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Research Article

Blow-Up Phenomena for Certain Nonlocal Evolution Equations and Systems

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We provide sufficient conditions for the nonexistence of global positive solutions to the nonlocal evolution equation $u_{tt}(x,t) = (J * u - u)(x,t) + u^p(x,t)$, $(x,t) \in \mathbb{R}^N \times (0,\infty)$, $(u(x,0), u_t(x,0)) = (u_0(x), u_1(x))$, $x \in \mathbb{R}^N$, where $J : \mathbb{R}^N \to \mathbb{R}_+$, $p > 1$, and $(u_0, u_1) \in L^1_{\text{loc}}(\mathbb{R}^N; \mathbb{R}_+) \times L^1_{\text{loc}}(\mathbb{R}^N; \mathbb{R}_+)$. Next, we deal with global nonexistence for certain nonlocal evolution systems. Our method of proof is based on a duality argument.

1. Introduction

In [1], García-Melián and Quirós considered the nonlocal diffusion problem:

$$\begin{aligned} u_t(x,t) &= (J * u - u)(x,t) + u^p(x,t), \\ &\qquad (x,t) \in \mathbb{R}^N \times (0,\infty), \quad (1) \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}^N, \end{aligned}$$

where $J : \mathbb{R}^N \to \mathbb{R}_+$ is a compactly supported nonnegative function with unit integral, $p > 1$, and $u_0 \in L^1(\mathbb{R}^N; \mathbb{R}_+) \cap L^\infty(\mathbb{R}^N; \mathbb{R}_+)$. Equation (1) may model a variety of biological, epidemiological, ecological, and physical phenomena involving media with properties varying in space [2, 3]; similar equations appear, for example, in Ising systems with Glauber dynamics [4]. In [1] the authors proved that (1) has a critical exponent:

$$p_c = 1 + \frac{2}{N}, \quad (2)$$

which is the Fujita exponent for the classical nonlinear heat equation $u_t = \Delta u + u^p$ [5]. More precisely, they proved that if $1 < p \le p_c$, the solution blows up in finite time for any nonnegative and nontrivial initial data $u_0 \in L^1(\mathbb{R}^N; \mathbb{R}_+) \cap L^\infty(\mathbb{R}^N; \mathbb{R}_+)$; if $p > p_c$, there exist global solutions for small initial data $u_0 \in L^1(\mathbb{R}^N; \mathbb{R}_+) \cap L^\infty(\mathbb{R}^N; \mathbb{R}_+)$. Very recently, Yang [6] considered the nonlinear coupled nonlocal diffusion system:

$$\begin{aligned} u_t(x,t) &= (J * u - u)(x,t) + v^p(x,t), \\ &\qquad (x,t) \in \mathbb{R}^N \times (0,\infty), \\ v_t(x,t) &= (J * v - v)(x,t) + u^q(x,t), \quad (3) \\ &\qquad (x,t) \in \mathbb{R}^N \times (0,\infty), \\ (u(x,0), v(x,0)) &= (u_0(x), v_0(x)), \quad x \in \mathbb{R}^N, \end{aligned}$$

where $p, q > 1$ and $(u_0, v_0) \in L^\infty(\mathbb{R}^N; \mathbb{R}_+) \times L^\infty(\mathbb{R}^N; \mathbb{R}_+)$. Equation (3) can serve as a model for the processes of heat diffusion and combustion in two-component continua with nonlinear heat conduction and volumetric release [7]. In this case, Yang established that the critical Fujita curve is given by

$$(pq)^* = 1 + \frac{2}{N} \max\{p+1, q+1\}, \quad (4)$$

which is also the Fujita curve for the coupled heat system $u_t = \Delta u + v^p$ and $v_t = \Delta v + u^q$, obtained by Escobedo and Herrero [8].

