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*Research Article*

# On the Slowly Decreasing Sequences of Fuzzy Numbers

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We introduce the slowly decreasing condition for sequences of fuzzy numbers. We prove that this is a Tauberian condition for the statistical convergence and the Cesáro convergence of a sequence of fuzzy numbers.

## 1. Introduction

The concept of statistical convergence was introduced by Fast [1]. A sequence $(x_k)_{k\in\mathbb{N}}$ of real numbers is said to be *statistically convergent* to some number $l$ if for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty}\frac{1}{n+1}\left|\left\{k \le n : \left|x_k - l\right| \ge \varepsilon\right\}\right| = 0, \quad (1)$$

where by $|S|$ and $\mathbb{N}$, we denote the number of the elements in the set $S$ and the set of natural numbers, respectively. In this case, we write $st\text{-}\lim_{k\to\infty}x_k = l$.

A sequence $(x_k)$ of real numbers is said to be $(C,1)$-convergent to $l$ if its Cesàro transform $\{(C_1x)_n\}$ of order one converges to $l$ as $n \to \infty$, where

$$(C_1x)_n = \frac{1}{n+1}\sum_{k=0}^{n}x_k \quad \forall n \in \mathbb{N}. \quad (2)$$

In this case, we write $(C,1)\text{-}\lim_{k\to\infty}x_k = l$.

We recall that a sequence $(x_k)$ of real numbers is said to be *slowly decreasing* according to Schmidt [2] if

$$\lim_{\lambda\to1^+}\liminf_{n\to\infty}\min_{n<k\le\lambda_n}\left(x_k - x_n\right) \ge 0, \quad (3)$$

where we denote by $\lambda_n$ the integral part of the product $\lambda n$, in symbol $\lambda_n := [\lambda n]$.

It is easy to see that (3) is satisfied if and only if for every $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon)$ and $\lambda = \lambda(\varepsilon) > 1$, as close to 1 as we wish, such that

$$x_k - x_n \ge -\varepsilon \quad \text{whenever } n_0 \le n < k \le \lambda_n. \quad (4)$$

**Lemma 1** (see [3, Lemma 1]). *Let $(x_k)$ be a sequence of real numbers. Condition* (3) *is equivalent to the following relation:*

$$\lim_{\lambda\to1^-}\liminf_{n\to\infty}\min_{\lambda_n<k\le n}\left(x_n - x_k\right) \ge 0. \quad (5)$$

A sequence $(x_k)$ of real numbers is said to be *slowly increasing* if

$$\lim_{\lambda\to1^+}\limsup_{n\to\infty}\max_{n<k\le\lambda_n}\left(x_k - x_n\right) \le 0. \quad (6)$$

Clearly, it is trivial that $(x_k)$ is slowly increasing if and only if the sequence $(-x_k)$ is slowly decreasing.

Furthermore, if a sequence $(x_k)$ of real numbers satisfies Landau's one-sided Tauberian condition (see [4, page 121])

$$k\left(x_k - x_{k-1}\right) \ge -H \quad \text{for some } H > 0 \text{ and every } k, \quad (7)$$

then $(x_k)$ is slowly decreasing.

Móricz [3, Lemma 6] proved that if a sequence $(x_k)$ is slowly decreasing, then

$$st\text{-}\lim_{k\to\infty}x_k = l \Longrightarrow \lim_{k\to\infty}x_k = l. \quad (8)$$

Also, Hardy [4, Theorem 68] proved that if a sequence $(x_k)$ is slowly decreasing, then

$$(C,1)\text{-}\lim_{k\to\infty}x_k = l \Longrightarrow \lim_{k\to\infty}x_k = l. \quad (9)$$

Maddox [5] defined a slowly decreasing sequence in an ordered linear space and proved implication (9) for slowly decreasing sequences in an ordered linear space.

We recall in this section the basic definitions dealing with fuzzy numbers. In 1972, Chang and Zadeh [6] introduced the concept of fuzzy number which is commonly used in fuzzy analysis and in many applications.

A *fuzzy number* is a fuzzy set on the real axis, that is, a mapping $u : \mathbb{R} \to [0,1]$ which satisfies the following four conditions

(i) $u$ is normal; that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

(ii) $u$ is fuzzy convex; that is, $u[\lambda x + (1-\lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$.

