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Research Article

Initial Boundary Value Problem of the General Three-Component Camassa-Holm Shallow Water System on an Interval

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We study the initial boundary value problem of the general three-component Camassa-Holm shallow water system on an interval subject to inhomogeneous boundary conditions. First we prove a local in time existence theorem and present a weak-strong uniqueness result. Then, we establish a asymptotic stabilization of this system by a boundary feedback. Finally, we obtain a result of blow-up solution with certain initial data and boundary profiles.

1. Introduction

It is well known that the Camassa-Holm equation has attracted much attention in the past decade. It is a nonlinear dispersive wave equation that models the propagation of unidirectional irrotational shallow water waves over a flat bed, as well as water waves moving over an underlying shear flow. It was first introduced by Fokas and Fuchssteiner as a bi-Hamiltonian model. Cauchy problem and initial boundary value problem for Camassa-Holm equation have been studied extensively in a number of papers (see [1–15] and the references within).

Fu and Qu in [16] proposed a coupled Camassa-Holm equation,

$$\begin{aligned} m_t &= 2mu_x + m_x u + (mv)_x + nv_x, \\ n_t &= 2nv_x + n_x v + (nu)_x + mu_x; \end{aligned} \tag{1}$$

with $m = u - u_{xx}$, $n = v - v_{xx}$, which has peakon solitons in the form of a superposition of multipeakons and may as well be integrable. They investigated the local well-posedness and blow-up solutions of system (1) by means of Kato's semigroup approach to nonlinear hyperbolic evolution equation and obtained a criterion and condition on the initial data guaranteeing the development of singularities in finite time for strong solutions of system (1) by energy estimates. Recently the initial boundary value problem for the system (1) has been established in [17]; moreover, the local well-posedness and blow-up phenomena for the coupled Camassa-Holm equation were also established in [16, 18–32]. In [33], Tian and Xu obtained the compact and bounded absorbing set and the existence of the global attractor for viscous system (1) with the periodic boundary condition in by uniform prior estimate.

Recently, Fu and Qu in [34] introduced a general three-component Camassa-Holm equation as follows:

$$\begin{aligned} m_t &= 2mu_x + m_x u + (mv + mw)_x + nv_x + lw_x, \\ n_t &= 2nv_x + n_x v + (nu + nw)_x + mu_x + lw_x, \\ l_t &= 2lw_x + l_x w + (lu + lv)_x + mu_x + nv_x, \end{aligned} \tag{2}$$

where $m = u - u_{xx}$, $n = v - v_{xx}$, and $l = w - w_{xx}$. Equation (2) also has peakon solitons in the form of a superposition of multipeakons. Such system also conserves the H^1-norm conservation law. Moreover, the well-posedness and blow-up phenomena for system (2) with peakons have been established in [35]. To our knowledge, the initial boundary value problem of (2) has not been studied yet. The first aim of this paper is to consider an initial boundary value problem of the following

$$\begin{aligned} m_t &= 2mu_x + m_x u + (mv + mw)_x + nv_x + lw_x, \\ n_t &= 2nv_x + n_x v + (nu + nw)_x + mu_x + lw_x, \\ l_t &= 2lw_x + l_x w + (lu + lv)_x + mu_x + nv_x, \\ m(0,\cdot) &= m_0, \quad n(0,\cdot) = n_0, \quad l(0,\cdot) = l_0, \\ m(\cdot,0)_{|\Gamma_l} &= m_l, \quad n(\cdot,0)_{|\Gamma_l} = n_l, \quad l(\cdot,0)_{|\Gamma_l} = l_l, \\ m(\cdot,1)_{|\Gamma_r} &= m_r, \quad n(\cdot,1)_{|\Gamma_r} = n_r, \quad l(\cdot,1)_{|\Gamma_r} = l_r, \end{aligned} \tag{3}$$

where $\Gamma_l = \{t \in [0,T] \mid (u+v+w)(t,x) < 0\}$, $\Gamma_r = \{t \in [0,T] \mid (u+v+w)(t,x) > 0\}$.

Then, we will consider the asymptotic stabilization of (3) by means of a stationary feedback law acting on the inhomogeneous boundary condition. Following the step in [11], we convert the initial boundary value problem of (3) on the interval into an ODE system and two PDE systems. Then, we can consider the system (3) easily. Consequently, we obtain a local in time existence theorem, a weak-strong uniqueness result, asymptotic stabilization result on the interval, and a result of blow-up solution, respectively.

Our paper is organized as follows. In Section 2, we will consider an initial boundary value problem and the uniqueness of the solution to (3). By using the feedback law enjoyed by (3), the asymptotic stabilization on an interval is considered in Section 3. Finally, in Section 4, a result of blow-up solution with certain initial data and boundary profiles will be established.

First, we begin with a general remark that will be used many times later.

Remark 1. Let T be a positive number and $\Omega_T = [0,T] \times [0,1]$. Changing $u(t,x)$ in $-u(t,1-x)$, $v(t,x)$ in $-v(t,1-x)$, $w(t,x)$ in $-w(t,1-x)$, and t in $T-t$, and it will be more convenient for us to analysis the system, if we define the following sets

$$\begin{aligned} P_l &= \{t \in [0,T] \mid (u+v+w)(t,0) = 0\}, \\ P_r &= \{t \in [0,T] \mid (u+v+w)(t,1) = 0\}. \end{aligned} \tag{4}$$

Let $\Lambda = (1-\partial_x^2)^{1/2}$, then the operator Λ^{-2} can be expressed as

$$\Lambda^{-2} f(x) = G * f(x) = \frac{1}{2}\int_0^1 e^{-|x-y|} f(y)\,dy, \tag{5}$$

where $G = (1/2)e^{-|x|}$. Now, let $A_i = \Lambda^{-2}B_i = G*B_i$, $i = 1,2,3$, where B_i is an auxiliary function which lifts the boundary values m_l, m_r, n_l, n_r, and l_l defined by

$$\begin{gathered} B_i(t,x) = 0, \quad (t,x) \in [0,T]\times[0,1], \\ G * B_i(t,0) = v_{l_i}(t), \quad G * B_i(t,1) = v_{r_i}(t), \quad \forall t \in [0,T], \end{gathered} \tag{6}$$

where $i = 1,2,3$.