In this paper, we are first concerned with the following evolution problem:

$$u_{tt}(x,t) = (J * u - u)(x,t) + u^p(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \tag{5}$$
$$(u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), \quad x \in \mathbb{R}^N,$$

where $J : \mathbb{R}^N \to \mathbb{R}_+$, $p > 1$, and $(u_0, u_1) \in L^1_{\text{loc}}(\mathbb{R}^N; \mathbb{R}_+) \times L^1_{\text{loc}}(\mathbb{R}^N; \mathbb{R}_+)$. We provide a sufficient condition for the nonexistence of global positive solutions to (5). Next, we consider the following two systems:

$$\begin{aligned} u_{tt}(x,t) &= (J * u - u)(x,t) + v^p(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \\ v_{tt}(x,t) &= (J * v - v)(x,t) + u^q(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \\ (u(x,0), v(x,0)) &= (u_0(x), v_0(x)), \quad x \in \mathbb{R}^N, \\ (u_t(x,0), v_t(x,0)) &= (u_1(x), v_1(x)), \quad x \in \mathbb{R}^N, \end{aligned} \tag{6}$$

$$\begin{aligned} u_{tt}(x,t) &= (J * v - v)(x,t) + u^p(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \\ v_{tt}(x,t) &= (J * u - u)(x,t) + v^q(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \\ (u(x,0), v(x,0)) &= (u_0(x), v_0(x)), \quad x \in \mathbb{R}^N, \\ (u_t(x,0), v_t(x,0)) &= (u_1(x), v_1(x)), \quad x \in \mathbb{R}^N, \end{aligned} \tag{7}$$

where $p, q > 1$. For each system, we find a bound on N leading to the absence of global nontrivial solutions. Our method of proof is based on a duality argument developed by Mitidieri and Pokhozhaev [9, 10].

2. Main Results

Through this paper, $\mathbb{R}_+ = [0,\infty)$, $Q = \mathbb{R}^N \times (0,\infty)$, and $J : \mathbb{R}^N \to \mathbb{R}_+$ is a continuous function satisfying the following conditions:

(J1) J is symmetric; that is, $J(z) = J(-z)$, for every $z \in \mathbb{R}^N$.

(J2) $\int_{\mathbb{R}^N} J(z)dz = 1$.

(J3) $A(J) := \int_{\mathbb{R}^N} J(z)|z|^2 \, dz < \infty$.

The following lemmas will be used later.

Lemma 1. *Let $a, b, \varepsilon > 0$ and $p > 1$. Then*

$$ab \leq \varepsilon a^p + c_\varepsilon b^{p/(p-1)}, \tag{8}$$

where $c_\varepsilon = (p-1)p^{-1}(\varepsilon p)^{-1/(p-1)}$.

Lemma 2 (see [11]). *Let X, Y, A, B, C, and D be nonnegative functions and let α_i and θ_i, $i = 1, 2, 3$, be positive reals such that $\alpha_2 < \alpha_1$, $\theta_2 < \theta_1$, $\theta_1, \theta_3 \geq 1$, and $\alpha_3\theta_3 < \alpha_1\theta_1$. If*

$$\begin{aligned} X^{\alpha_1} &\leq AX^{\alpha_2} + BY^{\theta_3}, \\ Y^{\theta_1} &\leq CX^{\alpha_3} + DY^{\theta_2}, \end{aligned} \tag{9}$$

then

$$X^{\alpha_1\theta_1} \leq L\left[A^{\alpha_1\theta_1/(\alpha_1-\alpha_2)} + D^{\theta_1\theta_3/(\theta_1-\theta_2)}B^{\theta_1} + \left(B^{\theta_1}C^{\theta_3}\right)^{\alpha_1\theta_1/(\alpha_1\theta_1-\alpha_3\theta_3)}\right], \tag{10}$$

for some constant $L > 0$.

2.1. A Nonexistence Result for (5). The definition of solutions we adopt for (5) is as follows.

Definition 3. Let $(u_0, u_1) \in L^1_{\text{loc}}(\mathbb{R}^N; \mathbb{R}_+) \times L^1_{\text{loc}}(\mathbb{R}^N; \mathbb{R}_+)$. We say that u is a global weak solution to (5) if $u \in L^p_{\text{loc}}(Q; \mathbb{R}_+)$, $J * u \in L^1_{\text{loc}}(Q; \mathbb{R}_+)$, and

$$\begin{aligned} &\int_Q u^p \varphi \, dx \, dt + \int_{\mathbb{R}^N} u_1(x) \varphi(x,0) \, dx \\ &= \int_Q u \varphi_{tt} \, dx \, dt - \int_Q (J * u - u) \varphi \, dx \, dt \\ &\quad + \int_{\mathbb{R}^N} u_0(x) \varphi_t(x,0) \, dx, \end{aligned} \tag{11}$$

for every regular test function $\varphi \geq 0$ with $\varphi(\cdot, t \geq T) \equiv 0$.