(iii) $u$ is upper semicontinuous.

(iv) The set $[u]_0 := \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, where $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$.

We denote the set of all fuzzy numbers on $\mathbb{R}$ by $E^1$ and call it *the space of fuzzy numbers*. *α-level set* $[u]_\alpha$ of $u \in E^1$ is defined by

$$[u]_\alpha := \begin{cases} \{t \in \mathbb{R} : x(t) \geq \alpha\}, & (0 < \alpha \leq 1), \\ \overline{\{t \in \mathbb{R} : x(t) > \alpha\}}, & (\alpha = 0). \end{cases} \tag{10}$$

The set $[u]_\alpha$ is closed, bounded, and nonempty interval for each $\alpha \in [0,1]$ which is defined by $[u]_\alpha := [u^-(\alpha), u^+(\alpha)]$. $\mathbb{R}$ can be embedded in $E^1$ since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(x) := \begin{cases} 1, & (x = r), \\ 0, & (x \neq r). \end{cases} \tag{11}$$

Let $u, v, w \in E^1$ and $k \in \mathbb{R}$. Then the operations addition and scalar multiplication are defined on $E^1$ by

$$\begin{gathered} u + v = w \Longleftrightarrow [w]_\alpha = [u]_\alpha + [v]_\alpha \quad \forall \alpha \in [0,1] \\ \Longleftrightarrow w^-(\alpha) = u^-(\alpha) + v^-(\alpha), \\ w^+(\alpha) = u^+(\alpha) + v^+(\alpha) \quad \forall \alpha \in [0,1], \\ [ku]_\alpha = k[u]_\alpha \quad \forall \alpha \in [0,1] \end{gathered} \tag{12}$$

(cf. Bede and Gal [7]).

**Lemma 2** (see [7]). *The following statements hold.*

(i) *$\overline{0} \in E^1$ is neutral element with respect to $+$, that is, $u + \overline{0} = \overline{0} + u = u$ for all $u \in E^1$.*

(ii) *With respect to $\overline{0}$, none of $u \neq \overline{r}$, $r \in \mathbb{R}$ has opposite in $E^1$.*

(iii) *For any $a, b \in \mathbb{R}$ with $a, b \geq 0$ or $a, b \leq 0$ and any $u \in E^1$, we have $(a+b)u = au + bu$. For general $a, b \in \mathbb{R}$, the above property does not hold.*

(iv) *For any $a \in \mathbb{R}$ and any $u, v \in E^1$, we have $a(u+v) = au + av$.*

(v) *For any $a, b \in \mathbb{R}$ and any $u \in E^1$, we have $a(bu) = (ab)u$.*

Notice that $E^1$ is not a linear space over $\mathbb{R}$.

Let $W$ be the set of all closed bounded intervals $A$ of real numbers with endpoints $\underline{A}$ and $\overline{A}$; that is, $A := [\underline{A}, \overline{A}]$. Define the relation $d$ on $W$ by

$$d(A, B) := \max\left\{\left|\underline{A} - \underline{B}\right|, \left|\overline{A} - \overline{B}\right|\right\}. \tag{13}$$

Then, it can be easily observed that $d$ is a metric on $W$ and $(W, d)$ is a complete metric space (cf. Nanda [8]). Now, we may define the metric $D$ on $E^1$ by means of the Hausdorff metric $d$ as follows:

$$\begin{aligned} D(u, v) &:= \sup_{\alpha \in [0,1]} d([u]_\alpha, [v]_\alpha) \\ &:= \sup_{\alpha \in [0,1]} \max\left\{\left|u^-(\alpha) - v^-(\alpha)\right|, \left|u^+(\alpha) - v^+(\alpha)\right|\right\}. \end{aligned} \tag{14}$$

One can see that

$$\begin{aligned} D(u, \overline{0}) &= \sup_{\alpha \in [0,1]} \max\left\{\left|u^-(\alpha)\right|, \left|u^+(\alpha)\right|\right\} \\ &= \max\left\{\left|u^-(0)\right|, \left|u^+(0)\right|\right\}. \end{aligned} \tag{15}$$

Now, we may give the following.

