A}\mathbf{q}_o + \mathbf{B}\mathbf{u}_o, \\ \mathbf{y}_o &= \mathbf{C}\mathbf{q}_o, \end{aligned} \tag{37}$$

$$\begin{aligned} \mathbf{0} &= \mathbf{J}\mathbf{p}_o + \mathbf{L}\mathbf{C}\mathbf{q}_o, \\ \mathbf{u}_o &= \mathbf{M}\mathbf{p}_o + \mathbf{N}\mathbf{C}\mathbf{q}_o + \mathbf{W}\mathbf{w}_o, \end{aligned} \tag{38}$$

where $\mathbf{q}_o$, $\mathbf{u}_o$, $\mathbf{y}_o$, $\mathbf{p}_o$, and $\mathbf{w}_o$ are steady-state values of the vectors $\mathbf{q}(t)$, $\mathbf{u}(t)$, $\mathbf{y}(t)$, $\mathbf{p}(t)$, and $\mathbf{w}(t)$, respectively.

Since in a steady state (38) implies

$$\mathbf{u}_o = \left(-\mathbf{M}\mathbf{J}^{-1}\mathbf{L}\mathbf{C} + \mathbf{N}\mathbf{C}\right)\mathbf{q}_o + \mathbf{W}\mathbf{w}_o, \tag{39}$$

then the substitution of (39) into (37) leads to the equation

$$\mathbf{0} = \left(\mathbf{A} - \mathbf{B}\mathbf{M}\mathbf{J}^{-1}\mathbf{L}\mathbf{C} + \mathbf{B}\mathbf{N}\mathbf{C}\right)\mathbf{q}_o + \mathbf{B}\mathbf{W}\mathbf{w}_o. \tag{40}$$

Then,

$$\mathbf{q}_o = -\left(\mathbf{A} - \mathbf{B}\mathbf{M}\mathbf{J}^{-1}\mathbf{L}\mathbf{C} + \mathbf{B}\mathbf{N}\mathbf{C}\right)^{-1}\mathbf{B}\mathbf{W}\mathbf{w}_o \tag{41}$$

and, according to (37) and (41),

$$\mathbf{y}_o = -\mathbf{C}\left(\mathbf{A} - \mathbf{B}\mathbf{M}\mathbf{J}^{-1}\mathbf{L}\mathbf{C} + \mathbf{B}\mathbf{N}\mathbf{C}\right)^{-1}\mathbf{B}\mathbf{W}\mathbf{w}_o. \tag{42}$$

Thus, considering $\mathbf{y}_o = \mathbf{w}_o$, (42) implies (36). This concludes the proof. □

The matrix **W** is nothing else than the inverse of the closed-loop static gain matrix. Note that the static gain realized by the **W** matrix is ideal in control if the plant parameters, on which the value of **W** depends, are known and do not vary with time. The forced regime is basically designed for constant references and is very closely related to shift of origin. If the command value $\mathbf{w}(t)$ is changed "slowly enough," the above scheme can do a reasonable job of tracking, that is, making $\mathbf{y}(t)$ follow $\mathbf{w}(t)$ [40].

Remark 5. Since the input and output matrix rank conditions of existence of FTC with actuator faults and VA generally mean that rank $\mathbf{C} \leq$ rank $\mathbf{B}_f <$ rank $\mathbf{B}$, then (36) gives

$$\mathbf{W} = -\left(\mathbf{C}\left(\mathbf{A} - \mathbf{BMJ}^{-1}\mathbf{LC} + \mathbf{BNC}\right)^{-1}\mathbf{B}\right)^{\ominus 1}, \quad (43)$$

where (43) is the pseudoinverse of $\mathbf{C}(\mathbf{A}-\mathbf{BMJ}^{-1}\mathbf{LC}+\mathbf{BNC})^{-1}\mathbf{B}$ [41].

3. Dynamic Virtual Actuators

The state-space description of the system with a single actuator fault is considered as follows:

$$\dot{\mathbf{q}}_{\mathrm{fa}}(t) = \mathbf{A}\mathbf{q}_{\mathrm{fa}}(t) + \mathbf{B}_f\mathbf{u}_{\mathrm{fa}}(t) + \mathbf{V}\mathbf{v}(t), \quad (44)$$

$$\mathbf{y}_{\mathrm{fa}}(t) = \mathbf{C}\mathbf{q}_{\mathrm{fa}}(t), \quad (45)$$

where $\mathbf{q}_{\mathrm{fa}}(t) \in \mathbb{R}^n$, $\mathbf{u}_{\mathrm{fa}}(t) \in \mathbb{R}^r$, and $\mathbf{y}_{\mathrm{fa}}(t) \in \mathbb{R}^m$ denote the faulty system state variables vector, the vector of the acting control input variables, and the vector of faulty output variables, respectively, and the matrix $\mathbf{B}_f \in \mathbb{R}^{n\times r}$ is finite valued, while rank$(\mathbf{B}_f) <$ rank$(\mathbf{B})$. Moreover, it is supposed that the pair $(\mathbf{A}, \mathbf{B}_f)$ is controllable and the input vector $\mathbf{u}_{\mathrm{fa}}(t)$ is available for reconfiguration (all inputs to the plant are available as they use the nominal controller, but one associated with the faulty actuator is broken).

Analogously, using the same system variable notations, the state-space description of DOC, acting on the system with a single actuator fault, but without DVA, is of the form

$$\dot{\mathbf{p}}_{\mathrm{fa}}(t) = \mathbf{J}\mathbf{p}_{\mathrm{fa}}(t) + \mathbf{L}\mathbf{y}_{\mathrm{fa}}(t), \quad (46)$$

$$\mathbf{u}_c(t) = \mathbf{M}\mathbf{p}_{\mathrm{fa}}(t) + \mathbf{N}\mathbf{y}_{\mathrm{fa}}(t), \quad (47)$$

where $\mathbf{p}_{\mathrm{fa}}(t) \in \mathbb{R}^p$ denotes the controller state variables vector in the faulty system control.

To obtain the DVA state-space description, the following theorem is proven at first.

Theorem 6 (separation principle). *The dynamic virtual actuator for the system with a single actuator fault ((44) and (45)) takes the form*

$$\dot{\mathbf{e}}_{fa}(t) = \left(\mathbf{A} + \mathbf{B}_f\mathbf{S}\right)\mathbf{e}_{fa}(t) + \mathbf{B}_f\mathbf{R}\mathbf{k}_{fa}(t) - \mathbf{B}\mathbf{u}_c(t), \quad (48)$$

$$\dot{\mathbf{k}}_{fa}(t) = \mathbf{O}\mathbf{k}_{fa}(t) + \mathbf{P}\mathbf{e}_{fa}(t), \quad (49)$$

where

$$\mathbf{e}_{fa}(t) = \mathbf{q}_{fa}(t) - \mathbf{q}(t), \quad (50)$$

$\mathbf{k}_{fa}(t) \in \mathbb{R}^k$ *is the state vector of DVA, k is the order of DVA, and $\mathbf{O} \in \mathbb{R}^{k\times k}$, $\mathbf{P} \in \mathbb{R}^{k\times n}$, $\mathbf{R} \in \mathbb{R}^{r\times k}$, and $\mathbf{S} \in \mathbb{R}^{r\times n}$ are real matrices.*

Proof. Using (1), (2), and (3) describing the dynamics of the system and DOC in the fault-free working conditions and (2), (44), and (46) describing the dynamics of the system and DOC in the faulty operating conditions and proposing the outlining dynamic equation of the DVA as follows:

$$\dot{\mathbf{k}}_{\mathrm{fa}}(t) = \mathbf{O}\mathbf{k}_{\mathrm{fa}}(t) + \mathbf{P}\left(\mathbf{q}_{\mathrm{fa}}(t) - \mathbf{q}(t)\right), \quad (51)$$

then the expression of the common system variable model is

$$\begin{bmatrix} \dot{\mathbf{q}}_{\mathrm{fa}}(t) \\ \dot{\mathbf{q}}(t) \\ \dot{\mathbf{p}}_{\mathrm{fa}}(t) \\ \dot{\mathbf{p}}(t) \\ \dot{\mathbf{k}}_{\mathrm{fa}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{LC} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{LC} & \mathbf{0} & \mathbf{J} & \mathbf{0} \\ \mathbf{P} & -\mathbf{P} & \mathbf{0} & \mathbf{0} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{\mathrm{fa}}(t) \\ \mathbf{q}(t) \\ \mathbf{p}_{\mathrm{fa}}(t) \\ \mathbf{p}(t) \\ \mathbf{k}_{\mathrm{fa}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\mathrm{fa}}(t) \\ \mathbf{u}_c(t) \end{bmatrix} + \begin{bmatrix} \mathbf{V} \\ \mathbf{V} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{v}(t). \quad (52)$$

Since it is possible to define the transform matrix $\mathbf{T}$ of the form

$$\mathbf{T} = \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_n & -\mathbf{I}_n & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_p & -\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_k \end{bmatrix}, \quad (53)$$

then

$$\mathbf{T}\begin{bmatrix} \mathbf{q}_{\mathrm{fa}}(t) \\ \mathbf{q}(t) \\ \mathbf{p}_{\mathrm{fa}}(t) \\ \mathbf{p}(t) \\ \mathbf{k}_{\mathrm{fa}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{q}_{\mathrm{fa}}(t) \\ \mathbf{e}_{\mathrm{fa}}(t) \\ \mathbf{p}_{\mathrm{fa}}(t) \\ \mathbf{e}_{p\mathrm{fa}}(t) \\ \mathbf{k}_{\mathrm{fa}}^T(t) \end{bmatrix},$$

$$\mathbf{T}\begin{bmatrix} \mathbf{B}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_f & \mathbf{0} \\ \mathbf{B}_f & -\mathbf{B} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{T}\begin{bmatrix} \mathbf{V} \\ \mathbf{V} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{V} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

$$\mathbf{T}\begin{bmatrix}\mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{LC} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{LC} & \mathbf{0} & \mathbf{J} & \mathbf{0}\\ \mathbf{P} & -\mathbf{P} & \mathbf{0} & \mathbf{0} & \mathbf{O}\end{bmatrix}\mathbf{T}^{-1} = \begin{bmatrix}\mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{LC} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{LC} & \mathbf{0} & \mathbf{J} & \mathbf{0}\\ \mathbf{0} & \mathbf{P} & \mathbf{0} & \mathbf{0} & \mathbf{O}\end{bmatrix}, \tag{54}$$

where

$$\mathbf{e}_{p\mathrm{fa}}(t) = \mathbf{p}_{\mathrm{fa}}(t) - \mathbf{p}(t) \tag{55}$$

and (52) can be rewritten as

$$\begin{bmatrix}\dot{\mathbf{q}}_{\mathrm{fa}}(t)\\ \dot{\mathbf{e}}_{\mathrm{fa}}(t)\\ \dot{\mathbf{p}}_{\mathrm{fa}}(t)\\ \dot{\mathbf{e}}_{p\mathrm{fa}}(t)\\ \dot{\mathbf{k}}_{\mathrm{fa}}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{LC} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{LC} & \mathbf{0} & \mathbf{J} & \mathbf{0}\\ \mathbf{0} & \mathbf{P} & \mathbf{0} & \mathbf{0} & \mathbf{O}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{\mathrm{fa}}(t)\\ \mathbf{e}_{\mathrm{fa}}(t)\\ \mathbf{p}_{\mathrm{fa}}(t)\\ \mathbf{e}_{p\mathrm{fa}}(t)\\ \mathbf{k}_{\mathrm{fa}}(t)\end{bmatrix} + \begin{bmatrix}\mathbf{B}_f & \mathbf{0}\\ \mathbf{B}_f & -\mathbf{B}\\ \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{u}_{\mathrm{fa}}(t)\\ \mathbf{u}_c(t)\end{bmatrix} + \begin{bmatrix}\mathbf{V}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}\mathbf{v}(t). \tag{56}$$