Our first main result is given by the following theorem.

Theorem 4. *Suppose that one of the following conditions hold:*

$$N = 1 < p \tag{12}$$

or

$$\begin{aligned} &N \geq 2, \\ &1 < p < \frac{N+1}{N-1}. \end{aligned} \tag{13}$$

Then (5) admits no global weak solutions other than the trivial one.

Proof. Suppose that u is a nontrivial global weak solution to (5). As a test function, we take

$$\varphi(x,t) = \xi_R(x) \phi^\omega\left(\frac{t}{R}\right), \quad (x,t) \in Q, \tag{14}$$

where $R > 0$ is large enough,

$$\xi_R(x) = \exp\left(\frac{-|x|^2}{R^2}\right), \quad x \in \mathbb{R}^N, \tag{15}$$

$\omega \gg 1$, and $\phi : \mathbb{R}_+ \to [0,1]$ is given by

$$\phi(\sigma) = \begin{cases} 1 & \text{if } 0 \le \sigma \le 1, \\ 0 & \text{if } \sigma \ge 2. \end{cases} \tag{16}$$

From the definition of ϕ, clearly we have

$$\varphi_t(x,0) = 0, \quad x \in \mathbb{R}^N, \tag{17}$$

which yields

$$\int_{\mathbb{R}^N} u_0(x)\,\varphi_t(x,0)\,dx = 0. \tag{18}$$

Writing

$$\int_Q u\varphi_{tt}\,dx\,dt = \int_Q \left(u\varphi^{1/p}\right)\left(\varphi^{-1/p}\varphi_{tt}\right)dx\,dt \tag{19}$$

and applying Lemma 1, we obtain

$$\int_Q u\varphi_{tt}\,dx\,dt \le \varepsilon \int_Q u^p \varphi\,dx\,dt + c_\varepsilon \int_Q \varphi^{-p'/p} \left|\varphi_{tt}\right|^{p'} dx\,dt, \tag{20}$$

for some $\varepsilon > 0$, where $p' = p/(p-1)$. On the other hand,

$$\int_Q \varphi^{-p'/p} \left|\varphi_{tt}\right|^{p'} dx\,dt = \omega^{p'} R^{-2p'} \int_Q \exp\left(\frac{-|x|^2}{R^2}\right) \cdot \phi^{\omega-2p/(p-1)}\left(\frac{t}{R}\right)\left|h\left(\frac{t}{R}\right)\right|^{p'} dx\,dt, \tag{21}$$

where

$$h(\sigma) := (\omega-1)\,\phi'(\sigma) + \phi(\sigma)\,\phi''(\sigma), \quad \sigma \ge 0. \tag{22}$$

Using the change of variable $x = Ry$ and $t = Rs$, we obtain

$$\int_Q \varphi^{-p'/p} \left|\varphi_{tt}\right|^{p'} dx\,dt = \omega^{p'} R^{N+1-2p'} \int_Q \exp\left(-|y|^2\right) \cdot \phi^{\omega-2p/(p-1)}(s)\,|h(s)|^{p'}\,dy\,ds. \tag{23}$$

The above equality with (20) yields

$$\int_Q u\varphi_{tt}\;dx\,dt \le \varepsilon \int_Q u^p \varphi\,dx\,dt + c_1 R^{N+1-2p'}, \tag{24}$$

for some constant $c_1 > 0$. Next, we have

$$\int_Q (J * u)\,\varphi\,dx\,dt = \int_0^\infty \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} J(x-y)\,u(y,t)\,dy\right)\varphi(x,t)\,dx\,dt. \tag{25}$$