Setting $u = p + G*B_1$, $v = q + G*B_2$, and $w = r + G*B_3$, we can further rewrite the system (3) as

$$\begin{gathered} p = G*m, \qquad q = G*n, \qquad r = G*l, \\ p(t,0) = p(t,1) = 0, \qquad q(t,0) = q(t,1) = 0, \\ r(t,0) = r(t,1) = 0, \end{gathered} \tag{7}$$

$$\begin{aligned} m_t &= (p + G*B_1 + v + w)\,m_x \\ &\quad + [2(p + G*B_1)_x + v_x + w_x]\,m + nv_x + lw_x, \\ n_t &= (q + G*B_2 + u + w)\,n_x \\ &\quad + [2(q + G*B_2)_x + u_x + w_x]\,n + mu_x + lw_x, \\ l_t &= (r + G*B_3 + u + v)\,l_x \\ &\quad + [2(r + G*B_3)_x + u_x + v_x]\,l + mu_x + nv_x, \\ m(0,\cdot) &= m_0, \quad m(\cdot,0)_{|\Gamma_l} = m_l, \quad m(\cdot,1)_{|\Gamma_r} = m_r, \\ n(0,\cdot) &= n_0, \quad n(\cdot,0)_{|\Gamma_l} = n_l, \quad n(\cdot,1)_{|\Gamma_r} = n_r, \\ l(0,\cdot) &= l_0, \quad l(\cdot,0)_{|\Gamma_l} = l_l, \quad l(\cdot,1)_{|\Gamma_r} = l_r, \end{aligned} \tag{8}$$

where functions m_l, m_r, n_l, n_r, l_l, and l_r in $C^0([0,1],R)$ are the boundary values and m_0, n_0, and l_0 in $L^\infty(0,1)$ are the initial datum.

Lemma 2. *We have* $A_i = G*B_i \in C^0([0,T];C^\infty[0,1]) \in C^0([0,T];C^\infty[0,1])$ *and* $\tilde{p},\tilde{q},\tilde{r} \in L^\infty((0,T),C^{1,1}([0,1])) \cap \mathrm{Lip}((0,T),H_0^1(0,1))$, $i = 1,2,3, m,n,l \in L^\infty(\Omega_T) \cap \mathrm{Lip}((0,T),H^{-1}(0,1))$. *Moreover, we also have the bounds*

$$\begin{aligned} &\|A_i\|_{L^\infty((0,T);\,C^{1,1}[0,1])} \\ &\quad \le \frac{\cosh(1)}{\sinh(1)}\left(\|v_{r_i}\|_{L^\infty(0,T)} + \|v_{l_i}\|_{L^\infty(0,T)}\right), \\ &\qquad (i = 1,2,3), \end{aligned} \tag{9}$$

$$\begin{aligned} &\|\tilde{p}\|_{L^\infty((0,T);\,C^{1,1}[0,1])} \le 2(1+\cosh(1))\,\|m\|_{L^\infty(\Omega_T)}, \\ &\|\partial_t\tilde{p}\|_{L^\infty((0,T);\,H_0^1[0,1])} \le \|\partial_t m\|_{L^\infty((0,T);\,H^{-1}[0,1])}, \\ &\|\tilde{q}\|_{L^\infty((0,T);\,C^{1,1}[0,1])} \le 2(1+\cosh(1))\,\|n\|_{L^\infty(\Omega_T)}, \\ &\|\partial_t\tilde{q}\|_{L^\infty((0,T);\,H_0^1[0,1])} \le \|\partial_t n\|_{L^\infty((0,T);\,H^{-1}[0,1])}, \\ &\|\tilde{r}\|_{L^\infty((0,T);\,C^{1,1}[0,1])} \le 2(1+\cosh(1))\,\|l\|_{L^\infty(\Omega_T)}, \\ &\|\partial_t\tilde{r}\|_{L^\infty((0,T);\,H_0^1[0,1])} \le \|\partial_t l\|_{L^\infty((0,T);\,H^{-1}[0,1])}. \end{aligned} \tag{10}$$

Proof. $A_i, \tilde{p}, \tilde{q}, \tilde{r}, (i = 1, 2, 3)$ can be expressed, respectively, as

$$\begin{aligned} A_i(t,x) &= G * B_i(t,x) \\ &= \frac{1}{\sinh(1)}\left(\sinh(x)\, v_{r_i}(t) + \sinh(1-x)\, v_{l_i}(t)\right), \\ &\qquad (i = 1,2,3), \\ \tilde{p}(t,x) &= -\int_0^x \sinh(x-\tilde{x})\, m(t,\tilde{x})\, d\tilde{x} \\ &\quad + \frac{\sinh(x)}{\sinh(1)}\int_0^1 \sinh(1-\tilde{x})\, m(t,\tilde{x})\, d\tilde{x}, \\ \tilde{q}(t,x) &= -\int_0^x \sinh(x-\tilde{x})\, n(t,\tilde{x})\, d\tilde{x} \\ &\quad + \frac{\sinh(x)}{\sinh(1)}\int_0^1 \sinh(1-\tilde{x})\, n(t,\tilde{x})\, d\tilde{x}, \\ \tilde{r}(t,x) &= -\int_0^x \sinh(x-\tilde{x})\, l(t,\tilde{x})\, d\tilde{x} \\ &\quad + \frac{\sinh(x)}{\sinh(1)}\int_0^1 \sinh(1-\tilde{x})\, l(t,\tilde{x})\, d\tilde{x}. \end{aligned} \tag{11}$$