Defining the covering of the faulty control input as follows:

$$\begin{aligned}\mathbf{u}_{\mathrm{fa}}(t) &= \mathbf{R}\mathbf{k}_{\mathrm{fa}}(t) + \mathbf{S}\left(\mathbf{p}_{\mathrm{fa}}(t) - \mathbf{p}(t)\right)\\ &= \mathbf{R}\mathbf{k}_{\mathrm{fa}}(t) + \mathbf{S}\mathbf{e}_{\mathrm{fa}}(t),\end{aligned} \tag{57}$$

then the substitution of (57) into (56) leads to

$$\begin{bmatrix}\dot{\mathbf{q}}_{\mathrm{fa}}(t)\\ \dot{\mathbf{e}}_{\mathrm{fa}}(t)\\ \dot{\mathbf{p}}_{\mathrm{fa}}(t)\\ \dot{\mathbf{e}}_{p\mathrm{fa}}(t)\\ \dot{\mathbf{k}}_{\mathrm{fa}}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{LC} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{LC} & \mathbf{0} & \mathbf{J} & \mathbf{0}\\ \mathbf{0} & \mathbf{P} & \mathbf{0} & \mathbf{0} & \mathbf{O}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{\mathrm{fa}}(t)\\ \mathbf{e}_{\mathrm{fa}}(t)\\ \mathbf{p}_{\mathrm{fa}}(t)\\ \mathbf{e}_{p\mathrm{fa}}(t)\\ \mathbf{k}_{\mathrm{fa}}(t)\end{bmatrix} - \begin{bmatrix}\mathbf{0}\\ \mathbf{B}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}\mathbf{u}_c(t) + \begin{bmatrix}\mathbf{B}_f\mathbf{S} & \mathbf{B}_f\mathbf{R}\\ \mathbf{B}_f\mathbf{S} & \mathbf{B}_f\mathbf{R}\\ \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{\mathrm{fa}}(t)\\ \mathbf{k}_{\mathrm{fa}}(t)\end{bmatrix} + \begin{bmatrix}\mathbf{V}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}\mathbf{v}(t), \tag{58}$$

which implies

$$\begin{bmatrix}\dot{\mathbf{q}}_{\mathrm{fa}}(t)\\ \dot{\mathbf{e}}_{\mathrm{fa}}(t)\\ \dot{\mathbf{p}}_{\mathrm{fa}}(t)\\ \dot{\mathbf{e}}_{p\mathrm{fa}}(t)\\ \dot{\mathbf{k}}_{\mathrm{fa}}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}_f\mathbf{S} & \mathbf{0} & \mathbf{0} & \mathbf{B}_f\mathbf{R}\\ \mathbf{0} & \mathbf{A}+\mathbf{B}_f\mathbf{S} & \mathbf{0} & \mathbf{0} & \mathbf{B}_f\mathbf{R}\\ \mathbf{LC} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{LC} & \mathbf{0} & \mathbf{J} & \mathbf{0}\\ \mathbf{0} & \mathbf{P} & \mathbf{0} & \mathbf{0} & \mathbf{O}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{\mathrm{fa}}(t)\\ \mathbf{e}_{\mathrm{fa}}(t)\\ \mathbf{p}_{\mathrm{fa}}(t)\\ \mathbf{e}_{p\mathrm{fa}}(t)\\ \mathbf{k}_{\mathrm{fa}}(t)\end{bmatrix} - \begin{bmatrix}\mathbf{0}\\ \mathbf{B}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}\mathbf{u}_c(t) + \begin{bmatrix}\mathbf{V}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}\mathbf{v}(t). \tag{59}$$

Thus, the second and the fifth rows of (59) imply (48) and (49).

Obviously, in view of the block structure of the extended system matrix of the system (59), the separation principle yields, and the parameters of DVA, $\mathbf{O}$, $\mathbf{P}$, $\mathbf{R}$, and $\mathbf{S}$ can be designed independently of the faulty system model, if the condition of the controllability for the pair $(\mathbf{A}, \mathbf{B}_f)$ is satisfied. □

Lemma 7. *The state-space description of the closed-loop faulty system with activated DVA is as follows:*

$$\dot{\mathbf{q}}^{\circ}_{fa}(t) = \left(\mathbf{A}^{\circ} + \mathbf{B}^{\circ}_f\mathbf{G}^{\circ}\right)\mathbf{q}^{\circ}_{fa}(t) + \mathbf{V}^{\circ}_{fa}\mathbf{d}^{\circ}_{fa}(t), \tag{60}$$

$$\mathbf{y}_{fa}(t) = \mathbf{C}^{\circ}\mathbf{q}^{\circ}_{fa}(t), \tag{61}$$

where

$$\mathbf{q}^{\circ}_{fa}(t) = \begin{bmatrix}\mathbf{q}_{fa}(t)\\ \mathbf{k}_{fa}(t)\end{bmatrix}, \quad \mathbf{d}^{\circ}_{fa}(t) = \begin{bmatrix}\mathbf{S}\mathbf{q}(t)\\ \mathbf{v}(t)\\ \mathbf{P}\mathbf{q}(t)\end{bmatrix}, \tag{62}$$

$$\mathbf{V}^{\circ}_{fa} = \begin{bmatrix}-\mathbf{B}_f & \mathbf{V} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_k\end{bmatrix}, \quad \mathbf{A}^{\circ} = \begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}, \quad \mathbf{B}^{\circ}_f = \begin{bmatrix}\mathbf{0} & \mathbf{B}_f\\ \mathbf{I}_k & \mathbf{0}\end{bmatrix}, \quad \mathbf{G}^{\circ} = \begin{bmatrix}\mathbf{P} & \mathbf{O}\\ \mathbf{S} & \mathbf{R}\end{bmatrix}, \quad \mathbf{C}^{\circ} = \begin{bmatrix}\mathbf{C} & \mathbf{0}\end{bmatrix}, \tag{63}$$

and $\mathbf{q}^{\circ}_{fa}(t) \in \mathbb{R}^{(n+k)}$, $\mathbf{d}^{\circ}_{fa}(t) \in \mathbb{R}^{(n+r_v+k)}$, $\mathbf{A}^{\circ} \in \mathbb{R}^{(n+k)\times(n+k)}$, $\mathbf{B}^{\circ} \in \mathbb{R}^{(n+k)\times(r+k)}$, $\mathbf{C}^{\circ} \in \mathbb{R}^{m\times(n+k)}$, $\mathbf{G}^{\circ} \in \mathbb{R}^{(r+k)\times(n+k)}$, *and* $\mathbf{V}^{\circ}_{fa} \in \mathbb{R}^{(n+k)\times(r+r_v+k)}$.

The system with an actuator fault, under control of DOC covered by the DVA, operates in the reconfiguration regime along with the unknown input disturbance $\mathbf{d}^{\circ}_{fa}(t)$. *Moreover, the stability of the closed-loop system in the reconfiguration regime is determined by the system matrix*

$$\mathbf{A}^{\circ}_{c} = \mathbf{A}^{\circ} + \mathbf{B}^{\circ}_{f}\mathbf{G}^{\circ}. \tag{64}$$

Proof. Since the first and the fifth rows of (59) give

$$\begin{aligned}\begin{bmatrix}\dot{\mathbf{q}}_{\mathrm{fa}}(t)\\ \dot{\mathbf{k}}_{\mathrm{fa}}(t)\end{bmatrix} &= \begin{bmatrix}\mathbf{A} & \mathbf{B}_f\mathbf{R}\\ \mathbf{0} & \mathbf{O}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{\mathrm{fa}}(t)\\ \mathbf{k}_{\mathrm{fa}}(t)\end{bmatrix}\\ &+ \begin{bmatrix}\mathbf{B}_f\mathbf{S} & \mathbf{V}\\ \mathbf{P} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{\mathrm{fa}}(t)\\ \mathbf{v}(t)\end{bmatrix},\end{aligned} \tag{65}$$

then, by substituting (50) into (65), they can obtain

$$\begin{aligned}\begin{bmatrix}\dot{\mathbf{q}}_{\mathrm{fa}}(t)\\ \dot{\mathbf{k}}_{\mathrm{fa}}(t)\end{bmatrix} &= \begin{bmatrix}\mathbf{A}+\mathbf{B}_f\mathbf{S} & \mathbf{B}_f\mathbf{R}\\ \mathbf{P} & \mathbf{O}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{\mathrm{fa}}(t)\\ \mathbf{k}_{\mathrm{fa}}(t)\end{bmatrix}\\ &+ \begin{bmatrix}-\mathbf{B}_f & \mathbf{V} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_k\end{bmatrix}\begin{bmatrix}\mathbf{S}\mathbf{q}(t)\\ \mathbf{v}(t)\\ \mathbf{P}\mathbf{q}(t)\end{bmatrix}.\end{aligned} \tag{66}$$