Using the symmetry of J and Fubini's theorem, we obtain

$$\begin{aligned} &\int_Q (J * u)\,\varphi\,dx\,dt \\ &= \int_0^\infty \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} J(x-y)\,u(y,t)\,dy\right)\varphi(x,t)\,dx\,dt \\ &= \int_0^\infty \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} J(y-x)\,u(y,t)\,dy\right)\varphi(x,t)\,dx\,dt \\ &= \int_0^\infty \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} J(y-x)\,\varphi(x,t)\,dx\right)u(y,t)\,dy \\ &= \int_Q (J * \varphi)\,u\,dx\,dt. \end{aligned} \tag{26}$$

Therefore,

$$\int_Q (J * u - u)\,\varphi\,dx\,dt = \int_Q (J * \varphi - \varphi)\,u\,dx\,dt. \tag{27}$$

Using the property (J1), we obtain

$$\begin{aligned} &(J * \varphi - \varphi)(x,t) \\ &= \phi^\omega\left(\frac{t}{R}\right)\left(\int_{\mathbb{R}^N} J(x-y)\,\xi_R(y)\,dy - \xi_R(x)\right) \\ &= \phi^\omega\left(\frac{t}{R}\right)\left(\int_{\mathbb{R}^N} J(y-x)\,\xi_R(y)\,dy - \xi_R(x)\right) \\ &= \phi^\omega\left(\frac{t}{R}\right)\left(\int_{\mathbb{R}^N} J(z)\,\xi_R(x+z)\,dz - \xi_R(x)\right). \end{aligned} \tag{28}$$

By the property (J2) and the definition of φ, we have

$$\begin{aligned} &(J * \varphi - \varphi)(x,t) = \phi^\omega\left(\frac{t}{R}\right)\int_{\mathbb{R}^N} J(z) \\ &\cdot \left(\xi_R(x+z) - \xi(x)\right)dz = \phi^\omega\left(\frac{t}{R}\right)\int_{\mathbb{R}^N} J(z) \\ &\cdot \left(\exp\left(-\frac{|x+z|^2}{R^2}\right) - \exp\left(-\frac{|x|^2}{R^2}\right)\right)dz \\ &= \phi^\omega\left(\frac{t}{R}\right)\exp\left(-\frac{|x|^2}{R^2}\right)\int_{\mathbb{R}^N} J(z) \\ &\cdot \left(\exp\left(\frac{-|x+z|^2 + |x|^2}{R^2}\right) - 1\right)dz = \varphi(x,t) \\ &\cdot \int_{\mathbb{R}^N} J(z) \\ &\cdot \left(\exp\left(\frac{-1}{R^2}\left(2\sum_{i=1}^N x_i z_i + |z|^2\right)\right) - 1\right)dz. \end{aligned} \tag{29}$$

The property (J3) and the inequality $e^x \geq x + 1$ yield

$$\begin{aligned}(J * \varphi - \varphi)(x,t) &\geq \frac{-\varphi(x,t)}{R^2}\int_{\mathbb{R}^N} J(z)\left(2\sum_{i=1}^{N} x_i z_i + |z|^2\right) dz \\ &= \frac{-\varphi(x,t)}{R^2}\int_{\mathbb{R}^N} J(z)|z|^2\,dz = -A(J)R^{-2}\varphi(x,t).\end{aligned} \tag{30}$$

From this, we have

$$-\int_Q (J * u - u)\varphi\,dx\,dt \leq A(J)R^{-2}\int_Q u\varphi\,dx\,dt. \tag{31}$$

Writing

$$\int_Q u\varphi\,dx\,dt = \int_Q u\varphi^{1/p}\varphi^{1/p'}\,dx\,dt, \tag{32}$$

using Hölder's inequality and Lemma 1, we obtain

$$\begin{aligned}-\int_Q (J * u - u)\varphi\,dx\,dt &\leq A(J) \\ &\cdot R^{-2}\left(\int_Q u^p\varphi\,dx\,dt\right)^{1/p}\left(\int_Q \varphi\,dx\,dt\right)^{1/p'} \\ &\leq \delta\int_Q u^p\varphi\,dx\,dt + c_\delta A(J)^{p'}R^{-2p'}\int_Q\varphi\,dx\,dt \\ &\leq \delta\int_Q u^p\varphi\,dx\,dt + c_\delta A(J)^{p'} \\ &\cdot R^{-2p'}\left(\int_{\mathbb{R}^N}\exp\left(\frac{-|x|^2}{R^2}\right)dx\right)\left(\int_0^\infty \phi^\omega\left(\frac{t}{R}\right)dt\right) \\ &\leq \delta\int_Q u^p\varphi\,dx\,dt + c_\delta A(J)^{p'} \\ &\cdot R^{N+1-2p'}\left(\int_{\mathbb{R}^N}\exp\left(-|y|^2\right)dx\right)\left(\int_0^\infty \phi^\omega(s)\,ds\right),\end{aligned} \tag{33}$$