**Proposition 3** (see [7]). *Let $u, v, w, z \in E^1$ and $k \in \mathbb{R}$. Then, the following statements hold.*

(i) *$(E^1, D)$ is a complete metric space.*

(ii) *$D(ku, kv) = |k| D(u, v)$.*

(iii) *$D(u + v, w + v) = D(u, w)$.*

(iv) *$D(u + v, w + z) \leq D(u, w) + D(v, z)$.*

(v) *$|D(u, \overline{0}) - D(v, \overline{0})| \leq D(u, v) \leq D(u, \overline{0}) + D(v, \overline{0})$.*

One can extend the natural order relation on the real line to intervals as follows:

$$A \preceq B \quad \text{iff } \underline{A} \leq \underline{B}, \ \overline{A} \leq \overline{B}. \tag{16}$$

Also, the partial ordering relation on $E^1$ is defined as follows:

$$\begin{gathered} u \preceq v \Longleftrightarrow [u]_\alpha \preceq [v]_\alpha \Longleftrightarrow u^-(\alpha) \leq v^-(\alpha), \\ u^+(\alpha) \leq v^+(\alpha) \quad \forall \alpha \in [0,1]. \end{gathered} \tag{17}$$

We say that $u \prec v$ if $u \preceq v$ and there exists $\alpha_0 \in [0,1]$ such that $u^-(\alpha_0) < v^-(\alpha_0)$ or $u^+(\alpha_0) < v^+(\alpha_0)$ (cf. Aytar et al. [9]).

**Lemma 4** (see [9, Lemma 6]). *Let $u, v \in E^1$ and $\varepsilon > 0$. The following statements are equivalent.*

(i) *$D(u, v) \leq \varepsilon$.*

(ii) *$u - \overline{\varepsilon} \preceq v \preceq u + \overline{\varepsilon}$.*

**Lemma 5** (see [10, Lemma 5]). *Let $\mu, \nu \in E^1$. If $\mu \preceq \nu + \overline{\varepsilon}$ for every $\varepsilon > 0$, then $\mu \preceq \nu$.*

**Lemma 6** (see [11, Lemma 3.4]). *Let $u, v, w \in E^1$. Then, the following statements hold.*

(i) *If $u \preceq v$ and $v \preceq w$, then $u \preceq w$.*

(ii) *If $u \prec v$ and $v \prec w$, then $u \prec w$.*

**Theorem 7** (see [11, Teorem 4.9]). *Let $u, v, w, e \in E^1$. Then, the following statements hold*

(i) *If $u \preceq w$ and $v \preceq e$, then $u + v \preceq w + e$.*

(ii) *If $u \succeq \overline{0}$ and $v \succ w$, then $uv \succeq uw$.*

Following Matloka [12], we give some definitions concerning sequences of fuzzy numbers. Nanda [8] introduced the concept of Cauchy sequence of fuzzy numbers and showed that every convergent sequence of fuzzy numbers is Cauchy.

A sequence $u = (u_k)$ of fuzzy numbers is a function $u$ from the set $\mathbb{N}$ into the set $E^1$. The fuzzy number $u_k$ denotes the value of the function at $k \in \mathbb{N}$ and is called the $k$th *term* of the sequence. We denote by $\omega(F)$, the set of all sequences of fuzzy numbers.

A sequence $(u_n) \in \omega(F)$ is called *convergent* to the limit $\mu \in E^1$ if and only if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$D(u_n, \mu) < \varepsilon \quad \forall n \geq n_0. \tag{18}$$

We denote by $c(F)$, the set of all convergent sequences of fuzzy numbers.

A sequence $u = (u_k)$ of fuzzy numbers is said to be Cauchy if for every $\varepsilon > 0$ there exists a positive integer $n_0$ such that

$$D(u_k, u_m) < \varepsilon \quad \forall k, m > n_0. \tag{19}$$

We denote by $C(F)$, the set of all Cauchy sequences of fuzzy numbers.

If $u_k \preceq u_{k+1}$ for every $k \in \mathbb{N}$, then $(u_k)$ is said to be a monotone increasing sequence.