Writing (45) as

$$\mathbf{y}_{\mathrm{fa}}(t) = \mathbf{C}\mathbf{q}_{\mathrm{fa}}(t) = \begin{bmatrix}\mathbf{C} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{\mathrm{fa}}(t)\\ \mathbf{k}_{\mathrm{fa}}(t)\end{bmatrix} \tag{67}$$

and using the relation

$$\begin{bmatrix}\mathbf{A}+\mathbf{B}_f\mathbf{S} & \mathbf{B}_f\mathbf{R}\\ \mathbf{P} & \mathbf{O}\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} + \begin{bmatrix}\mathbf{0} & \mathbf{B}_f\\ \mathbf{I}_k & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{P} & \mathbf{O}\\ \mathbf{S} & \mathbf{R}\end{bmatrix}, \tag{68}$$

with the notations (62) and (63) then (66) and (67) imply (60) and (61), respectively. This concludes the proof. □

Lemma 8. *The state-space description ((48) and (49)) of the DVA with the covering of the faulty control input (57) is as follows:*

$$\begin{aligned}\dot{\mathbf{e}}^{\circ}_{fa}(t) &= \left(\mathbf{A}^{\circ} + \mathbf{B}^{\circ}_{f}\mathbf{G}^{\circ}\right)\mathbf{e}^{\circ}_{fa}(t) + \mathbf{B}^{\circ}\mathbf{u}_c(t),\\ \mathbf{u}_c(t) &= \mathbf{I}^{\circ}\mathbf{G}^{\circ}\mathbf{e}^{\circ}_{fa}(t),\end{aligned} \tag{69}$$

where

$$\begin{aligned}\mathbf{e}^{\circ}_{fa}(t) &= \begin{bmatrix}\mathbf{e}_{fa}(t)\\ \mathbf{k}_{fa}(t)\end{bmatrix},\\ \mathbf{B}^{\circ} &= \begin{bmatrix}-\mathbf{B}\\ \mathbf{0}\end{bmatrix},\\ \mathbf{I}^{\circ} &= \begin{bmatrix}\mathbf{0} & \mathbf{I}_r\end{bmatrix}.\end{aligned} \tag{70}$$

In the autonomous regime, the stability of the DVA is determined by the same system matrix (64) as stability of the closed-loop system in the reconfiguration regime.

Proof. Writing (48), (49), and (57) in the following form:

$$\begin{aligned}\begin{bmatrix}\dot{\mathbf{e}}_{\mathrm{fa}}(t)\\ \dot{\mathbf{k}}_{\mathrm{fa}}(t)\end{bmatrix} &= \begin{bmatrix}\mathbf{A}+\mathbf{B}_f\mathbf{S} & \mathbf{B}_f\mathbf{R}\\ \mathbf{P} & \mathbf{O}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{\mathrm{fa}}(t)\\ \mathbf{k}_{\mathrm{fa}}(t)\end{bmatrix}\\ &+ \begin{bmatrix}-\mathbf{B}\\ \mathbf{0}\end{bmatrix}\mathbf{u}_c(t),\end{aligned} \tag{71}$$

$$\begin{aligned}\mathbf{u}_{\mathrm{fa}}(t) &= \begin{bmatrix}\mathbf{S} & \mathbf{R}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{\mathrm{fa}}(t)\\ \mathbf{k}_{\mathrm{fa}}(t)\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{0} & \mathbf{I}_r\end{bmatrix}\begin{bmatrix}\mathbf{P} & \mathbf{O}\\ \mathbf{S} & \mathbf{R}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{\mathrm{fa}}(t)\\ \mathbf{k}_{\mathrm{fa}}(t)\end{bmatrix},\end{aligned} \tag{72}$$

respectively, and using notations (70) as well as the relation (68), then (71) and (72) imply (69). This concludes the proof. □

Lemma 9. *The state-space description of DOC masked in inputs by DVA and acting on the system with a single actuator fault is of the form*

$$\begin{bmatrix}\dot{\mathbf{p}}_{fa}(t)\\ \mathbf{u}_c(t)\end{bmatrix} = \begin{bmatrix}\mathbf{J} & \mathbf{L}\\ \mathbf{M} & \mathbf{N}\end{bmatrix}\begin{bmatrix}\mathbf{p}_{fa}(t)\\ \mathbf{y}_{fa}(t)\end{bmatrix} - \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{M} & \mathbf{N}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{pfa}(t)\\ \mathbf{C}\mathbf{e}_{fa}(t)\end{bmatrix}, \tag{73}$$

where $\mathbf{y}_{fa}(t)$ *is the measurable output of the closed-loop faulty system.*

Proof. Using (45) and (50), then (2) can be rewritten as

$$\begin{aligned}\mathbf{y}(t) &= \mathbf{C}\mathbf{q}(t) = \mathbf{C}\left(\mathbf{q}_{\mathrm{fa}}(t) - \left(\mathbf{q}_{\mathrm{fa}}(t) - \mathbf{q}(t)\right)\right)\\ &= \mathbf{y}_{\mathrm{fa}}(t) - \mathbf{C}\mathbf{e}_{\mathrm{fa}}(t).\end{aligned} \tag{74}$$

Considering (74) as the input to the nominal DOC which masks an actuator fault, by using (55) and (74), then (3) and (4) imply

$$\begin{aligned}\begin{bmatrix}\dot{\mathbf{p}}(t)\\ \mathbf{u}_c(t)\end{bmatrix} &= \begin{bmatrix}\mathbf{J} & \mathbf{L}\\ \mathbf{M} & \mathbf{N}\end{bmatrix}\begin{bmatrix}\mathbf{p}(t)\\ \mathbf{y}(t)\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{J} & \mathbf{L}\\ \mathbf{M} & \mathbf{N}\end{bmatrix}\begin{bmatrix}\mathbf{p}_{\mathrm{fa}}(t) - \mathbf{e}_{p\mathrm{fa}}(t)\\ \mathbf{y}_{\mathrm{fa}}(t) - \mathbf{C}\mathbf{e}_{\mathrm{fa}}(t)\end{bmatrix},\end{aligned} \tag{75}$$

$$\begin{bmatrix}\dot{\mathbf{p}}(t)\\ \mathbf{u}_c(t)\end{bmatrix} = \begin{bmatrix}\mathbf{J} & \mathbf{L}\\ \mathbf{M} & \mathbf{N}\end{bmatrix}\begin{bmatrix}\mathbf{p}_{\mathrm{fa}}(t)\\ \mathbf{y}_{\mathrm{fa}}(t)\end{bmatrix} - \begin{bmatrix}\mathbf{J} & \mathbf{L}\\ \mathbf{M} & \mathbf{N}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{p\mathrm{fa}}(t)\\ \mathbf{C}\mathbf{e}_{\mathrm{fa}}(t)\end{bmatrix}, \tag{76}$$

respectively. Separating the equation given by the forth row of (59) as follows:

$$\dot{\mathbf{e}}_{p\mathrm{fa}}(t) = \mathbf{J}\mathbf{e}_{p\mathrm{fa}}(t) + \mathbf{L}\mathbf{C}\mathbf{e}_{\mathrm{fa}}(t), \tag{77}$$

(76) can be rewritten as

$$\begin{aligned}\begin{bmatrix}\dot{\mathbf{p}}(t)\\ \mathbf{u}_c(t)\end{bmatrix} &= \begin{bmatrix}\mathbf{J} & \mathbf{L}\\ \mathbf{M} & \mathbf{N}\end{bmatrix}\begin{bmatrix}\mathbf{p}_{\mathrm{fa}}(t)\\ \mathbf{y}_{\mathrm{fa}}(t)\end{bmatrix} - \begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix}\dot{\mathbf{e}}_{p\mathrm{fa}}(t)\\ &- \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{M} & \mathbf{N}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{p\mathrm{fa}}(t)\\ \mathbf{C}\mathbf{e}_{\mathrm{fa}}(t)\end{bmatrix}.\end{aligned} \tag{78}$$

Since the time derivative of (55) takes the form

$$\dot{\mathbf{e}}_{pfa}(t) = \dot{\mathbf{p}}_{\mathrm{fa}}(t) - \dot{\mathbf{p}}(t), \qquad (79)$$

using (79), then (78) implies the equations of DOC covered by DVA (73). This concludes the proof. □

4. Design of Dynamic Virtual Actuators

If the pair $(\mathbf{A}^{\circ}, \mathbf{B}_f^{\circ})$ is controllable, for the given structure of DVAs (48) and (49), the form of the unknown input disturbance $\mathbf{d}_{\mathrm{fa}}^{\circ}(t)$ (62), and the system matrix parameters (63), the conditions for design of DVA are given by the following theorems.

Theorem 10 (bounded real lemma). *The closed-loop system, consisting of the plant with a single actuator fault (44), (45), DOC (6), and DVAs (48) and (49), is stable with the quadratic performance γ° if there exist a symmetric positive definite matrix $\mathbf{X}^{\circ} \in \mathbb{R}^{(n+k)\times(n+k)}$, a matrix $\mathbf{Y}^{\circ} \in \mathbb{R}^{(r+k)\times(n+k)}$, and a positive scalar $\gamma^{\circ} \in \mathbb{R}$ such that*

$$\mathbf{X}^{\circ} = \mathbf{X}^{\circ T} > 0, \quad \gamma^{\circ} > 0, \qquad (80)$$

$$\begin{bmatrix} \mathbf{A}^{\circ}\mathbf{X}^{\circ} + \mathbf{X}^{\circ}\mathbf{A}^{\circ T} + \mathbf{B}_f^{\circ}\mathbf{Y}^{\circ} + \mathbf{Y}^{\circ T}\mathbf{B}_f^{\circ T} & * & * \\ \mathbf{V}_{fa}^{\circ T} & -\gamma^{\circ}\mathbf{I}_{r+r_v+k} & * \\ \mathbf{C}^{\circ}\mathbf{X}^{\circ} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0, \qquad (81)$$

where the generalized system matrices are defined in (62) and (63).