for some $\delta > 0$. We get

$$\begin{aligned}&-\int_Q (J * u - u)\varphi\,dx\,dt \\ &\quad\leq \delta\int_Q u^p\varphi\,dx\,dt + c_2R^{N+1-2p'},\end{aligned} \tag{34}$$

for some constant $c_2 > 0$. Consequently, it follows from (11), (18), (24), and (34) that

$$(1 - \varepsilon - \delta)\int_Q u^p\varphi\,dx\,dt \leq (c_1 + c_2)R^{N+1-2p'}. \tag{35}$$

For $\varepsilon = \delta = 1/4$, we obtain

$$\int_Q u^p\varphi\,dx\,dt \leq cR^{N+1-2p'}, \tag{36}$$

where $c = 2(c_1 + c_2)$. Observe that if one of conditions (12) or (13) is satisfied, then $N + 1 - 2p' < 0$. In this case, letting $R \to \infty$ in the above inequality and using the monotone convergence theorem, we obtain

$$\int_Q u^p\,dx\,dt = 0, \tag{37}$$

which is a contradiction. The proof is finished. □

2.2. A Nonexistence Result for System (6). The definition of solutions we adopt for (6) is as follows.

Definition 5. Let $(u_i, v_i) \in L^1_{\text{loc}}(\mathbb{R}^N;\mathbb{R}_+) \times L^1_{\text{loc}}(\mathbb{R}^N;\mathbb{R}_+)$, $i = 0, 1$. We say that the pair (u,v) is a global weak solution to (6) if $(u,v) \in L^q_{\text{loc}}(Q;\mathbb{R}_+) \times L^p_{\text{loc}}(Q;\mathbb{R}_+)$, $(J * u, J * v) \in L^1_{\text{loc}}(Q;\mathbb{R}_+) \times L^1_{\text{loc}}(Q;\mathbb{R}_+)$, and

$$\begin{aligned}&\int_Q v^p\varphi\,dx\,dt + \int_{\mathbb{R}^N} u_1(x)\varphi(x,0)\,dx \\ &\quad= \int_Q u\varphi_{tt}\,dx\,dt - \int_Q (J * u - u)\varphi\,dx\,dt \\ &\quad\quad+ \int_{\mathbb{R}^N} u_0(x)\varphi_t(x,0)\,dx;\end{aligned} \tag{38}$$

$$\begin{aligned}&\int_Q u^q\varphi\,dx\,dt + \int_{\mathbb{R}^N} v_1(x)\varphi(x,0)\,dx \\ &\quad= \int_Q v\varphi_{tt}\,dx\,dt - \int_Q (J * v - v)\varphi\,dx\,dt \\ &\quad\quad+ \int_{\mathbb{R}^N} v_0(x)\varphi_t(x,0)\,dx,\end{aligned} \tag{39}$$

for every regular test function $\varphi \geq 0$ with $\varphi(\cdot, t \geq T) \equiv 0$.

We have the following result.

Theorem 6. *Let $p, q > 1$. Suppose that*

$$1 \leq N < 1 + \frac{2}{pq - 1}\max\{p + 1, q + 1\}. \tag{40}$$

Then (6) admits no global weak solutions other than the trivial one.