Estimates (9) and (10) can be easily obtained from the above expressions. □

2. Initial Boundary Value Problem

First, we define what we mean by a weak solution to (8). Our test functions will be in the space:

$$\begin{aligned} \operatorname{Adm}(\Omega_T) = \{&\varphi \in C^1(\Omega_T) \mid \varphi(t,x) = 0 \text{ on } [0,T]\setminus\Gamma_l \\ &\times\{0\} \cup [0,T]\setminus\Gamma_r \times \{0\} \cup \{T\}\times[0,1]\}. \end{aligned} \tag{12}$$

Definition 3. When $(p,q,r) \in L^\infty((0,T);\mathrm{Lip}[0,1]) \times L^\infty((0,T);\mathrm{Lip}[0,1]) \times L^\infty((0,T);\mathrm{Lip}[0,1])$, the function $(m,n,l) \in L^\infty(\Omega_T)\times L^\infty(\Omega_T)\times L^\infty(\Omega_T)$ is the weak solution to (8) if for all $\varphi \in \operatorname{Adm}(\Omega_T)$:

$$\begin{aligned} &\iint_{\Omega_T} m\left(\varphi_t - (u+v+w)\varphi_x + \left(p_x + \partial_x G * B_1\right)\varphi\right) dt\,dx \\ &\quad = -\iint_{\Omega_T} (nv_x + lw_x)\,\varphi(t,x)\,dt\,dx - \int_0^1 m_0(x)\,\varphi(0,x)\,dt \\ &\qquad + \int_0^T \big((u+v+w)(t,0)\,\varphi(t,0)\,m(t,0) \\ &\qquad\quad - (u+v+w)(t,1)\,\varphi(t,1)\,m(t,1)\big)\,dt, \\ &\iint_{\Omega_T} n\left(\varphi_t - (u+v+w)\varphi_x + \left(q_x + \partial_x G * B_2\right)\varphi\right) dt\,dx \\ &\quad = -\iint_{\Omega_T} (mu_x + lw_x)\,\varphi(t,x)\,dt\,dx - \int_0^1 n_0(x)\,\varphi(0,x)\,dx \\ &\qquad + \int_0^T \big((u+v+w)(t,0)\,\varphi(t,0)\,n(t,0) \\ &\qquad\quad - (u+v+w)(t,1)\,\varphi(t,1)\,n(t,1)\big)\,dt, \\ &\iint_{\Omega_T} l\left(\varphi_t - (u+v+w)\varphi_x + \left(r_x + \partial_x G * B_3\right)\varphi\right) dt\,dx \\ &\quad = -\iint_{\Omega_T} (mu_x + nv_x)\,\varphi(t,x)\,dt\,dx - \int_0^1 l_0(x)\,\varphi(0,x)\,dx \\ &\qquad + \int_0^T \big((u+v+w)(t,0)\,\varphi(t,0)\,l(t,0) \\ &\qquad\quad - (u+v+w)(t,1)\,\varphi(t,1)\,l(t,1)\big)\,dt. \end{aligned} \tag{13}$$

It is obvious that $C_0^1(\Omega_T) \subset \operatorname{Adm}(\Omega_T)$; therefore, a weak solution to system (8) is also a solution to (8) in the distribution sense. And it is clear that a regular weak solution is a classical solution.

Definition 4. For $(t,x) \in \Omega_T$, we consider $\omega(\cdot,t,x)$ the maximal solution satisfying

$$\begin{aligned} \omega_t &= -(u+v+w)(t,\omega(t,x)), \\ \omega(0,x) &= x. \end{aligned} \tag{14}$$

We consider that ω is the flow of $(u(t,x), v(t,x), w(t,x))$. For $(t,x) \in \Omega_T$, $\omega(\cdot,t,x)$ is defined on a set $[e(t,x), h(t,x)]$. Here $e(t,x)$ is basically the entrance time in Ω_T of the characteristic curve going through (t,x).

Remark 5. Obviously $e(t,x) > 0$ implies that $\omega(e(t,x),t,x) \in \{0,1\}$.

In the following, we consider a partition of Ω_T, which allows us to distinguish the different influence zones in Ω_T.

Definition 6. Let $P = \{(t,x) \in \Omega_T \mid \exists s \in [e(t,x), h(t,x)]$ such that $\omega \in \{0,1\}$ and $(u+v+w)(s,\omega(s,t,x)) = 0\} \cup \{(s,\omega(s,0,0)) \mid \text{for all } s \in [0,T]\} \cup \{(s,\omega(s,0,1)) \mid \text{for all } s \in [0,T]\}$,

$$\begin{aligned} I &= \{(t,x) \in \Omega_T \setminus p \mid e(t,x) = 0\}, \\ L &= \{(t,x) \in \Omega_T \setminus p \mid \omega(e(t,x),t,x) = 0\}, \\ R &= \{(t,x) \in \Omega_T \setminus p \mid \omega(e(t,x),t,x) = 1\}. \end{aligned} \tag{15}$$

Those points of the set P are tangent to the boundary, which are precisely the singular points of e and h. It's obviously that the sets P, I, L, and R constitute a partition of Ω_T. Furthermore, if $(t,x) \in L$, then $e(t,x) \in \Gamma_l$, and if $(t,x) \in R$, then $e(t,x) \in \Gamma_r$.