When the above conditions hold,

$$\mathbf{G}^{\circ} = \mathbf{Y}^{\circ}(\mathbf{X}^{\circ})^{-1}. \qquad (82)$$

Proof. Considering the Lyapunov function candidate as follows:

$$v(\mathbf{q}_{\mathrm{fa}}^{\circ}(t)) = \mathbf{q}_{\mathrm{fa}}^{\circ T}(t)\mathbf{P}^{\circ}\mathbf{q}_{\mathrm{fa}}^{\circ}(t) + \int_0^t \left(\mathbf{y}_{\mathrm{fa}}^{T}(\tau)\mathbf{y}_{\mathrm{fa}}(\tau) - \gamma^{\circ}\mathbf{d}_{\mathrm{fa}}^{\circ T}(\tau)\mathbf{d}_{\mathrm{fa}}^{\circ}(\tau)\right)\mathrm{d}\tau > 0, \qquad (83)$$

where $\mathbf{P}^{\circ} \in \mathbb{R}^{(n+k)\times(n+k)}$ is symmetric positive definite matrix and $\gamma^{\circ} \in \mathbb{R}$ is square of the H_{∞} norm of the transfer function matrix of the disturbance $\mathbf{d}_{\mathrm{fa}}^{\circ}$, then

$$\dot{v}(\mathbf{q}_{\mathrm{fa}}^{\circ}(t)) = \dot{\mathbf{q}}_{\mathrm{fa}}^{\circ T}(t)\mathbf{P}^{\circ}\mathbf{q}_{\mathrm{fa}}^{\circ}(t) + \mathbf{q}_{\mathrm{fa}}^{\circ T}(t)\mathbf{P}^{\circ}\dot{\mathbf{q}}_{\mathrm{fa}}^{\circ}(t) + \mathbf{y}_{\mathrm{fa}}^{T}(t)\mathbf{y}_{\mathrm{fa}}(t) - \gamma^{\circ}\mathbf{d}_{\mathrm{fa}}^{\circ T}(t)\mathbf{d}_{\mathrm{fa}}^{\circ}(t) < 0. \qquad (84)$$

Substituting (69) with (64) in (84), we have

$$\begin{aligned} \dot{v}(\mathbf{q}_{\mathrm{fa}}^{\circ}(t)) = {} & \mathbf{d}_{\mathrm{fa}}^{\circ T}(t)\mathbf{B}_f^{\circ T}\mathbf{P}^{\circ}\mathbf{q}_{\mathrm{fa}}^{\circ}(t) + \mathbf{q}_{\mathrm{fa}}^{\circ T}(t)\mathbf{P}^{\circ}\mathbf{B}_f^{\circ}\mathbf{d}_{\mathrm{fa}}^{\circ}(t) \\ & + \mathbf{q}_{\mathrm{fa}}^{\circ T}(t)\left(\mathbf{A}^{\circ} + \mathbf{B}_f^{\circ}\mathbf{G}^{\circ}\right)^T\mathbf{P}^{\circ}\mathbf{q}_{\mathrm{fa}}^{\circ}(t) \\ & + \mathbf{q}_{\mathrm{fa}}^{\circ T}(t)\mathbf{C}^{\circ T}\mathbf{C}^{\circ}\mathbf{q}_{\mathrm{fa}}^{\circ}(t) \\ & + \mathbf{q}_{\mathrm{fa}}^{\circ T}(t)\mathbf{P}^{\circ}\left(\mathbf{A}^{\circ} + \mathbf{B}_f^{\circ}\mathbf{G}^{\circ}\right)\mathbf{q}_{\mathrm{fa}}^{\circ}(t) \\ & - \gamma^{\circ}\mathbf{d}_{\mathrm{fa}}^{\circ T}(t)\mathbf{d}_{\mathrm{fa}}^{\circ}(t) < 0 \end{aligned} \qquad (85)$$

and with the notation

$$\mathbf{q}_{c\mathrm{fa}}^{\circ T}(t) = \left[\mathbf{q}_{\mathrm{fa}}^{\circ T}(t) \quad \mathbf{d}_{\mathrm{fa}}^{\circ T}(t)\right], \qquad (86)$$

then (85) can be rewritten as

$$\dot{v}(\mathbf{q}_{c\mathrm{fa}}^{\circ}(t)) = \mathbf{q}_{c\mathrm{fa}}^{\circ T}(t)\mathbf{P}_{\mathrm{fa}}^{\circ}\mathbf{q}_{\mathrm{fa}}^{\circ}(t) < 0, \qquad (87)$$

where

$$\mathbf{P}_{\mathrm{fa}}^{\circ} = \begin{bmatrix} \mathbf{A}_c^{\circ T}\mathbf{P}^{\circ} + \mathbf{P}^{\circ}\mathbf{A}_c^{\circ} + \mathbf{C}^{\circ T}\mathbf{C}^{\circ} & * \\ \mathbf{V}_{\mathrm{fa}}^{\circ T}\mathbf{P}^{\circ} & -\gamma^{\circ}\mathbf{I}_{r+r_v+k} \end{bmatrix} < 0 \qquad (88)$$

and, using the Schur complement property, (88) can be written as follows:

$$\begin{bmatrix} \mathbf{A}_c^{\circ}\mathbf{P}^{\circ} + \mathbf{P}^{\circ}\mathbf{A}_c^{\circ} & * & * \\ \mathbf{V}_{\mathrm{fa}}^{\circ T}\mathbf{P}^{\circ} & -\gamma^{\circ}\mathbf{I}_{r+r_v+k} & * \\ \mathbf{C}^{\circ} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0. \qquad (89)$$

Introducing the transform matrix of the form

$$\mathbf{T}_f = \mathrm{diag}\left[\mathbf{X}^{\circ} \quad \mathbf{I}_{r+r_v+k} \quad \mathbf{I}_m\right], \quad \mathbf{X}^{\circ} = (\mathbf{P}^{\circ})^{-1}, \qquad (90)$$

premultiplying and postmultiplying (89) by $\mathbf{T}_f$, and inserting (64), then it yields

$$\begin{bmatrix} \mathbf{X}^{\circ}\left(\mathbf{A}^{\circ} + \mathbf{B}_f^{\circ}\mathbf{G}^{\circ}\right)^T + \left(\mathbf{A}^{\circ} + \mathbf{B}_f^{\circ}\mathbf{G}^{\circ}\right)\mathbf{X}^{\circ} & * & * \\ \mathbf{V}_{\mathrm{fa}}^{\circ T} & -\gamma^{\circ}\mathbf{I}_{r+r_v+k} & * \\ \mathbf{C}^{\circ}\mathbf{X}^{\circ} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0. \qquad (91)$$

Thus, using the notation

$$\mathbf{Y}^{\circ} = \mathbf{G}^{\circ}\mathbf{X}^{\circ}, \qquad (92)$$

then (91) and (92) imply (81) and (82), respectively. This concludes the proof. □

In order to adjust fault detection and isolation time to the dynamics of the closed-loop system, not only is selection of the order p of DOC provided with a free tuning parameter in control design step, but also the selection of the order k of DVA could be offered with a further tuning parameter in virtual actuator design. This augmentation is reflected in the following theorem.

Theorem 11 (enhanced bounded real lemma). *The closed-loop system, consisting of the plant with a single actuator fault (44), (45), DOC (6), and DVAs (48) and (49), is stable with the quadratic performance γ° if for given positive $\delta^{\circ} \in \mathbb{R}$ there exist symmetric positive definite matrices $\mathbf{R}^{\circ}, \mathbf{U}^{\circ} \in \mathbb{R}^{(n+k)\times(n+k)}$, a matrix $\mathbf{Y}^{\circ} \in \mathbb{R}^{(r+k)\times(n+k)}$, and a positive scalar $\gamma^{\circ} \in \mathbb{R}$ such that*

$$\begin{aligned} &\mathbf{R}^{\circ} = \mathbf{R}^{\circ T} > 0, \\ &\mathbf{U}^{\circ} = \mathbf{U}^{\circ T} > 0, \\ &\gamma^{\circ} > 0, \end{aligned} \tag{93}$$

$$\begin{bmatrix} \mathbf{A}^{\circ}\mathbf{R}^{\circ} + \mathbf{R}^{\circ}\mathbf{A}^{\circ T} + \mathbf{B}_f^{\circ}\mathbf{Y}^{\circ} + \mathbf{Y}^{\circ T}\mathbf{B}_f^{\circ T} & * & * & * \\ \mathbf{U}^{\circ} - \mathbf{R}^{\circ} + \delta^{\circ}\mathbf{A}^{\circ}\mathbf{R}^{\circ} + \delta^{\circ}\mathbf{B}_f^{\circ}\mathbf{Y}^{\circ} & -2\delta^{\circ}\mathbf{R}^{\circ} & * & * \\ \mathbf{V}_{fa}^{\circ T} & \delta^{\circ}\mathbf{V}_{fa}^{\circ T} & -\gamma^{\circ}\mathbf{I}_{r+r_v+p_v} & * \\ \mathbf{C}^{\circ}\mathbf{R}^{\circ} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0, \tag{94}$$

where the generalized system matrices are defined in (62) and (63) and positive δ° is the tuning parameter.

When the above conditions hold,

$$\mathbf{G}^{\circ} = \mathbf{Y}^{\circ}\left(\mathbf{R}^{\circ}\right)^{-1}. \tag{95}$$

Proof. Since (60) with (64) implies

$$\mathbf{A}_c^{\circ}\mathbf{q}_{\text{fa}}^{\circ}(t) + \mathbf{V}_{\text{fa}}^{\circ}\mathbf{d}_{\text{fa}}^{\circ}(t) - \dot{\mathbf{q}}_{\text{fa}}^{\circ}(t) = \mathbf{0}, \tag{96}$$

then it yields

$$\begin{aligned} &\left(\mathbf{q}_{\text{fa}}^{\circ T}(t)\,\mathbf{S}^{\circ} + \delta^{\circ}\dot{\mathbf{q}}_{\text{fa}}^{\circ T}(t)\,\mathbf{S}^{\circ}\right) \\ &\quad \cdot \left(\mathbf{A}_c^{\circ}\mathbf{q}_{\text{fa}}^{\circ}(t) + \mathbf{V}_{\text{fa}}^{\circ}\mathbf{d}_{\text{fa}}^{\circ}(t) - \dot{\mathbf{q}}_{\text{fa}}^{\circ}(t)\right) = 0, \end{aligned} \tag{97}$$

where $\mathbf{S}^{\circ} \in \mathbb{R}^{(n+k)\times(n+k)}$ is a symmetric positive definite matrix and $\delta^{\circ} \in \mathbb{R}$ is a positive scalar.