Proof. Suppose that (u,v) is a nontrivial global weak solution to (6). As a test function, we take the function φ defined by (14). From the definition of φ, we have

$$\int_{\mathbb{R}^N} u_0(x)\varphi_t(x,0)\,dx = \int_{\mathbb{R}^N} v_0(x)\varphi_t(x,0)\,dx = 0. \tag{41}$$

Writing

$$\int_Q u\varphi_{tt}\,dx\,dt = \int_Q \left(u\varphi^{1/q}\right)\left(\varphi^{-1/q}\varphi_{tt}\right)dx\,dt \tag{42}$$

and using Hölder's inequality, we obtain

$$\begin{aligned}&\int_Q u\varphi_{tt}\,dx\,dt \\ &\quad\leq \left(\int_Q u^q\varphi\,dx\,dt\right)^{1/q}\left(\int_Q \varphi^{-q'/q}|\varphi_{tt}|^{q'}\,dx\,dt\right)^{1/q'},\end{aligned} \tag{43}$$

where $q' = q/(q-1)$. On the other hand, from (23), we have

$$\int_Q \varphi^{-q'/q} |\varphi_{tt}|^{q'} \, dx\, dt = \omega^{q'} R^{N+1-2q'} \int_Q \exp\left(-|y|^2\right) \cdot \phi^{\omega-2q/(q-1)}(s) \, |h(s)|^{q'} \, dy\, ds, \tag{44}$$

which yields

$$\int_Q u\varphi_{tt}\, dx\, dt \le c_1 R^{(N+1)/q'-2} \left(\int_Q u^q \varphi \, dx\, dt\right)^{1/q}, \tag{45}$$

for some constant $c_1 > 0$. Using (33), we obtain

$$\begin{aligned} -\int_Q (J*u-u)\,\varphi\, dx\, dt &\le A(J) \cdot R^{-2} \left(\int_Q u^q\varphi\, dx\, dt\right)^{1/q} \left(\int_Q \varphi\, dx\, dt\right)^{1/q'} \le A(J) \\ &\cdot R^{-2} \left(\int_Q u^q\varphi\, dx\, dt\right)^{1/q} \\ &\cdot \left(\int_{\mathbb{R}^N} \exp\left(\frac{-|x|^2}{R^2}\right) dx\right)^{1/q'} \\ &\cdot \left(\int_0^\infty \phi^\omega\left(\frac{t}{R}\right) dt\right)^{1/q'} \le A(J) \\ &\cdot R^{-2+(N+1)/q'} \left(\int_{\mathbb{R}^N} \exp\left(-|y|^2\right) dx\right)^{1/q'} \\ &\cdot \left(\int_0^\infty \phi^\omega(s)\, ds\right)^{1/q'} \left(\int_Q u^q\varphi\, dx\, dt\right)^{1/q}, \end{aligned} \tag{46}$$

which yields

$$-\int_Q (J*u-u)\,\varphi\, dx\, dt \le c_2 R^{-2+(N+1)/q'} \left(\int_Q u^q\varphi\, dx\, dt\right)^{1/q}, \tag{47}$$

for some constant $c_2 > 0$. As consequence, from (38), (45), and (47), it follows that

$$\int_Q v^p \varphi\, dx\, dt \le C_1 R^{-2+(N+1)/q'} \left(\int_Q u^q\varphi\, dx\, dt\right)^{1/q}, \tag{48}$$

where $C_1 = c_1 + c_2$. Similarly, we have

$$\int_Q u^q \varphi\, dx\, dt \le C_2 R^{-2+(N+1)/p'} \left(\int_Q v^p\varphi\, dx\, dt\right)^{1/p}, \tag{49}$$

for some constant $C_2 > 0$. Combining (48) with (49), we obtain

$$\begin{aligned} \left(\int_Q v^p\varphi\, dx\, dt\right)^{1-1/pq} &\le CR^{\lambda_1}, \\ \left(\int_Q u^q\varphi\, dx\, dt\right)^{1-1/pq} &\le CR^{\lambda_2}, \end{aligned} \tag{50}$$

for some constant $C > 0$, where

$$\begin{aligned} \lambda_1 &= -2 - \frac{2}{q} + \frac{N+1}{pq}(qp-1), \\ \lambda_2 &= -2 - \frac{2}{p} + \frac{N+1}{pq}(qp-1). \end{aligned} \tag{51}$$

Observe that (40) is equivalent to $\lambda_i < 0$, $i = 1, 2$. Under this condition, letting $R \to \infty$ in (50), we get

$$\int_Q v^p\, dx\, dt = \int_Q u^q\, dx\, dt = 0, \tag{52}$$

which is a contradiction. □

Remark 7. Taking $u = v$ and $p = q$ in Theorem 6, we obtain the result given by Theorem 4 for (5).