Statistical convergence of a sequence of fuzzy numbers was introduced by Nuray and Savaş [13]. A sequence $(u_k)$ of fuzzy numbers is said to be *statistically convergent* to some number $\mu_0$ if for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{n+1} \left|\{k \leq n : D(u_k, \mu_0) \geq \varepsilon\}\right| = 0. \tag{20}$$

Nuray and Savaş [13] proved that if a sequence $(u_k)$ is convergent, then $(u_k)$ is statistically convergent. However, the converse is false, in general.

**Lemma 8** (see [14, Remark 3.7]). *If $(u_k) \in \omega(F)$ is statistically convergent to some $\mu$, then there exists a sequence $(v_k)$ which is convergent (in the ordinary sense) to $\mu$ and*

$$\lim_{n\to\infty} \frac{1}{n+1} \left|\{k \leq n : u_k \neq v_k\}\right| = 0. \tag{21}$$

Basic results on statistical convergence of sequences of fuzzy numbers can be found in [10, 15–17].

The Cesàro convergence of a sequence of fuzzy numbers is defined in [18] as follows. The sequence $(u_k)$ is said to be *Cesàro convergent* (written $(C,1)$-convergent) to a fuzzy number $\mu$ if

$$\lim_{n\to\infty} (C_1 u)_n = \mu. \tag{22}$$

Talo and Çakan [19, Theorem 2.1] have recently proved that if a sequence $(u_k)$ of fuzzy numbers is convergent, then $(u_k)$ is $(C,1)$-convergent. However, the converse is false, in general.

*Definition 9* (see [14]). A sequence $(u_k)$ of fuzzy numbers is said to be slowly oscillating if

$$\inf_{\lambda>1} \limsup_{n\to\infty} \max_{n<k\leq\lambda_n} D(u_k, u_n) = 0. \tag{23}$$

It is easy to see that (23) is satisfied if and only if for every $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon)$ and $\lambda = \lambda(\varepsilon) > 1$, as close to 1 as wished, such that $D(u_k, u_n) \leq \varepsilon$ whenever $n_0 \leq n < k \leq \lambda_n$.

Talo and Çakan [19, Corollary 2.7] proved that if a sequence $(u_k)$ of fuzzy numbers is slowly oscillating, then the implication (9) holds.

In this paper, we define the slowly decreasing sequence over $E^1$ which is partially ordered and is not a linear space. Also, we prove that if $(u_k) \in \omega(F)$ is slowly decreasing, then the implications (8) and (9) hold.

## 2. The Main Results

*Definition 10.* A sequence $(u_k)$ of fuzzy numbers is said to be slowly decreasing if for every $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon)$ and $\lambda = \lambda(\varepsilon) > 1$, as close to 1 as wished, such that for every $n > n_0$

$$u_k \succeq u_n - \overline{\varepsilon} \quad \text{whenever } n < k \leq \lambda_n. \tag{24}$$

Similarly, $(u_k)$ is said to be slowly increasing if for every $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon)$ and $\lambda = \lambda(\varepsilon) > 1$, as close to 1 as wished, such that for every $n > n_0$

$$u_k \preceq u_n + \overline{\varepsilon} \quad \text{whenever } n < k \leq \lambda_n. \tag{25}$$

*Remark 11.* Each slowly oscillating sequence of fuzzy numbers is slowly decreasing. On the other hand, we define the sequence $(u_n) = (\sum_{k=0}^{n} v_k)$, where

$$v_k(t) = \begin{cases} 1 - t\sqrt{k+1}, & \left(0 \leq t \leq \dfrac{1}{\sqrt{k+1}}\right), \\ 0, & (\text{otherwise}). \end{cases} \tag{26}$$

Then, for each $\alpha \in [0,1]$, since

$$u_n^-(\alpha) = 0, \qquad u_n^+(\alpha) = (1-\alpha)\sum_{k=0}^{n} \frac{1}{\sqrt{k+1}}, \tag{27}$$

$(u_n)$ is increasing. Therefore, $(u_n)$ is slowly decreasing. However, it is not slowly oscillating because for each $n \in \mathbb{N}$ and $\lambda > 1$ we get for $\alpha = 0$ and $k = \lambda_n$ the statements $k \le \lambda n$ and

$$\begin{aligned} u_k^+(0) - u_n^+(0) &= \sum_{j=n+1}^{k} \frac{1}{\sqrt{j+1}} \ge \frac{k-n}{\sqrt{k+1}} \\ &\ge \frac{\lambda n - 1 - n}{\sqrt{\lambda n + 1}} \\ &\ge \frac{n(\lambda - 1)}{\sqrt{\lambda n + 1}} - \frac{1}{\sqrt{\lambda n + 1}} \longrightarrow \infty \quad (n \longrightarrow \infty) \end{aligned} \tag{28}$$

hold.