Adding (97) as well as its transpose to (84) and then inserting (61), it can be seen that

$$\begin{aligned} \dot{v}\left(\mathbf{q}_{\text{fa}}^{\circ}(t)\right) &= \dot{\mathbf{q}}_{\text{fa}}^{\circ T}(t)\,\mathbf{P}^{\circ}\mathbf{q}_{\text{fa}}^{\circ}(t) + \mathbf{q}_{\text{fa}}^{\circ T}(t)\,\mathbf{P}^{\circ}\dot{\mathbf{q}}_{\text{fa}}^{\circ}(t) \\ &+ \left(\mathbf{q}_{\text{fa}}^{\circ T}(t)\,\mathbf{S}^{\circ} + \delta^{\circ}\dot{\mathbf{q}}_{\text{fa}}^{\circ T}(t)\,\mathbf{S}^{\circ}\right) \\ &\cdot \left(\mathbf{A}_c^{\circ}\mathbf{q}_{\text{fa}}^{\circ}(t) + \mathbf{V}_{\text{fa}}^{\circ}\mathbf{d}_{\text{fa}}^{\circ}(t) - \dot{\mathbf{q}}_{\text{fa}}^{\circ}(t)\right) \\ &+ \left(\mathbf{A}_c^{\circ}\mathbf{q}_{\text{fa}}^{\circ}(t) + \mathbf{V}_{\text{fa}}^{\circ}\mathbf{d}_{\text{fa}}^{\circ}(t) - \dot{\mathbf{q}}_{\text{fa}}^{\circ}(t)\right)^T \\ &\cdot \left(\mathbf{S}^{\circ}\mathbf{q}_{\text{fa}}^{\circ}(t) + \delta^{\circ}\mathbf{S}^{\circ}\dot{\mathbf{q}}_{\text{fa}}^{\circ}(t)\right) + \mathbf{q}_{\text{fa}}^{\circ T}(t)\,\mathbf{C}^{\circ T}\mathbf{C}^{\circ}\mathbf{q}_{\text{fa}}^{\circ}(t) \\ &- \gamma^{\circ}\mathbf{d}_{\text{fa}}^{\circ T}(t)\,\mathbf{d}_{\text{fa}}^{\circ} < 0 \end{aligned} \tag{98}$$

and with the notation

$$\mathbf{q}_{ce}^{\circ T}(t) = \left[\mathbf{q}_{\text{fa}}^{\circ T}(t) \;\; \dot{\mathbf{q}}_{\text{fa}}^{\circ T}(t) \;\; \mathbf{d}_{\text{fa}}^{\circ T}\right] \tag{99}$$

inequality (98) can be written as

$$\dot{v}\left(\mathbf{q}_c^{\circ}(t)\right) = \mathbf{q}_{ce}^{\circ T}(t)\,\mathbf{P}_{ce}^{\circ}\mathbf{q}_{ce}^{\circ}(t) < 0, \tag{100}$$

where

$$\mathbf{P}_{ce}^{\circ} = \begin{bmatrix} \mathbf{S}^{\circ}\mathbf{A}_c^{\circ} + \mathbf{A}_c^{\circ T}\mathbf{S}^{\circ} + \mathbf{C}^{\circ T}\mathbf{C}^{\circ} & * & * \\ \mathbf{P}^{\circ} - \mathbf{S}^{\circ} + \delta^{\circ}\mathbf{S}^{\circ}\mathbf{A}_c^{\circ} & -2\delta^{\circ}\mathbf{S}^{\circ} & * \\ \mathbf{V}_{\text{fa}}^{\circ T}\mathbf{S}^{\circ} & \delta^{\circ}\mathbf{V}_{\text{fa}}^{\circ T}\mathbf{S}^{\circ} & -\gamma^{\circ}\mathbf{I}_{r+r_v+k} \end{bmatrix}. \tag{101}$$

Since, using the Schur complement property, (101) can be rewritten as

$$\begin{bmatrix} \mathbf{S}^{\circ}\mathbf{A}_c^{\circ} + \mathbf{A}^{\circ T}\mathbf{S}^{\circ} & * & * & * \\ \mathbf{P}^{\circ} - \mathbf{S}^{\circ} + \delta^{\circ}\mathbf{S}^{\circ}\mathbf{A}_c^{\circ} & -2\delta^{\circ}\mathbf{S}^{\circ} & * & * \\ \mathbf{V}_{\text{fa}}^{\circ T}\mathbf{S}^{\circ} & \delta^{\circ}\mathbf{V}_{\text{fa}}^{\circ T}\mathbf{S}^{\circ} & -\gamma^{\circ}\mathbf{I}_{r+r_v+k} & * \\ \mathbf{C}^{\circ} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0, \tag{102}$$

inserting (64) in (102) and then premultiplying and postmultiplying the result by the transform matrix

$$\mathbf{T}_{fe}^{\circ} = \operatorname{diag}\left[\mathbf{R}^{\circ} \;\; \mathbf{R}^{\circ} \;\; \mathbf{I}_{r+r_v+k} \;\; \mathbf{I}_m\right], \quad \mathbf{R}^{\circ} = \left(\mathbf{S}^{\circ}\right)^{-1}, \tag{103}$$

it can be obtained that

$$\begin{bmatrix} \left(\mathbf{A}^{\circ} + \mathbf{B}_f^{\circ}\mathbf{G}^{\circ}\right)\mathbf{R}^{\circ} + \mathbf{R}^{\circ}\left(\mathbf{A}^{\circ} + \mathbf{B}_f^{\circ}\mathbf{G}^{\circ}\right)^T & * & * & * \\ \mathbf{R}^{\circ}\mathbf{P}^{\circ}\mathbf{R}^{\circ} - \mathbf{R}^{\circ} + \delta^{\circ}\left(\mathbf{A}^{\circ} + \mathbf{B}_f^{\circ}\mathbf{G}^{\circ}\right)\mathbf{R}^{\circ} & -2\delta^{\circ}\mathbf{R}^{\circ} & * & * \\ \mathbf{V}_{\text{fa}}^{\circ T} & \delta^{\circ}\mathbf{V}_{\text{fa}}^{\circ T} & -\gamma^{\circ}\mathbf{I}_{r+r_v+k} & * \\ \mathbf{C}^{\circ}\mathbf{R}^{\circ} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0. \tag{104}$$

Introducing the notations

$$\begin{aligned} &\mathbf{U}^{\circ} = \mathbf{R}^{\circ}\mathbf{P}^{\circ}\mathbf{R}^{\circ}, \\ &\mathbf{Y}^{\circ} = \mathbf{G}^{\circ}\mathbf{R}^{\circ}, \end{aligned} \tag{105}$$

then (104) and (105) imply (94) and (95), respectively. This concludes the proof. □

When control with DOC is implemented in the forced mode, DVA also must have a forced mode. Using (49) and

(57), DVA extended by the block defining the forced mode can be considered in the form

$$\dot{\mathbf{k}}_{\text{fa}}(t) = \mathbf{O}\mathbf{k}_{\text{fa}}(t) + \mathbf{P}\mathbf{e}_{\text{fa}}(t), \quad (106)$$

$$\mathbf{u}_{\text{fa}}(t) = \mathbf{R}\mathbf{k}_{\text{fa}}(t) + \mathbf{S}\mathbf{e}_{\text{fa}}(t) + \mathbf{F}\mathbf{u}_c(t), \quad (107)$$

where $\mathbf{F} \in \mathbb{R}^{r\times r}$ and (48) is changed as follows:

$$\dot{\mathbf{e}}_{\text{fa}}(t) = \left(\mathbf{A} + \mathbf{B}_f\mathbf{S}\right)\mathbf{e}_{\text{fa}}(t) + \mathbf{B}_f\mathbf{R}\mathbf{k}_{\text{fa}}(t) + \left(\mathbf{B}_f\mathbf{F} - \mathbf{B}\right)\mathbf{u}_c(t). \quad (108)$$

Moreover, (74) prescribes that

$$\mathbf{y}(t) = \mathbf{y}_{\text{fa}}(t) - \mathbf{C}\mathbf{e}_{\text{fa}}(t). \quad (109)$$

If the pair $(\mathbf{A}^{\bullet}, \mathbf{B}^{\bullet}_f)$, given in (8), is stabilizable by DOC with the gain matrix $\mathbf{G}^{\bullet}$ of the structure (5) and rank $\mathbf{B}^{\circ}_f \geq$ rank $\mathbf{C}^{\circ}$, then the following theorem is applicable.

Theorem 12. *Under the above given conditions, a forced mode in the closed-loop system consisting of the plant with a single actuator fault (44), (45), DOC (6), and DVAs (48) and (49) can be achieved if there exists a matrix* $\mathbf{F} \in \mathbb{R}^{r\times r}$ *of the form*

$$\mathbf{F} = \mathbf{F}_{\circ}\left(\mathbf{C}\left(\mathbf{A} - \mathbf{B}_f\mathbf{R}\mathbf{O}^{-1}\mathbf{P} + \mathbf{B}_f\mathbf{S}\right)^{-1}\mathbf{B}\right), \quad (110)$$

where

$$\mathbf{F}_{\circ} = \left(\mathbf{C}\left(\mathbf{A} - \mathbf{B}_f\mathbf{R}\mathbf{O}^{-1}\mathbf{P} + \mathbf{B}_f\mathbf{S}\right)^{-1}\mathbf{B}_f\right)^{\ominus 1} \quad (111)$$

is the pseudoinverse of $\mathbf{C}(\mathbf{A} - \mathbf{B}_f\mathbf{R}\mathbf{O}^{-1}\mathbf{P} + \mathbf{B}_f\mathbf{S})^{-1}\mathbf{B}_f$.

Proof. In a steady-state, DVA equations (106), (108), and (109) imply

$$\mathbf{O}\mathbf{k}_{\text{fa}\circ} + \mathbf{P}\mathbf{e}_{\text{fa}\circ} = \mathbf{0}, \quad (112)$$

$$\left(\mathbf{A} + \mathbf{B}_f\mathbf{S}\right)\mathbf{e}_{\text{fa}\circ} + \mathbf{B}_f\mathbf{R}\mathbf{k}_{\text{fa}\circ} + \left(\mathbf{B}_f\mathbf{F} - \mathbf{B}\right)\mathbf{u}_{c\circ} = \mathbf{0}, \quad (113)$$

$$\mathbf{y}_{\circ} = \mathbf{y}_{\text{fa}\circ} - \mathbf{C}\mathbf{e}_{\text{fa}\circ}, \quad (114)$$

where $\mathbf{e}_{\text{fa}\circ}$, $\mathbf{k}_{\text{fa}\circ}$, $\mathbf{u}_{c\circ}$, $\mathbf{y}_{\circ}$, and $\mathbf{y}_{\text{fa}\circ}$ are steady-state values of the vectors $\mathbf{e}_{\text{fa}}(t)$, $\mathbf{k}_{\text{fa}}(t)$, $\mathbf{u}_c(t)$, $\mathbf{y}(t)$, and $\mathbf{y}_{\text{fa}}(t)$, respectively.