2.3. A Nonexistence Result for System (7). The definition of solutions we adopt for (7) is as follows.

Definition 8. Let $(u_i, v_i) \in L^1_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}_+) \times L^1_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}_+)$, $i = 0, 1$. We say that the pair (u, v) is a global weak solution to (6) if $(u,v) \in L^p_{\mathrm{loc}}(Q;\mathbb{R}_+) \times L^q_{\mathrm{loc}}(Q;\mathbb{R}_+)$, $u \not\equiv 0$, $v \not\equiv 0$, $(J*u, J*v) \in L^1_{\mathrm{loc}}(Q;\mathbb{R}_+) \times L^1_{\mathrm{loc}}(Q;\mathbb{R}_+)$, and

$$\begin{aligned} \int_Q u^p\varphi\, dx\, dt + \int_{\mathbb{R}^N} u_1(x)\,\varphi(x,0)\, dx &= \int_Q u\varphi_{tt}\, dx\, dt - \int_Q (J*v-v)\,\varphi\, dx\, dt \\ &+ \int_{\mathbb{R}^N} u_0(x)\,\varphi_t(x,0)\, dx; \end{aligned} \tag{53}$$

$$\begin{aligned} \int_Q v^q\varphi\, dx\, dt + \int_{\mathbb{R}^N} v_1(x)\,\varphi(x,0)\, dx &= \int_Q v\varphi_{tt}\, dx\, dt - \int_Q (J*u-u)\,\varphi\, dx\, dt \\ &+ \int_{\mathbb{R}^N} v_0(x)\,\varphi_t(x,0)\, dx, \end{aligned} \tag{54}$$

for every regular test function $\varphi \ge 0$ with $\varphi(\cdot, t \ge T) \equiv 0$.

We have the following result.

Theorem 9. *Let $p, q > 1$. If*

$$1 \le N < \max\{\Theta_1, \Theta_2\}, \tag{55}$$

where

$$\begin{aligned} \Theta_1 &= \min\left\{\frac{p+1}{p-1}, \frac{q+1}{q-1}, \frac{pq+2p+1}{pq-1}\right\}, \\ \Theta_2 &= \min\left\{\frac{p+1}{p-1}, \frac{q+1}{q-1}, \frac{pq+2q+1}{pq-1}\right\}, \end{aligned} \tag{56}$$

then (7) admits nonglobal weak solutions.

Proof. As before, we argue by contradiction. Suppose that (u, v) is a nontrivial global weak solution to (7). As a test function, we take the function φ defined by (14). From (45), we have

$$\int_Q u\varphi_{tt}\,dx\,dt \le c_1 R^{(N+1)/p'-2}\left(\int_Q u^p\varphi\,dx\,dt\right)^{1/p}. \quad (57)$$

From (47), we have

$$-\int_Q (J * v - v)\,\varphi\,dx\,dt \le c_2 R^{-2+(N+1)/q'}\left(\int_Q v^q\varphi\,dx\,dt\right)^{1/q}. \quad (58)$$

Using (53), (57), and (58), we get

$$\int_Q u^p\varphi\,dx\,dt \le c_1 R^{(N+1)/p'-2}\left(\int_Q u^p\varphi\,dx\,dt\right)^{1/p} + c_2 R^{-2+(N+1)/q'}\left(\int_Q v^q\varphi\,dx\,dt\right)^{1/q}. \quad (59)$$

Similarly, using (54), (57), and (58), we get

$$\int_Q v^q\varphi\,dx\,dt \le d_1 R^{(N+1)/q'-2}\left(\int_Q v^q\varphi\,dx\,dt\right)^{1/q} + d_2 R^{-2+(N+1)/p'}\left(\int_Q u^p\varphi\,dx\,dt\right)^{1/p}. \quad (60)$$