Definition 7. Here, we consider the case of data $(u,v,w) \in L^\infty([0,T];C^1([0,1])) \times L^\infty([0,T];C^1([0,1])) \times$

$L^\infty([0,T];C^1([0,1]))$, $(m_l,n_l,l_l) \in C_c^1(\Gamma_l)\times C_c^1(\Gamma_l)\times C_c^1(\Gamma_l)$; $(m_r,n_r,l_r) \in C_c^1(\Gamma_r)\times C_c^1(\Gamma_r)\times C_c^1(\Gamma_r)$, $(m_0,n_0,l_0) \in C_c^1(0,1)\times C_c^1(0,1)\times C_c^1(0,1)$. We define the functions m, n, and l in the following way.

When $(t,x)\in P$, $m(t,x)=0$, $n(t,x)=0$, and $l(t,x)=0$, when $(t,x)\in I$,

$$
\begin{aligned}
m(t,x) &= m_0(\omega(0,t,x)) \\
&\quad\times \exp\left(\int_0^t [2(p_x+\partial_x G * B_1)+v_x+w_x] \times (s,\omega(s,t,x))\,ds\right) \\
&\quad+ \int_0^t (nv_x+lw_x)(s,\omega(s,t,x)) \\
&\quad\times \exp\left(\int_s^t [2(p_x+\partial_x G * B_1)+v_x+w_x] \times (s',\omega(s',t,x))\,ds'\right) ds, \\
n(t,x) &= n_0(\omega(0,t,x)) \\
&\quad\times \exp\left(\int_0^t [2(q_x+\partial_x G * B_2)+u_x+w_x] \times (s,\omega(s,t,x))\,ds\right) \\
&\quad+ \int_0^t (mu_x+lw_x)(s,\omega(s,t,x)) \\
&\quad\times \exp\left([2(q_x+\partial_x G * B_2)+u_x+w_x] \times (s',\omega(s',t,x))\,ds'\right) ds, \\
l(t,x) &= l_0(\omega(0,t,x)) \\
&\quad\times \exp\left(\int_0^t [2(r_x+\partial_x G * B_3)+u_x+v_x] \times (s,\omega(s,t,x))\,ds\right) \\
&\quad+ \int_0^t (mu_x+nv_x)(s,\omega(s,t,x)) \\
&\quad\times \exp\left(\int_s^t [2(r_x+\partial_x G * B_3)+u_x+v_x] \times (s',\omega(s',t,x))\,ds'\right) ds,
\end{aligned}
\tag{16}
$$

when $(t,x)\in L$,

$$
\begin{aligned}
m(t,x) &= m_l(e(t,x)) \\
&\quad\times \exp\left(\int_{e(t,x)}^t [2(p_x+\partial_x G * B_1)+v_x+w_x] \times (s,\omega(s,t,x))\,ds\right) \\
&\quad+ \int_{e(t,x)}^t (nv_x+lw_x)(r,\omega(r,t,x)) \\
&\quad\times \exp\left(\int_s^t [2(p_x+\partial_x G * B_1)+v_x+w_x] \times (s',\omega(s',t,x))\,ds'\right) dr, \\
n(t,x) &= n_l(e(t,x)) \\
&\quad\times \exp\left(\int_{e(t,x)}^t [2(q_x+\partial_x G * B_2)+u_x+w_x] \times (s,\omega(s,t,x))\,ds\right) \\
&\quad+ \int_{e(t,x)}^t (mu_x+lw_x)(s,\omega(s,t,x)) \\
&\quad\times \exp\left(\int_s^t [2(q_x+\partial_x G * B_2)+u_x+w_x] \times (s',\omega(s',t,x))\,ds'\right) ds, \\
l(t,x) &= l_l(e(t,x)) \\
&\quad\times \exp\left(\int_{e(t,x)}^t [2(r_x+\partial_x G * B_3)+u_x+v_x] \times (s,\omega(s,t,x))\,ds\right) \\
&\quad+ \int_{e(t,x)}^t (mu_x+nv_x)(s,\omega(s,t,x)) \\
&\quad\times \exp\left(\int_s^t [2(r_x+\partial_x G * B_3)+u_x+v_x] \times (s',\omega(s',t,x))\,ds'\right) ds,
\end{aligned}
\tag{17}
$$

when $(t,x)\in R$,

$$
\begin{aligned}
m(t,x) &= m_l(e(t,x)) \\
&\quad\times \exp\left(\int_{e(t,x)}^t [2(p_x+\partial_x G * B_1)+v_x+w_x] \times (s,\omega(s,t,x))\,ds\right) \\
&\quad+ \int_{e(t,x)}^t (nv_x+lw_x)(r,\omega(r,t,x)) \\
&\quad\times \exp\left(\int_s^t [2(p_x+\partial_x G * B_1)+v_x+w_x] \times (s',\omega(s',t,x))\,ds'\right) dr,
\end{aligned}
$$

$$\begin{aligned}
n(t,x) &= n_l(e(t,x)) \\
&\quad \times \exp\left(\int_{e(t,x)}^{t} [2(q_x + \partial_x G * B_2) + u_x + w_x] \times (s, \omega(s,t,x))\, ds\right) \\
&\quad + \int_{e(t,x)}^{t} (mu_x + lw_x)(s, \omega(s,t,x)) \\
&\quad \times \exp\left(\int_{s}^{t} [2(q_x + \partial_x G * B_2) + u_x + w_x] \times \left(s', \omega\left(s',t,x\right)\right) ds'\right) ds, \\
l(t,x) &= l_l(e(t,x)) \\
&\quad \times \exp\left(\int_{e(t,x)}^{t} [2(r_x + \partial_x G * B_3) + u_x + v_x] \times (s, \omega(s,t,x))\, ds\right) \\
&\quad + \int_{e(t,x)}^{t} (mu_x + nv_x)(s, \omega(s,t,x)) \\
&\quad \times \exp\left(\int_{s}^{t} [2(r_x + \partial_x G * B_3) + u_x + v_x] \times \left(s', \omega\left(s',t,x\right)\right) ds'\right) ds.
\end{aligned} \tag{18}$$