Since $\mathbf{O}$ is a regular matrix, (112) implies

$$\mathbf{k}_{\text{fa}\circ} = -\mathbf{O}^{-1}\mathbf{P}\mathbf{e}_{\text{fa}_\circ} \quad (115)$$

and, substituting (115) into (113), it yields

$$\left(\mathbf{A} - \mathbf{B}_f\mathbf{R}\mathbf{O}^{-1}\mathbf{P} + \mathbf{B}_f\mathbf{S}\right)\mathbf{e}_{\text{fa}_\circ} + \left(\mathbf{B}_f\mathbf{F} - \mathbf{B}\right)\mathbf{u}_{c_\circ} = \mathbf{0}, \quad (116)$$

which gives

$$\mathbf{e}_{\text{fa}_\circ} = -\left(\mathbf{A} - \mathbf{B}_f\mathbf{R}\mathbf{O}^{-1}\mathbf{P} + \mathbf{B}_f\mathbf{S}\right)^{-1}\left(\mathbf{B}_f\mathbf{F} - \mathbf{B}\right)\mathbf{u}_{c_\circ}, \quad (117)$$

$$\mathbf{C}\mathbf{e}_{\text{fa}_\circ} = -\mathbf{C}\left(\mathbf{A} - \mathbf{B}_f\mathbf{R}\mathbf{O}^{-1}\mathbf{P} + \mathbf{B}_f\mathbf{S}\right)^{-1}\left(\mathbf{B}_f\mathbf{F} - \mathbf{B}\right)\mathbf{u}_{c_\circ}, \quad (118)$$

respectively.

Considering that in a steady state $\mathbf{y}_{\circ} = \mathbf{y}_{\text{fa}\circ}$, then (115) implies $\mathbf{C}\mathbf{e}_{\text{fa}_\circ} = \mathbf{0}$; that is, (118) gives

$$-\mathbf{C}\left(\mathbf{A} - \mathbf{B}_f\mathbf{R}\mathbf{O}^{-1}\mathbf{P} + \mathbf{B}_f\mathbf{S}\right)^{-1}\left(\mathbf{B}_f\mathbf{F} - \mathbf{B}\right) = \mathbf{0}, \quad (119)$$

which implies (110). □

5. Illustrative Example

The considered system is represented by the model ((1) and (2)) with the model matrix parameters [42]

$$\mathbf{A} = \begin{bmatrix} 0.5432 & 0.0137 & 0 & 0.9778 & 0 \\ 0 & -0.1178 & 0.2215 & 0 & -0.9661 \\ 0 & -10.5130 & -0.9967 & 0 & 0.6176 \\ 2.6221 & -0.0030 & 0 & -0.5057 & 0 \\ 0 & 0.7075 & -0.0939 & 0 & -0.2120 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} -0.0318 & -0.0548 & -0.0548 & -0.0318 & 0.0004 \\ 0.0024 & 0.0095 & -0.0095 & 0.0024 & 0.0287 \\ -2.2849 & -1.9574 & 1.9574 & 2.2849 & 1.4871 \\ -0.4628 & -0.8107 & 0.8107 & -0.4628 & 0.0024 \\ 0.0944 & -0.1861 & -0.1861 & 0.0944 & -0.8823 \end{bmatrix}, \quad (120)$$

$$\mathbf{V} = \begin{bmatrix} 0.7593 \\ 0.4116 \\ 0.8793 \\ 0.0272 \\ 0.0389 \end{bmatrix},$$

and $\mathbf{v}(t)$ is noise with the variance $\sigma_v^2 = 7.1 \times 10^{-3}$.

The system is controlled by DOC (6), whose parameters were determined by using (15)–(17) for the DOC order $p = 1$ and by setting the tuning parameter as $\delta^{\bullet} = 10$. Using the SeDuMi package [43], the LMI variables take the values

$$\mathbf{R}^{\bullet} = \begin{bmatrix} 0.6975 & 0.0740 & -0.0000 & -0.0581 & -0.0188 & 0.0000 \\ 0.0740 & 0.6196 & 0.0000 & 0.0332 & 0.3265 & -0.0000 \\ -0.0000 & 0.0000 & 0.9569 & -0.0000 & 0.0000 & 0.0000 \\ -0.0581 & 0.0332 & -0.0000 & 0.5631 & 0.0008 & 0.0000 \\ -0.0188 & 0.3265 & 0.0000 & 0.0008 & 0.4883 & -0.0000 \\ 0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.7480 \end{bmatrix},$$

$$\mathbf{U}^{\bullet} = \begin{bmatrix} 8.8815 & 0.6615 & -0.3939 & -0.6145 & -0.2835 & 0.0000 \\ 0.6615 & 4.2501 & -1.2541 & 0.0617 & 1.8953 & -0.0000 \\ -0.3939 & -1.2541 & 9.2830 & -0.0156 & 0.8764 & 0.0000 \\ -0.6145 & 0.0617 & -0.0156 & 10.7212 & 0.3256 & 0.0000 \\ -0.2835 & 1.8953 & 0.8764 & 0.3256 & 13.7591 & 0.0000 \\ 0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & 9.9276 \end{bmatrix},$$

$$\mathbf{H}^{\bullet} = \begin{bmatrix} 0.7480 & 0.0000 & -0.0000 & 0.0000 & -0.0000 \\ 0.0000 & 0.6975 & 0.0740 & -0.0581 & -0.0188 \\ -0.0000 & 0.0740 & 0.6196 & 0.0332 & 0.3265 \\ 0.0000 & -0.0581 & 0.0332 & 0.5631 & 0.0008 \\ -0.0000 & -0.0188 & 0.3265 & 0.0008 & 0.4883 \end{bmatrix},$$

$$\mathbf{Y}^{\bullet} = \begin{bmatrix} -1.0298 & -0.0000 & -0.0000 & -0.0000 & -0.0000 \\ 0.0000 & 4.9096 & -5.5163 & 1.7685 & 1.5014 \\ 0.0000 & 6.9316 & 4.4012 & 2.4561 & -1.5329 \\ 0.0000 & 9.9788 & 1.3867 & 3.5236 & -0.4959 \\ 0.0000 & 4.3088 & 0.6154 & 1.5426 & 0.3277 \\ -0.0000 & -2.5551 & -1.1452 & -0.8279 & 2.5340 \end{bmatrix},$$

$$\gamma^{\bullet} = 13.4697. \tag{121}$$

Using (18), the DOC gain matrices are separated as follows:

$$\mathbf{J} = -1.3767,$$

$$\mathbf{L} = 10^{-8}\begin{bmatrix} -0.0673 & -0.0049 & -0.1764 & -0.0886 \end{bmatrix},$$

$$\mathbf{M} = 10^{-8}\begin{bmatrix} -0.2501 \\ 0.3877 \\ 0.0318 \\ 0.0714 \\ -0.0043 \end{bmatrix}, \tag{122}$$

$$\mathbf{N} = \begin{bmatrix} 9.9037 & -18.8161 & 5.2467 & 16.0287 \\ 8.8524 & 11.2393 & 4.6291 & -10.3231 \\ 14.8863 & 0.4394 & 7.7691 & -0.7511 \\ 6.5785 & -0.7130 & 3.4584 & 1.3946 \\ -2.8555 & -6.3480 & -1.4051 & 9.3270 \end{bmatrix}$$

and the signal gain matrix $\mathbf{W}$ is calculated by (43) as

$$\mathbf{W} = \begin{bmatrix} -5.1176 & -0.1008 & 0.0521 & -0.2320 \\ -7.0937 & 0.5788 & -0.4934 & -1.4523 \\ -11.5871 & -0.2253 & 1.4234 & 0.6632 \\ -3.0013 & -0.7243 & -1.0022 & 3.3029 \\ 2.6838 & 2.3520 & -0.6184 & -5.3107 \end{bmatrix}. \tag{123}$$

The closed-loop system is stable with the closed-loop system matrix eigenvalue spectrum

$$\rho\left(\mathbf{A}^{\bullet} + \mathbf{B}^{\bullet}\mathbf{G}^{\bullet}\mathbf{C}^{\bullet}\right) = \{-4.3447 \ \ -1.9657 \ \ -0.5713 \ \ -1.2897 \ \ -1.0802 \ \ -1.3767\}. \tag{124}$$

In Figures 1 and 2 are shown the time responses of the system output and control variables for the control law realized by DOC of order $p = 1$ in the controller forced mode, acting on the fault-free system. The initial condition was set to $\mathbf{q}_0 = 0$ and the desired output values were $\mathbf{w}(t) = [0.3 \ \ 0.4 \ \ 0.5 \ \ 0.6]$, which were changed within the interval $t \in \langle 20\,\mathrm{s}, 30\,\mathrm{s}\rangle$ to $\mathbf{w}(t) = [0.75 \ \ 0.70 \ \ 0.85 \ \ 1.0]$.

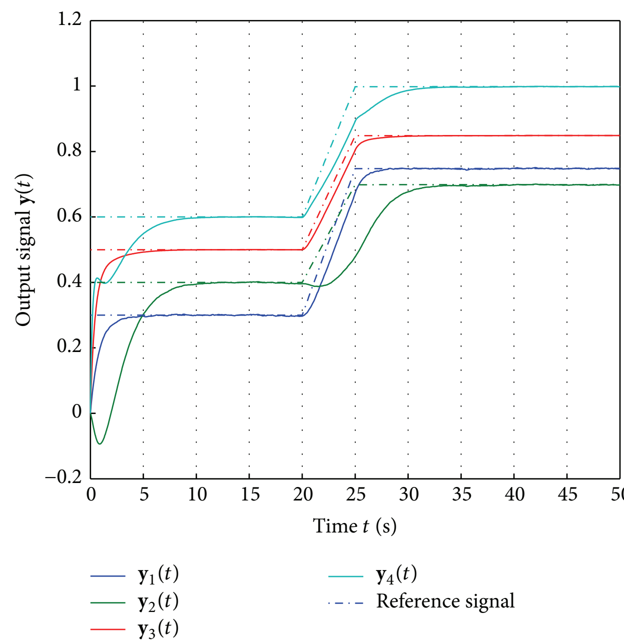

Figure 1: Output response.