Here, $c_i, d_i, i = 1, 2$, are some positive constants. Set

$$X = \left(\int_Q u^p\varphi\,dx\,dt\right)^{1/p}, \quad Y = \left(\int_Q v^q\varphi\,dx\,dt\right)^{1/q}; \quad (61)$$

we obtain from (59) and (60) the following system:

$$X^p \le c_1 R^{(N+1)/p'-2} X + c_2 R^{-2+(N+1)/q'} Y, \quad Y^q \le d_2 R^{-2+(N+1)/p'} X + d_1 R^{(N+1)/q'-2} Y. \quad (62)$$

Using Lemma 2, we obtain

$$X^{pq} \le c\left(R^{\lambda_1} + R^{\lambda_2} + R^{\lambda_3}\right), \quad (63)$$

where

$$\frac{p-1}{q}\lambda_1 = (N+1)(p-1) - 2p,$$
$$\frac{q-1}{q}\lambda_2 = (N+1)(q-1) - 2q,$$
$$\frac{pq-1}{pq}\lambda_3 = -2q + (N+1)(q-1) + \frac{(N+1)(p-1)-2p}{p}. \quad (64)$$

Similarly. we have

$$Y^{pq} \le c\left(R^{\mu_1} + R^{\mu_2} + R^{\mu_3}\right), \quad (65)$$

where

$$\frac{q-1}{p}\mu_1 = (N+1)(q-1) - 2q,$$
$$\frac{p-1}{p}\mu_2 = (N+1)(p-1) - 2p,$$
$$\frac{pq-1}{pq}\mu_3 = -2p + (N+1)(p-1) + \frac{(N+1)(q-1)-2q}{q}. \quad (66)$$

It is not difficult to observe that condition (55) is equivalent to $\lambda_i < 0, i = 1, 2, 3$, or $\mu_i < 0, i = 1, 2, 3$. In both cases, letting $R \to \infty$ in (63) or in (65), we obtain $XY = 0$, which is a contradiction. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] J. García-Melián and F. Quirós, "Fujita exponents for evolution problems with nonlocal diffusion," *Journal of Evolution Equations*, vol. 10, no. 1, pp. 147–161, 2010.

[2] O. Diekmann and H. G. Kaper, "On the bounded solutions of a nonlinear convolution equation," *Nonlinear Analysis—Theory, Methods & Applications*, vol. 2, no. 6, pp. 721–737, 1978.

[3] P. Fife, "Some nonclassical trends in parabolic and parabolic-like evolutions," in *Trends in Nonlinear Analysis*, pp. 153–191, Springer, Berlin, Germany, 2003.

[4] A. De Masi, E. Orlandi, E. Presutti, and L. Triolo, "Glauber evolution with Kac potentials. I. Mesoscopic and macroscopic limits, interface dynamics," *Nonlinearity*, vol. 7, no. 3, pp. 633–696, 1994.

[5] H. Fujita, "On the blowing up of solution of the Cauchy problem for $ut = \Delta u + u^{\alpha+1}$," *Journal of the Faculty of Science, the University of Tokyo*, vol. 13, pp. 109–124, 1966.

[6] J. Yang, "Fujita-type phenomenon of nonlinear coupled nonlocal diffusion system," *Journal of Mathematical Analysis and Applications*, vol. 428, no. 1, pp. 227–237, 2015.

[7] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, *Blow-up in Quasilinear Parabolic Equations*, vol. 19 of *De Gruyter Expositions in Mathematics*, Walter de Gruyter, Berlin, Germany, 1995.

[8] M. Escobedo and M. A. Herrero, "Boundedness and blow-up for a semilinear reaction-diffusion system," *Journal of Differential Equations*, vol. 89, no. 1, pp. 176–202, 1991.

[9] E. Mitidieri and S. I. Pokhozhaev, "A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities," *Trudy Matematicheskogo Instituta imeni V.A. Steklova*, vol. 234, pp. 3–383, 2001.

[10] S. I. Pokhozhaev, "Essentially nonlinear capacities induced by differential operators," *Doklady Akademii Nauk*, vol. 357, no. 5, pp. 592–594, 1997.

[11] M. Kirane and M. Qafsaoui, "Global nonexistence for the Cauchy problem of some nonlinear reaction-diffusion systems," *Journal of Mathematical Analysis and Applications*, vol. 268, no. 1, pp. 217–243, 2002.