Lemma 8. *Since* $(m,n,l) \in L^\infty(\Omega_T) \times L^\infty(\Omega_T) \times L^\infty(\Omega_T)$ *and satisfies* (8), *we immediately get that* (m,n,l) *is the weak solution of* (8) *and* $(m,n,l) \in W^{1,\infty}(0,T,H^{-1}(0,1)) \times W^{1,\infty}(0,T,H^{-1}(0,1)) \times W^{1,\infty}(0,T,H^{-1}(0,1))$. *However, the functions* $m, n,$ *and* l *satisfy the following estimates:*

$$\begin{aligned}
&\|m\|_{C^0(\Omega_T)} \\
&\quad \le \left[\max\left(\|m_0\|_{L^\infty}, \|m_l\|_{L^\infty}, \|m_r\|_{L^\infty}\right) + \|nv_x + lw_x\|_{L^\infty(0,1)} T\right] \\
&\qquad \times e^{T\|2(p+G*B_1)_x + v_x + w_x\|_{C^0(\Omega_T)}}, \\
&\|n\|_{C^0(\Omega_T)} \\
&\quad \le \left[\max\left(\|n_0\|_{L^\infty}, \|n_l\|_{L^\infty}, \|n_r\|_{L^\infty}\right) + \|mu_x + lw_x\|_{L^\infty(0,1)} T\right] \\
&\qquad \times e^{T\|2(q+G*B_2)_x + u_x + w_x\|_{C^0(\Omega_T)}}, \\
&\|l\|_{C^0(\Omega_T)} \\
&\quad \le \left[\max\left(\|m_0\|_{L^\infty}, \|m_l\|_{L^\infty}, \|m_r\|_{L^\infty}\right) + \|mu_x + nv_x\|_{L^\infty(0,1)} T\right] \\
&\qquad \times e^{T\|2(r+G*B_3)_x + u_x + v_x\|_{C^0(\Omega_T)}},
\end{aligned}$$

$$\begin{aligned}
&\|\partial_t m\|_{C^0(\Omega_T)} \\
&\quad \le \Big\{ 2\max\left(\|m_0\|_{L^\infty}, \|m_l\|_{L^\infty}, \|m_r\|_{L^\infty}\right) \\
&\qquad \times \left(\|p + G * B_1\|_{L^\infty((0,T);\mathrm{Lip}[0,1])} + \|v\|_{L^\infty((0,T);\mathrm{Lip}[0,1])} + \|w\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}\right) \\
&\qquad + \Big[\left(\|u\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 u\right\|_{L^\infty(\Omega_T)}\right)^2 \\
&\qquad\quad + \left(\|v\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 v\right\|_{L^\infty(\Omega_T)}\right)^2 \\
&\qquad\quad + \left(\|w\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 w\right\|_{L^\infty(\Omega_T)}\right)^2\Big] \\
&\qquad \times \left(1 + 2\left(\|p + G * B_1\|_{L^\infty((0,T);\mathrm{Lip}[0,1])} + \|v\|_{L^\infty((0,T);\mathrm{Lip}[0,1])} + \|w\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}\right)\right) T\Big\} \\
&\qquad \times \exp\left(2T\left(\|(p + G * B_1)_x\|_{L^\infty(\Omega_T)} + \|v_x\|_{L^\infty(\Omega_T)} + \|w_x\|_{L^\infty(\Omega_T)}\right)\right), \\
&\|\partial_t n\|_{C^0(\Omega_T)} \\
&\quad \le \Big\{ 2\max\left(\|n_0\|_{L^\infty}, \|n_l\|_{L^\infty}, \|n_r\|_{L^\infty}\right) \\
&\qquad \times \left(\|q + G * B_2\|_{L^\infty((0,T);\mathrm{Lip}[0,1])} + \|u\|_{L^\infty((0,T);\mathrm{Lip}[0,1])} + \|w\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}\right) \\
&\qquad + \Big[\left(\|u\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 u\right\|_{L^\infty(\Omega_T)}\right)^2 \\
&\qquad\quad + \left(\|v\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 v\right\|_{L^\infty(\Omega_T)}\right)^2 \\
&\qquad\quad + \left(\|w\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 w\right\|_{L^\infty(\Omega_T)}\right)^2\Big] \\
&\qquad \times \left(1 + 2\left(\|q + G * B_2\|_{L^\infty((0,T);\mathrm{Lip}[0,1])} + \|u\|_{L^\infty((0,T);\mathrm{Lip}[0,1])} + \|w\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}\right)\right) T\Big\} \\
&\qquad \times \exp\left(2T\left(\|(q + G * B_2)_x\|_{L^\infty(\Omega_T)} + \|p_x\|_{L^\infty(\Omega_T)} + \|w_x\|_{L^\infty(\Omega_T)}\right)\right), \\
&\|\partial_t l\|_{C^0(\Omega_T)} \\
&\quad \le \Big\{ 2\max\left(\|l_0\|_{L^\infty}, \|l_l\|_{L^\infty}, \|l_r\|_{L^\infty}\right) \\
&\qquad \times \left(\|r + G * B_3\|_{L^\infty((0,T);\mathrm{Lip}[0,1])} + \|u\|_{L^\infty((0,T);\mathrm{Lip}[0,1])} + \|v\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}\right) \\
&\qquad + \Big[\left(\|u\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 u\right\|_{L^\infty(\Omega_T)}\right)^2
\end{aligned}$$

$$+\left(\|v\|_{L^\infty(\Omega_T)}+\left\|\partial_{xx}^2 v\right\|_{L^\infty(\Omega_T)}\right)^2$$
$$+\left(\|w\|_{L^\infty(\Omega_T)}+\left\|\partial_{xx}^2 w\right\|_{L^\infty(\Omega_T)}\right)^2\Big]$$
$$\times\Big(1+2\Big(\|r+G*B_3\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}+\|p\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}+\|q\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}\Big)\Big)T\Big\}$$
$$\times\exp\Big(2T\Big(\|(r+G*B_3)_x\|_{L^\infty(\Omega_T)}+\|u_x\|_{L^\infty(\Omega_T)}+\|v_x\|_{L^\infty(\Omega_T)}\Big)\Big). \tag{19}$$