**Lemma 12.** *Let $(u_n)$ be a sequence of fuzzy numbers. If $(u_n)$ is slowly decreasing, then for every $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon)$ and $\lambda = \lambda(\varepsilon) < 1$, as close to 1 as wished, such that for every $n > n_0$*

$$u_n \succeq u_k - \bar{\varepsilon} \quad \text{whenever } \lambda_n < k \le n. \tag{29}$$

*Proof.* We prove the lemma by an indirect way. Assume that the sequence $(u_n)$ is slowly decreasing and there exists some $\varepsilon_0 > 0$ such that for all $\lambda < 1$ and $m \ge 1$ there exist integers $k$ and $n \ge m$ for which

$$u_n \not\succeq u_k - \bar{\varepsilon} \quad \text{whenever } \lambda_n < k \le n. \tag{30}$$

Therefore, there exists $\alpha_0 \in [0, 1]$ such that

$$u_n^-(\alpha_0) < u_k^-(\alpha_0) - \varepsilon \quad \text{or} \quad u_n^+(\alpha_0) < u_k^+(\alpha_0) - \varepsilon. \tag{31}$$

For the sake of definiteness, we only consider the case $u_n^-(\alpha_0) < u_k^-(\alpha_0) - \varepsilon$. Clearly, (5) is not satisfied by $\{u_n^-(\alpha_0)\}$. That is, $\{u_n^-(\alpha_0)\}$ is not slowly decreasing. This contradicts the hypothesis that $(u_n)$ is slowly decreasing. □

**Theorem 13.** *Let $(u_n)$ be a sequence of fuzzy number. If $(u_n)$ is statistically convergent to some $\mu \in E^1$ and slowly decreasing, then $(u_n)$ is convergent to $\mu$.*

*Proof.* Let us start by setting $n = l_m$ in (21), where $0 \le l_0 < l_1 < l_2 < \cdots$ is a subsequence of those indices $k$ for which $u_k = v_k$. Therefore, we have

$$\lim_{m\to\infty} \frac{1}{l_m + 1} \left|\{k \le l_m : u_k = v_k\}\right| = \lim_{m\to\infty} \frac{m+1}{l_m + 1} = 1. \tag{32}$$

Consequently, it follows that

$$\lim_{m\to\infty} \frac{l_{m+1}}{l_m} = \lim_{m\to\infty} \frac{l_{m+1}}{m+1} \times \frac{m+1}{m} \times \frac{m}{l_m} = 1. \tag{33}$$

By the definition of the subsequence $(l_m)$, we have

$$\lim_{m\to\infty} u_{l_m} = \lim_{m\to\infty} v_{l_m} = \mu. \tag{34}$$

Since $(u_n)$ is slowly decreasing for every $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon)$ and $\lambda = \lambda(\varepsilon) > 1$, as close to 1 as we wish, such that for every $n > n_0$

$$u_k \succeq u_n - \frac{\bar{\varepsilon}}{2} \quad \text{whenever } n < k \le \lambda_n. \tag{35}$$

For every large enough $m$

$$u_k \succeq u_{l_m} - \frac{\bar{\varepsilon}}{2} \quad \text{whenever } l_m < k \le \lambda l_m. \tag{36}$$

By (33), we have $l_{m+1} < \lambda l_m$ for every large enough $m$, whence it follows that

$$u_k \succeq u_{l_m} - \frac{\bar{\varepsilon}}{2} \quad \text{whenever } l_m < k < l_{m+1}. \tag{37}$$

By (34) and Lemma 4, for every large enough $m$ we have

$$\mu - \frac{\bar{\varepsilon}}{2} \prec u_{l_m} \prec \mu + \frac{\bar{\varepsilon}}{2}. \tag{38}$$

Combining (37) and (38) we can see that

$$u_k \succ \mu - \bar{\varepsilon} \quad \text{whenever } l_m < k < l_{m+1}. \tag{39}$$