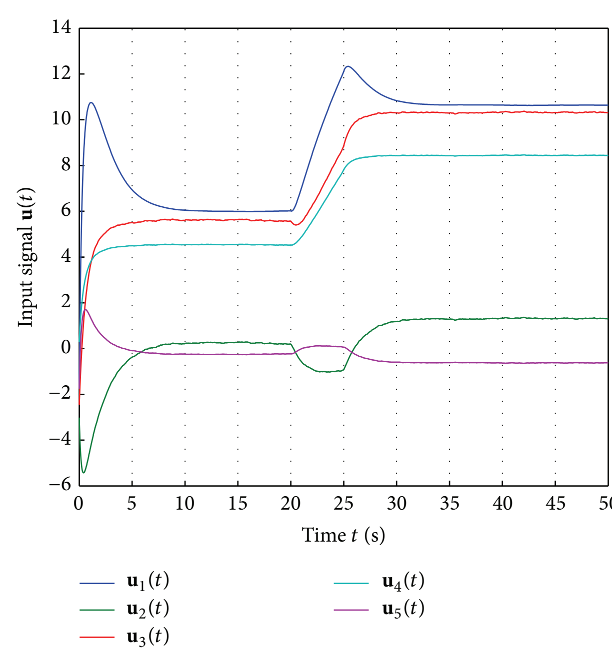

Figure 2: Control variables.

The control reconstruction by DVA is illustrated for single fault of the second actuator, which is modeled by the matrix $\mathbf{B}_f$ of the form

$$\mathbf{B}_f = \begin{bmatrix} -0.0318 & 0 & -0.0548 & -0.0318 & 0.0004 \\ 0.0024 & 0 & -0.0095 & 0.0024 & 0.0287 \\ -2.2849 & 0 & 1.9574 & 2.2849 & 1.4871 \\ -0.4628 & 0 & 0.8107 & -0.4628 & 0.0024 \\ 0.0944 & 0 & -0.1861 & 0.0944 & -0.8823 \end{bmatrix}. \tag{125}$$

For this fault scenario, with a total loss of the the second actuator gain, the closed-loop system consisting of (44), (45), and DOCs (3) and (4) with the above designed parameters is instable and DVA has to be activated to stabilize the faulty system.

The DVA parameters are determined by using (93) and (94) for the DVA order $k = 1$ and by setting the tuning parameter as $\delta^\circ = 1.834$. Using the SeDuMi package, the LMI variables take the values

$$\mathbf{R}^\circ = \begin{bmatrix} 0.7433 & 0.0235 & 0.0409 & -0.3245 & -0.0007 & 0.0000 \\ 0.0235 & 0.7999 & -0.3282 & 0.0094 & 0.5090 & -0.0000 \\ 0.0409 & -0.3282 & 3.6571 & 0.0899 & -0.0134 & 0.0002 \\ -0.3245 & 0.0094 & 0.0899 & 1.2167 & -0.0124 & 0.0000 \\ -0.0007 & 0.5090 & -0.0134 & -0.0124 & 1.2983 & 0.0001 \\ 0.0000 & -0.0000 & 0.0002 & 0.0000 & 0.0001 & 2.2724 \end{bmatrix},$$

$$\mathbf{U}^\circ = \begin{bmatrix} 2.4329 & 0.0396 & -0.0549 & -0.7109 & -0.0189 & -0.0001 \\ 0.0396 & 1.9899 & -0.0690 & -0.0632 & 0.5240 & 0.0000 \\ -0.0549 & -0.0690 & 5.5004 & 0.0779 & -0.4162 & 0.0010 \\ -0.7109 & -0.0632 & 0.0779 & 5.8210 & -0.0589 & -0.0001 \\ -0.0189 & 0.5240 & -0.4162 & -0.0589 & 6.2191 & 0.0008 \\ -0.0001 & 0.0000 & 0.0010 & -0.0001 & 0.0008 & 5.1268 \end{bmatrix}, \quad (126)$$

$$\mathbf{Y}^\circ = \begin{bmatrix} -0.0000 & -0.0000 & -0.0002 & 0.0000 & 0.0004 & -2.1667 \\ 12.2309 & -0.1429 & 1.0569 & 3.4036 & 0.0041 & -0.0001 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 7.6875 & 1.0336 & 0.3775 & 1.1029 & -0.1676 & -0.0000 \\ 5.5771 & 1.9340 & 0.1682 & 2.2672 & -0.3676 & -0.0001 \\ 0.2607 & 0.4858 & -1.7157 & 0.2977 & 4.4104 & 0.0001 \end{bmatrix},$$

$$\gamma^\circ = 7.4681.$$

Using (95), the common gain matrix of DVA is computed as

$$\mathbf{G}^\circ = \begin{bmatrix} 0.0000 & -0.0004 & -0.0000 & 0.0001 & 0.0005 & -0.9535 \\ 20.0899 & -1.3715 & -0.2580 & 8.1918 & 0.6273 & -0.0001 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 12.1035 & 1.2399 & -0.0244 & 4.1211 & -0.5695 & -0.0000 \\ 9.2730 & 3.0481 & 0.1051 & 4.2907 & -1.4311 & -0.0000 \\ 0.7123 & -2.4879 & -0.6979 & 0.5499 & 4.3709 & -0.0001 \end{bmatrix}, \quad (127)$$

from which are separated the matrix parameters of DVA as follows:

$$\mathbf{O} = -0.9535,$$

$$\mathbf{P} = 10^{-3}\begin{bmatrix} 0.0496 & -0.3536 & -0.0350 & 0.0801 & 0.4884 \end{bmatrix},$$

$$\mathbf{R} = 10^{-3}\begin{bmatrix} -0.1466 \\ 0 \\ -0.0475 \\ -0.0479 \\ -0.0709 \end{bmatrix},$$

$$\mathbf{S} = \begin{bmatrix} 20.0899 & -1.3715 & -0.2580 & 8.1918 & 0.6273 \\ 0 & 0 & 0 & 0 & 0 \\ 12.1035 & 1.2399 & -0.0244 & 4.1211 & -0.5695 \\ 9.2730 & 3.0481 & 0.1051 & 4.2907 & -1.4311 \\ 0.7123 & -2.4879 & -0.6979 & 0.5499 & 4.3709 \end{bmatrix}. \quad (128)$$

The eigenvalue spectrum of the matrix $\mathbf{A}_c^\circ = \mathbf{A}^\circ + \mathbf{B}_f^\circ \mathbf{G}^\circ$ is

$$\rho(\mathbf{A}_c^\circ) = \{-3.4345 \quad -2.6887 \quad -1.3070 \quad -1.0258 \pm 0.1850i \quad -0.9535\} \quad (129)$$

and this spectrum determines the dynamics of the closed loop system with DOC after DVA activation.

In Figures 3 and 4 are shown the time responses of the system output and control variables for the control realized by DOC of order $p = 1$ in the controller forced mode and DVA of order $k = 1$, acting on the faulty system. The single second actuator fault occurred at time instant $t = 15$ s and DVA was activated at time instant $t = 17$ s. In simulation, the initial condition was set as $\mathbf{q}_0 = 0$ and the desired output values $\mathbf{w}(t) = [0.3 \; 0.4 \; 0.5 \; 0.6]$ were changed stepwise at the time instants $t = 40$ s and $t = 70$ s to $\mathbf{w}(t) = [0.4 \; -0.2 \; -0.2 \; -0.1 \; 0.1]$ and $\mathbf{w}(t) = [0.45 \; 0.3 \; 0.35 \; 0.4 \; 0.1]$, respectively.

From the system response in Figure 3, it can be seen that the system outputs after intervention of DVA do not reach the desired values; therefore, it was supplemented the forced mode of DVA, too. The supplemented forced mode of DVA was realized by the additive parts $\mathbf{F}\mathbf{u}_c(t)$ in (107) with the signal gain matrix $\mathbf{F}$, computed from (110) and (111) as follows:

$$\mathbf{F} = \begin{bmatrix} 1.0000 & 1.4253 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.0080 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.3113 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.3820 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}. \quad (130)$$

Using the same conditions in simulation as those presented in the comment to Figures 3 and 4, the time responses of the system output and control variables for control realized by DOC of order $p = 1$ in the controller forced mode and DVA of order $k = 1$ in the virtual actuator forced mode are presented in Figures 5 and 6. Although the required values of output variables were achieved, the output and control variables peaks after the activation of DVA are excessively high.

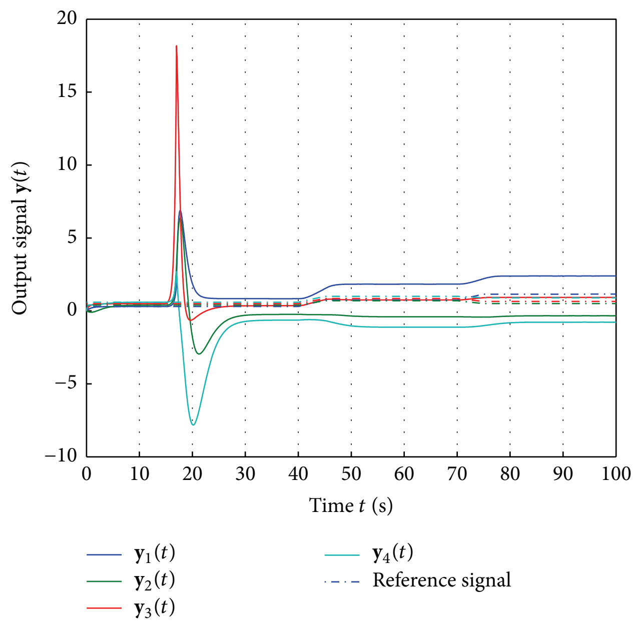

FIGURE 3: Output response.

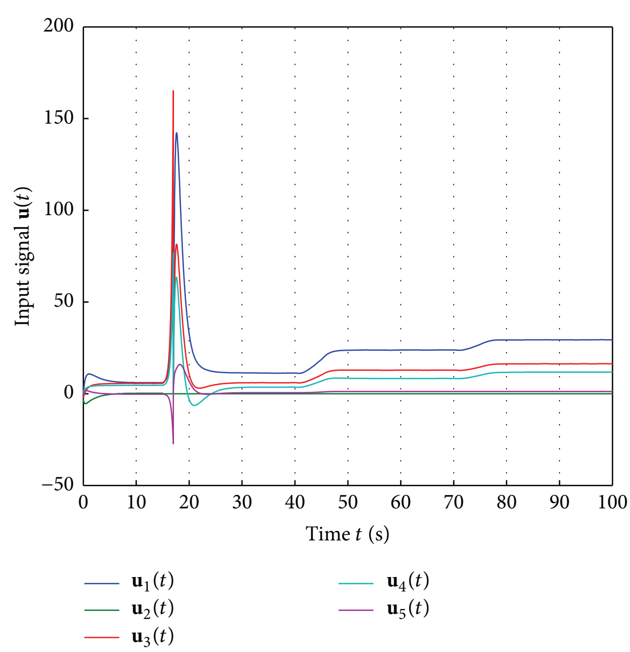

FIGURE 4: Control variables.