Definition 9. We can define operator E and a domain for the system (8) by: for all $p,q,r\in L^\infty((0,T);C^{1,1}([0,1]))\cap\mathrm{Lip}([0,T];H_0^1(0,1))$,

$$\begin{aligned}E(p)&=\widetilde{p}\in L^\infty\left((0,T);C^{1,1}([0,1])\right)\\&\quad\cap\mathrm{Lip}\left([0,T];H_0^1(0,1)\right),\\E(q)&=\widetilde{q}\in L^\infty\left((0,T);C^{1,1}([0,1])\right)\\&\quad\cap\mathrm{Lip}\left([0,T];H_0^1(0,1)\right),\\E(r)&=\widetilde{r}\in L^\infty\left((0,T);C^{1,1}([0,1])\right)\\&\quad\cap\mathrm{Lip}\left([0,T];H_0^1(0,1)\right),\\C_{M_0,M_1,T}&=\Big\{p,q,r\in L^\infty\left((0,T);C^{1,1}([0,1])\right)\\&\quad\cap\mathrm{Lip}\left([0,T];H_0^1(0,1)\right)\mid\|d\|_{L^\infty((0,T);C^{1,1}[0,1])}\\&\quad\le M_0,\|d\|_{\mathrm{Lip}((0,T);H_0^1(0,1))}\le M_1\Big\},\end{aligned}\tag{20}$$

where

$$d(t,x)=\max\left(p(t,x),q(t,x),r(t,x)\right),\quad (t,x)\in[0,T]\times[0,1]. \tag{21}$$

Obviously $C_{M_0,M_1,T}$ is convex and $C_{M_0,M_1,T}$ is compact with respect to the norm $\|\cdot\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$. We will endow $C_{M_0,M_1,T}$ with the norm $\|\cdot\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$. There exist positive numbers M_0,M_1, and T such that E maps $C_{M_0,M_1,T}$ into itself.

Theorem 10. *There exists $T>0$, and (m,n,l) is a weak solution of (8) with $p,q,r\in L^\infty((0,T);C^{1,1}([0,1]))\cap\mathrm{Lip}([0,T];H_0^1(0,1))$ and $m,n,l\in L^\infty(\Omega_T)$. Moreover, any such solution (p,q,r) is in fact in $C^0([0,T];W^{2,P}(0,1))\cap C^1([0,T];W_0^{1,p}(0,1))$, for all $p<+\infty$. Furthermore, the existence time of a maximal solution $T\ge\min(T^*,\widetilde{T})$, with*

$$\begin{aligned}T^*&=\max_{\alpha>0,\ \beta>0}\left(\frac{1}{6\alpha}\ln\left(\frac{|\alpha-C_1|}{4(1+\cosh(1))\alpha}\right)\right),\\\widetilde{T}&=\max_{\alpha>0,\ \beta>0}\left(\frac{1}{6\alpha}\ln\left(\frac{|\beta|}{12\alpha^2}\right)\right),\\C_1&=\max\left[\frac{\cosh(1)}{\sinh(1)}\left(\left\|v_{r_i}\right\|_{L^\infty(0,T)}+\left\|v_{l_i}\right\|_{L^\infty(0,T)}\right)\right],\\&\qquad i=1,2,3.\end{aligned}\tag{22}$$

Proof. For $\widetilde{T}>0$, we consider m_l,m_r,n_l,n_r,l_l, and l_r in $C^0([0,\widetilde{T}])$ such that the sets P_l and P_r have only a finite number of connected components.

Let $C_0=\max(\|m_i\|_{L^\infty(0,1)},\|n_i\|_{L^\infty(\Gamma_l)},\|l_i\|_{L^\infty(\Gamma_r)})$, where $i=0,r,l$ and

$$C_1=\max\left[\frac{\cosh(1)}{\sinh(1)}\left(\left\|v_{r_i}\right\|_{L^\infty(0,T)}+\left\|v_{l_i}\right\|_{L^\infty(0,T)}\right)\right],\quad i=1,2,3. \tag{23}$$

Now, if $u,v,w\in C_{M_0,M_1,T}$ (see (21)), we have

$$\begin{aligned}&\|2(p+G*B_1)_x+v_x+w_x\|_{C^\infty(\Omega_T)}\\&\quad\le 2\left(\|\partial_x u\|_{C^\infty(\Omega_T)}+\|\partial_x v\|_{C^\infty(\Omega_T)}+\|\partial_x w\|_{C^\infty(\Omega_T)}\right),\\&\|2(q+G*B_2)_x+u_x+w_x\|_{C^\infty(\Omega_T)}\\&\quad\le 2\left(\|\partial_x u\|_{C^\infty(\Omega_T)}+\|\partial_x v\|_{C^\infty(\Omega_T)}+\|\partial_x w\|_{C^\infty(\Omega_T)}\right),\\&\|2(r+G*B_3)_x+u_x+v_x\|_{C^\infty(\Omega_T)}\\&\quad\le 2\left(\|\partial_x u\|_{C^\infty(\Omega_T)}+\|\partial_x v\|_{C^\infty(\Omega_T)}+\|\partial_x w\|_{C^\infty(\Omega_T)}\right).\end{aligned}\tag{24}$$