On the other hand, by virtue of Lemma 12, for every $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon)$ and $\lambda = \lambda(\varepsilon) < 1$ such that for every $n > n_0$

$$u_n \succeq u_k - \frac{\bar{\varepsilon}}{2} \quad \text{whenever } \lambda_n < k \le n. \tag{40}$$

For every large enough $m$

$$u_{l_{m+1}} \succeq u_k - \frac{\bar{\varepsilon}}{2} \quad \text{whenever } \lambda l_{m+1} < k \le l_{m+1}. \tag{41}$$

By (33), we have $\lambda l_{m+1} < l_m$ for every large enough $m$, whence it follows that

$$u_{l_{m+1}} \succeq u_k - \frac{\bar{\varepsilon}}{2} \quad \text{whenever } l_m < k < l_{m+1}. \tag{42}$$

By (34) and Lemma 4, for every large enough $m$ we have

$$\mu - \frac{\bar{\varepsilon}}{2} \prec u_{l_{m+1}} \prec \mu + \frac{\bar{\varepsilon}}{2}. \tag{43}$$

Therefore, (42) and (43) lead us to the consequence that

$$u_k \prec \mu + \bar{\varepsilon} \quad \text{whenever } l_m < k < l_{m+1} \tag{44}$$

which yields with (39) for each $\varepsilon > 0$ and Lemma 4 that

$$D(u_k, \mu) \le \varepsilon \quad \text{whenever } l_m < k < l_{m+1}. \tag{45}$$

Therefore, (45) gives together with (34) that the whole sequence $(u_k)$ is convergent to $\mu$. □

**Lemma 14.** *Let $\mu, \nu, w \in E^1$. If $\mu + w \preceq \nu + w$, then $\mu \preceq \nu$.*

*Proof.* Let $\mu, \nu, w \in E^1$. If $\mu + w \preceq \nu + w$, then

$$\mu^-(\alpha) + w^-(\alpha) \leq \nu^-(\alpha) + w^-(\alpha), \quad \mu^+(\alpha) + w^+(\alpha) \leq \nu^+(\alpha) + w^+(\alpha) \tag{46}$$

for all $\alpha \in [0,1]$. Therefore, we have $\mu^-(\alpha) \leq \nu^-(\alpha)$ and $\mu^+(\alpha) \leq \nu^+(\alpha)$ for all $\alpha \in [0,1]$. This means that $\mu \preceq \nu$. □

**Theorem 15.** *Let $(u_n) \in \omega(F)$. If $(u_n)$ is $(C,1)$-convergent to some $\mu \in E^1$ and slowly decreasing, then $(u_n)$ is convergent to $\mu$.*

*Proof.* Assume that $(u_n) \in \omega(F)$ is satisfied (22) and is slowly decreasing. Then for every $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon)$ and $\lambda = \lambda(\varepsilon) > 1$, as close to 1 as we wish, such that for every $n > n_0$

$$u_k \succeq u_n - \frac{\overline{\varepsilon}}{3} \quad \text{whenever } n < k \leq \lambda_n. \tag{47}$$

If $n$ is large enough in the sense that $\lambda_n > n$, then

$$\frac{\lambda_n + 1}{\lambda_n - n}(C_1 u)_{\lambda_n} + (C_1 u)_n = \frac{\lambda_n + 1}{\lambda_n - n}(C_1 u)_n + \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n} u_k. \tag{48}$$

For every large enough $n$, since

$$\frac{\lambda_n + 1}{\lambda_n - n} \leq \frac{2\lambda}{\lambda - 1}, \tag{49}$$

we have

$$\begin{aligned} \lim_{n\to\infty} D\left[\frac{\lambda_n + 1}{\lambda_n - n}(C_1 u)_{\lambda_n}, \frac{\lambda_n + 1}{\lambda_n - n}(C_1 u)_n\right] &= \lim_{n\to\infty}\frac{\lambda_n + 1}{\lambda_n - n} D\left[(C_1 u)_{\lambda_n}, (C_1 u)_n\right] \\ &\leq \lim_{n\to\infty}\frac{2\lambda}{\lambda - 1} D\left[(C_1 u)_{\lambda_n}, (C_1 u)_n\right] = 0. \end{aligned} \tag{50}$$