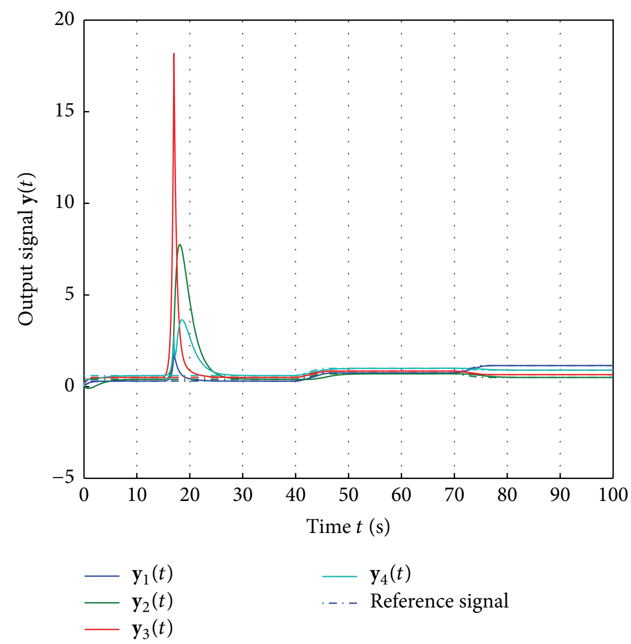

FIGURE 5: Output response.

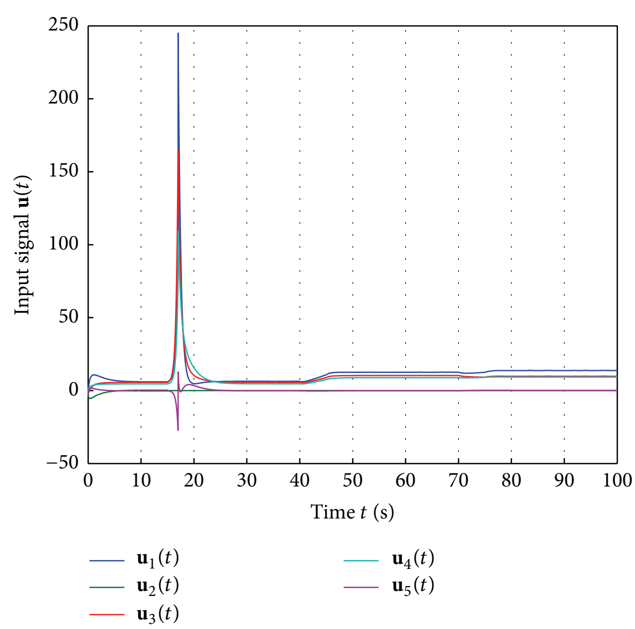

FIGURE 6: Control variables.

Within the same simulation conditions, Figures 7 and 8 show the time response of the system output and input variables for control realized by DOC of order $p = 4$ in the controller forced mode and DVA of order $k = 1$ in the virtual actuator forced mode and Figures 9 and 10 show the time response of the system output and input variables for control realized by DOC of order $p = 7$ in the controller forced mode and DVA of order $k = 1$ in the virtual actuator forced mode, respectively.

It is obvious that an appropriate conjunction of orders of DOC and DVA gives the possibility to significantly reduce the output and control variables peaks after activation of DVAs.

6. Concluding Remarks

A key contribution of the proposed approach is the blending of the virtual actuator technique and the output control principle in a unique dynamic scheme, able to provide fault tolerance against actuator faults with such acceptable responses of the system variables, primarily after activation of DVA, which cannot by reached by applying a static output controller on the exactly same plant.

The proposed model of the dynamic effect of a virtual actuator in the FTC structure relies on newly introduced generalized disturbance patterns, reflecting fading of the nominal

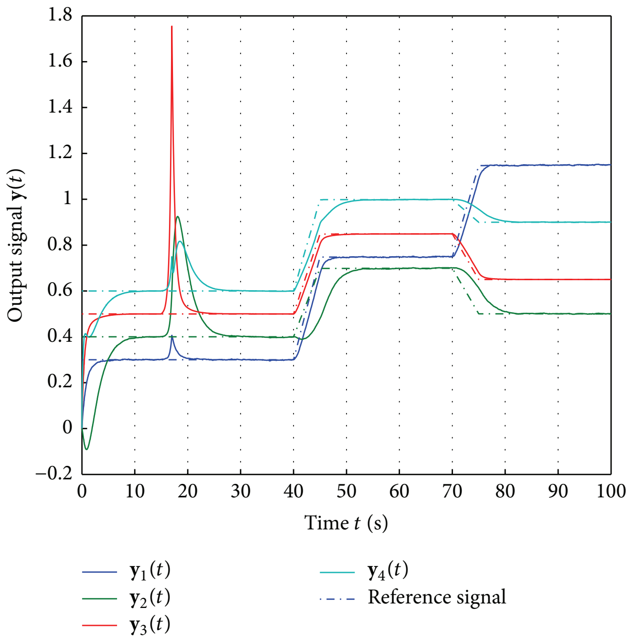

Figure 7: System output.

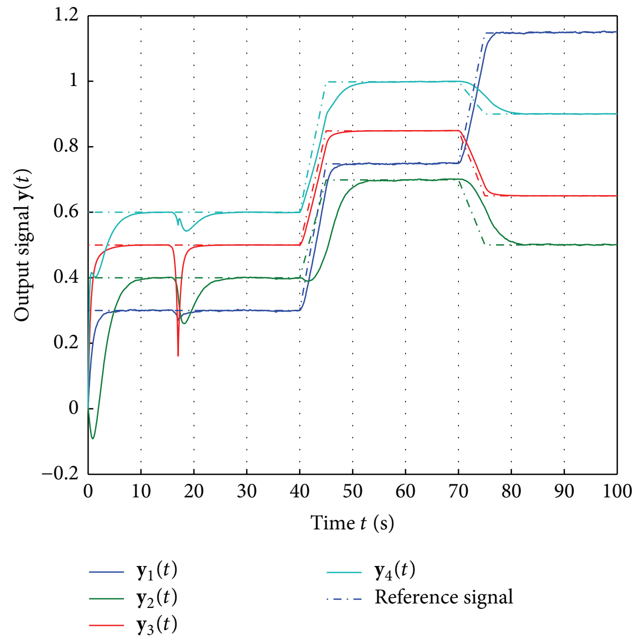

Figure 9: System output.

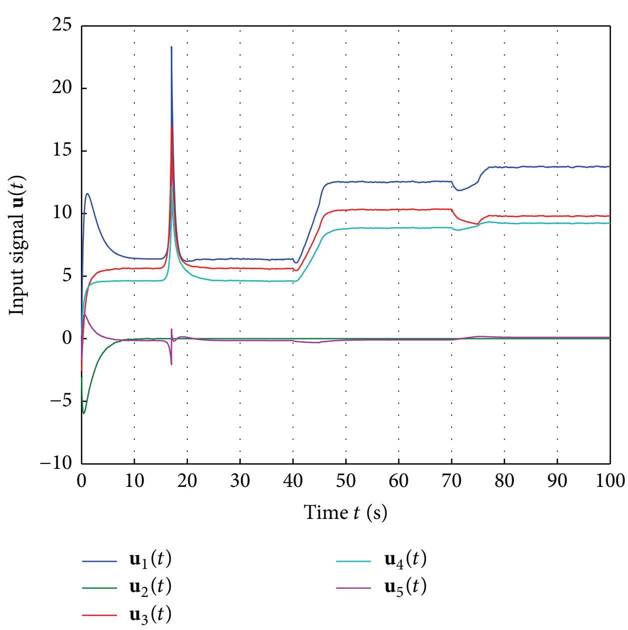

Figure 8: Control variables.

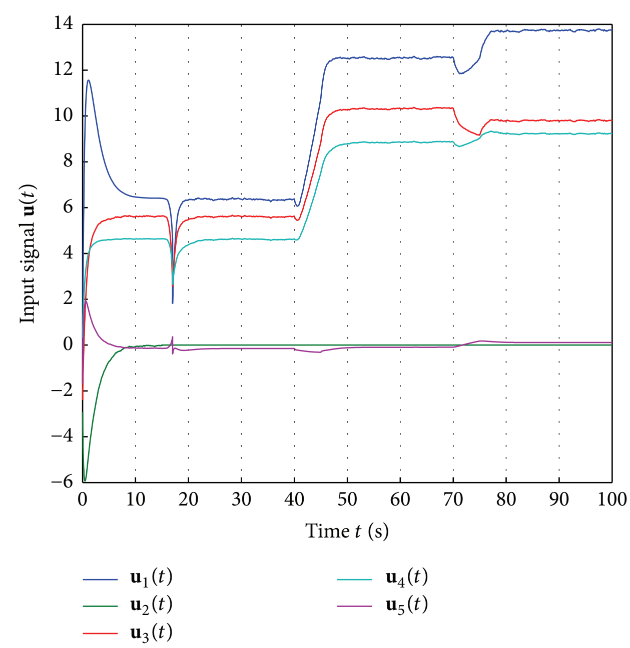

Figure 10: Control variables.

system state variables after DVA activation. This allows including in the DVA design conditions the disturbance input/system output model property by H_∞ norm approach.

Another important contribution presented in the paper consists in showing that the separation principle is valid for the proposed dynamic schemes of arbitrarily order and, as a consequence, these components of the dynamic structure can be designed separately, to achieve stability, as well as the desired performance under both nominal and faulty situations.

The application of the proposed approach requires that a fault detection and isolation subsystem is available. However, it becomes clear that the desired performances depend on the fault isolation time, but a suitable order conjunction of both dynamic components allows significantly extending the time limit of fault detection.

The authors believe the presented method, although partly interactive, can be useful in real context as a suitable and skilled way to set desired properties of FTC with DOC and DVA for linear systems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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