For all $u,v,w\in L^\infty((0,T);W^{2,\infty}(0,1))$, we have

$$\begin{aligned}\|\partial_x u\|_{L^\infty(\Omega_T)}&\le 2\sqrt{\|u\|_{L^\infty(\Omega_T)}\|\partial_{xx}^2 u\|_{L^\infty(\Omega_T)}}\\&\le\left(\|u\|_{L^\infty(\Omega_T)}+\left\|\partial_{xx}^2 u\right\|_{L^\infty(\Omega_T)}\right),\\\|\partial_x v\|_{L^\infty(\Omega_T)}&\le 2\sqrt{\|v\|_{L^\infty(\Omega_T)}\|\partial_{xx}^2 v\|_{L^\infty(\Omega_T)}}\\&\le\left(\|v\|_{L^\infty(\Omega_T)}+\left\|\partial_{xx}^2 v\right\|_{L^\infty(\Omega_T)}\right),\end{aligned}$$

$$\begin{aligned}
\|\partial_x w\|_{L^\infty(\Omega_T)} &\le 2\sqrt{\|w\|_{L^\infty(\Omega_T)}\|\partial^2_{xx} w\|_{L^\infty(\Omega_T)}} \\
&\le \left(\|w\|_{L^\infty(\Omega_T)} + \left\|\partial^2_{xx} w\right\|_{L^\infty(\Omega_T)}\right), \\
&\|nv_x + lw_x\|_{L^\infty(0,1)} \\
&\le \left[\left(\|u\|_{L^\infty(\Omega_T)} + \left\|\partial^2_{xx} u\right\|_{L^\infty(\Omega_T)}\right)^2 + \left(\|v\|_{L^\infty(\Omega_T)} + \left\|\partial^2_{xx} v\right\|_{L^\infty(\Omega_T)}\right)^2 + \left(\|w\|_{L^\infty(\Omega_T)} + \left\|\partial^2_{xx} w\right\|_{L^\infty(\Omega_T)}\right)^2\right], \\
&\|mu_x + lw_x\|_{L^\infty(0,1)} \\
&\le \left[\left(\|u\|_{L^\infty(\Omega_T)} + \left\|\partial^2_{xx} u\right\|_{L^\infty(\Omega_T)}\right)^2 + \left(\|v\|_{L^\infty(\Omega_T)} + \left\|\partial^2_{xx} v\right\|_{L^\infty(\Omega_T)}\right)^2 + \left(\|w\|_{L^\infty(\Omega_T)} + \left\|\partial^2_{xx} w\right\|_{L^\infty(\Omega_T)}\right)^2\right], \\
&\|mu_x + nv_x\|_{L^\infty(0,1)} \\
&\le \left[\left(\|u\|_{L^\infty(\Omega_T)} + \left\|\partial^2_{xx} u\right\|_{L^\infty(\Omega_T)}\right)^2 + \left(\|v\|_{L^\infty(\Omega_T)} + \left\|\partial^2_{xx} v\right\|_{L^\infty(\Omega_T)}\right)^2 + \left(\|w\|_{L^\infty(\Omega_T)} + \left\|\partial^2_{xx} w\right\|_{L^\infty(\Omega_T)}\right)^2\right].
\end{aligned} \tag{25}$$

We also define that $\widetilde{d}(t,x) = \max(\widetilde{p}(t,x), \widetilde{q}(t,x), \widetilde{r}(t,x))$, $(t,x) \in \Omega_T$. If $p,q,r \in C_{M_0,M_1,T}$, then from Lemmas 2 and 8, we derive that

$$\begin{aligned}
&\left\|\widetilde{d}\right\|_{L^\infty((0,T);C^{1,1}[0,1])} \\
&\le 2(1+\cosh(1))\left(C_0 + 12T^*(M_0+C_1)^2\right) \times \exp\left(6T^*(M_0+C_1)\right), \\
&\left\|\partial_t \widetilde{d}\right\|_{C^0(\Omega_T)} \\
&\le \left[6C_0(M_0+C_1) + 12(M_0+2C_1)^2 \times \left(1 + 6(M_0+C_1)\widetilde{T}\right)\right]\exp\left(6\widetilde{T}(M_0+C_1)\right).
\end{aligned} \tag{26}$$

Finally, to obtain $\widetilde{p}, \widetilde{q}, \widetilde{r} \in C_{M_0,M_1,T}$, it is sufficient to show that

$$\begin{aligned}
&2(1+\cosh(1))\left(C_0 + 12T^*(M_0+C_1)^2\right) \times \exp\left(6T^*(M_0+C_1)\right) \le M_0, \\
&M_0 + \left[6C_0(M_0+C_1) + 12(M_0+C_1)^2 \times \left(1 + 6\widetilde{T}(M_0+C_1)\right)\right] \times \exp\left(6\widetilde{T}(M_0+C_1)\right) \\
&\le M_1,
\end{aligned} \tag{27}$$

if we have chosen T and M_0; it is easy to choose M_1 to satisfy the second inequality. For the above two inequalities, we just choose M_0 and M_1 sufficiently large and then T close to 0. More precisely:

$$\begin{aligned}
&M_0 > 2(1+\cosh(1))C_0, \\
&T^* \le \frac{1}{6(M_0+C_1)}\ln\left(\frac{M_0}{4(1+\cosh(1))(M_0+C_1)}\right), \\
&\widetilde{T} \le \frac{1}{6(M_0+C_1)}\ln\left(\frac{M_1-M_0}{12(M_0+C_1)^2}\right).
\end{aligned} \tag{28}$$

Maximizing the bound of T, we can get minimum existence. Then, we get the result announced, where $\alpha = M_0 + C_1$, $\beta = M_1 - M_0$. □

Lemma 11. *The operator $E : C_{M_0,M_1,T} \to C_{M_0,M_1,T}$ is continuous with respect to $\|\cdot\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}$.*

Proof. The proof is omitted here; one can see a similar proof in [8, Proposition 2.4].