By Lemma 4, we obtain for large enough $n$ that

$$\frac{\lambda_n + 1}{\lambda_n - n}(C_1 u)_n - \frac{\overline{\varepsilon}}{3} \preceq \frac{\lambda_n + 1}{\lambda_n - n}(C_1 u)_{\lambda_n} \preceq \frac{\lambda_n + 1}{\lambda_n - n}(C_1 u)_n + \frac{\overline{\varepsilon}}{3}. \tag{51}$$

By (22), for large enough $n$ we obtain

$$\mu - \frac{\overline{\varepsilon}}{3} \preceq (C_1 u)_n \preceq \mu + \frac{\overline{\varepsilon}}{3}. \tag{52}$$

Since $(u_n)$ is slowly decreasing, we have

$$\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n} u_k \succeq u_n - \frac{\overline{\varepsilon}}{3}. \tag{53}$$

Combining (51), (52), and (53) we obtain by (48) for each $\varepsilon > 0$ that

$$\frac{\lambda_n + 1}{\lambda_n - n}(C_1 u)_n + \frac{\overline{\varepsilon}}{3} + \mu + \frac{\overline{\varepsilon}}{3} \succeq \frac{\lambda_n + 1}{\lambda_n - n}(C_1 u)_n + u_n - \frac{\overline{\varepsilon}}{3}. \tag{54}$$

By Lemma 14, we have

$$\mu + \overline{\varepsilon} \succeq u_n. \tag{55}$$

On the other hand, by virtue of Lemma 12, for every $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon)$ and $\lambda = \lambda(\varepsilon) < 1$ such that for every $n > n_0$

$$u_n \succeq u_k - \frac{\overline{\varepsilon}}{3} \quad \text{whenever } \lambda_n < k \leq n. \tag{56}$$

If $n$ is large enough in the sense that $\lambda_n < n$, then

$$\frac{\lambda_n + 1}{n - \lambda_n}(C_1 u)_{\lambda_n} + \frac{1}{n - \lambda_n}\sum_{k=\lambda_n+1}^{n} u_k = \left(\frac{\lambda_n + 1}{n - \lambda_n} + 1\right)(C_1 u)_n. \tag{57}$$

For large enough $n$, since

$$\frac{\lambda_n + 1}{n - \lambda_n} \leq \frac{2\lambda}{1 - \lambda}, \tag{58}$$

we have

$$\lim_{n\to\infty} D\left[\frac{\lambda_n + 1}{n - \lambda_n}(C_1 u)_{\lambda_n}, \frac{\lambda_n + 1}{n - \lambda_n}(C_1 u)_n\right] = 0. \tag{59}$$

Using the similar argument above, we conclude that

$$u_n \succeq \mu - \overline{\varepsilon}. \tag{60}$$

Therefore, combining (55) and (60) for each $\varepsilon \geq 0$ and large enough $n$, it is obtained that $D(u_n, \mu) \leq \varepsilon$. This completes the proof. □

Now, we define the Landau's one-sided Tauberian condition for sequences of fuzzy numbers.

**Lemma 16.** *If a sequence $(u_n) \in \omega(F)$ satisfies the one-sided Tauberian condition*

$$n u_n \succeq n u_{n-1} - \overline{H} \quad \text{for some } H > 0 \text{ and every } n, \tag{61}$$

*then $(u_n)$ is slowly decreasing.*

*Proof.* A sequence of fuzzy numbers $(u_k)$ satisfies

$$n u_n \succeq n u_{n-1} - \overline{H} \tag{62}$$

for $n \in \mathbb{N}$, where $H > 0$ is suitably chosen. Therefore, for all $\alpha \in [0,1]$ we have

$$u_n^-(\alpha) - u_{n-1}^-(\alpha) \geq \frac{-H}{n}, \qquad u_n^+(\alpha) - u_{n-1}^+(\alpha) \geq \frac{-H}{n}. \tag{63}$$

For all $n < k$ and $\alpha \in [0,1]$, we obtain

$$\begin{aligned} u_k^-(\alpha) - u_n^-(\alpha) &\geq \sum_{j=n+1}^{k}\left[u_j^-(\alpha) - u_{j-1}^-(\alpha)\right] \\ &\geq \sum_{j=n+1}^{k}\frac{-H}{j} \geq -H\left(\frac{k-n}{n}\right). \end{aligned} \tag{64}$$

Hence, for each $\varepsilon > 0$ and $1 < \lambda \leq 1 + \varepsilon/H$ we get for all $n < k \leq \lambda_n$

$$u_k^-(\alpha) - u_n^-(\alpha) \geq -H\left(\frac{k}{n} - 1\right) \geq -H(\lambda - 1) \geq -\varepsilon. \tag{65}$$

Similarly, for all $n < k \leq \lambda_n$ and $\alpha \in [0,1]$ we have

$$u_k^+(\alpha) - u_n^+(\alpha) \geq -\varepsilon. \tag{66}$$

Combining (65) and (66), one can see that $u_k \succeq u_n - \overline{\varepsilon}$ which proves that $(u_k)$ is slowly decreasing. □

By Theorems 13, 15 and Lemma 16, we derive the following two consequences.

**Corollary 17.** *Let $(u_k)$ be a sequence of fuzzy numbers which is statistically convergent to a fuzzy number $\mu_0$. If (61) is satisfied, then $\lim_{k\to\infty} u_k = \mu_0$.*

**Corollary 18.** *Let $(u_k)$ be a sequence of fuzzy numbers which is $(C,1)$-convergent to a fuzzy number $\mu_0$. If (61) is satisfied, then $\lim_{k\to\infty} u_k = \mu_0$.*

**Lemma 19.** *If the sequence $(u_n) \in \omega(F)$ satisfies (61), then*

$$n(C_1u)_n \succeq n(C_1u)_{n-1} - \overline{H} \quad \text{for some } H > 0 \text{ and every } n. \tag{67}$$

*Proof.* Assume that the sequence $(u_n) \in \omega(F)$ satisfies (61), then for all $\alpha \in [0,1]$ we have

$$n\left[u_n^-(\alpha) - u_{n-1}^-(\alpha)\right] \geq -H, \qquad n\left[u_n^+(\alpha) - u_{n-1}^+(\alpha)\right] \geq -H. \tag{68}$$

By the proof of Theorem 2.3 in [20], we obtain

$$\begin{aligned} n\left[(C_1u)_n^-(\alpha) - (C_1u)_{n-1}^-(\alpha)\right] &\geq -H, \\ n\left[(C_1u)_n^+(\alpha) - (C_1u)_{n-1}^+(\alpha)\right] &\geq -H. \end{aligned} \tag{69}$$

This means that $n(C_1u)_n \succeq n(C_1u)_{n-1} - \overline{H}$, as desired. □

**Corollary 20.** *If the sequence $(u_n) \in \omega(F)$ satisfies (61), then*

$$st\text{-}\lim_{n\to\infty}(C_1u)_n = \mu_0 \Longrightarrow \lim_{n\to\infty} u_n = \mu_0. \tag{70}$$

*Proof.* By Lemma 19, $n(C_1u)_n \succeq n(C_1u)_{n-1} - \overline{H}$ which is a Tauberian condition for statistical convergence by Corollary 17. Therefore, $st\text{-}\lim_{n\to\infty}(C_1u)_n = \mu_0$ implies that $\lim_{n\to\infty}(C_1u)_n = \mu_0$. Then, Corollary 18 yields that $\lim_{n\to\infty} u_n = \mu_0$. □

## 3. Conclusion

In the present paper, we introduce the slowly decreasing condition for a sequence of fuzzy numbers. This is a Tauberian condition from $st\text{-}\lim u_k = \mu_0$ to $\lim u_k = \mu_0$ and from $(C,1)$-$\lim u_k = \mu_0$ to $\lim u_k = \mu_0$.

Since we are not able to prove the fact that "$(C,1)$-*statistical convergence can be replaced by* $(C,1)$*-convergence as a weaker condition, if it is proved that* $\{(C_1u)_n\}$ *is slowly decreasing while* $(u_k) \in \omega(F)$ *is slowly decreasing,*" this problem is still open. So, it is meaningful to solve this problem.

Finally, we note that our results can be extended to Riesz means of sequences of fuzzy numbers which are introduced by Tripathy and Baruah in [21].

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