Now, we can apply Shauder's fixed point theorem to the operator E, and we get the result that there exist fixed points p,q,r such that $E(p)=p$, $E(q)=q$, and $E(r)=r$, so we know that there exists a wake solution of (9).

$$p,q,r \in L^\infty\left((0,T);C^{1,1}([0,1])\right) \cap \mathrm{Lip}\left([0,T];H^1_0(0,1)\right). \tag{29}$$

□

2.1. Uniqueness. We will prove the weak-strong uniqueness of weak solution of (8) in the following.

Theorem 12. *Let $(p,m),(q,n),(r,l) \in L^\infty((0,T);C^{1,1}([0,1])) \cap \mathrm{Lip}([0,T];H^1_0(0,1) \times L^\infty((0,T);\mathrm{Lip}([0,1]))$ be the weak solution of (7)-(8), then it is unique in $L^\infty((0,T);C^{1,1}([0,1])) \times L^\infty(\Omega_T)$.*

Proof. Define $\Phi = m - \widetilde{m}$, $\Psi = n - \widetilde{n}$, $\Upsilon = l - \widetilde{l}$ and $P = p - \widetilde{p}$, $Q = q - \widetilde{q}$, $H = r - \widetilde{r}$, then we have

$$\begin{aligned}
P(t,\cdot) &= G * \Phi(t,\cdot), \\
Q(t,\cdot) &= G * \Psi(t,\cdot), \\
\mathrm{H}(t,\cdot) &= G * \Upsilon(t,\cdot),
\end{aligned} \tag{30}$$

where $P, Q, H \in \mathrm{Lip}([0,T]; H_0^1(0,1))$, and $(\Phi, \Psi, \Upsilon) \in L^\infty(\Omega_T) \times L^\infty(\Omega_T) \times L^\infty(\Omega_T)$ is the unique weak solution of

$$\begin{aligned}
\Phi_t &= (p + G * B + v + w)\Phi_x \\
&\quad + \left[2(p + G * B_1)_x + v_x + w_x\right]\Phi \\
&\quad + (P + Q + H)\widetilde{m}_x + (2P_x + Q_x + H_x)\widetilde{m} \\
&\quad + \Psi v_x + \widetilde{n}Q_x + \Upsilon w_x + \widetilde{l}H_x, \\
\Psi_t &= (q + G * B + u + w)\Psi_x \\
&\quad + \left[2(q + G * B_2)_x + u_x + w_x\right]\Psi \\
&\quad + (P + Q + H)\widetilde{n}_x + (P_x + 2Q_x + H_x)\widetilde{n} \\
&\quad + \Phi u_x + \widetilde{m}\Phi_x + \Upsilon w_x + \widetilde{l}H_x, \\
\Upsilon_t &= (r + G * B + u + v)\Upsilon_x \\
&\quad + \left[2(r + G * B_3)_x + u_x + v_x\right]\Upsilon \\
&\quad + (P + Q + H)\widetilde{l}_x + (P_x + Q_x + 2H_x)\widetilde{l} \\
&\quad + \Phi u_x + \widetilde{m}\Phi_x + \Psi v_x + \widetilde{n}Q_x.
\end{aligned} \tag{31}$$

Let

$$\begin{aligned}
b_1 &= 2(p + G * B_1)_x + v_x + w_x, \\
b_2 &= 2(q + G * B_2)_x + u_x + w_x, \\
b_3 &= 2(r + G * B_3)_x + u_x + v_x, \\
f_1 &= (P + Q + H)\widetilde{m}_x + (2P_x + Q_x + H_x)\widetilde{m} \\
&\quad + \Psi v_x + \widetilde{n}Q_x + \Upsilon w_x + \widetilde{l}H_x, \\
f_2 &= (P + Q + H)\widetilde{n}_x + (P_x + 2Q_x + H_x)\widetilde{n} \\
&\quad + \Phi u_x + \widetilde{m}\Phi_x + \Upsilon w_x + \widetilde{l}H_x, \\
f_3 &= (P + Q + H)\widetilde{l}_x + (P_x + Q_x + 2H_x)\widetilde{l} \\
&\quad + \Phi u_x + \widetilde{m}\Phi_x + \Psi v_x + \widetilde{n}Q_x,
\end{aligned} \tag{32}$$

with $i_0 = 0$, $i_l = 0$, and $i_r = 0$, where $i = \Phi, \Psi, \Upsilon$.

For $(t,x) \in P$, we have $\Phi(t,x) = 0, \Psi(t,x) = 0$, and $\Upsilon(t,x) = 0$.

Then, we get the uniqueness result.

For $(t,x) \in I$, we have

$$\begin{aligned}
\Phi(t,x) &= \int_0^t f_1(s, \omega(s,t,x)) \\
&\qquad \times \exp\left(\int_r^t b_1(s', \omega(s',t,x))\, ds'\right) ds, \\
\Psi(t,x) &= \int_0^t f_2(s, \omega(s,t,x)) \\
&\qquad \times \exp\left(\int_r^t b_2(s', \omega(s',t,x))\, ds'\right) ds, \\
\Upsilon(t,x) &= \int_0^t f_3(s, \omega(s,t,x)) \\
&\qquad \times \exp\left(\int_r^t b_3(s', \omega(s',t,x))\, ds'\right) ds.
\end{aligned} \tag{33}$$

For $(t,x) \in L$, we have

$$\begin{aligned}
\Phi(t,x) &= \int_{e(t,x)}^t f_1(s, \omega(s,t,x)) \\
&\qquad \times \exp\left(\int_r^t b_1(s', \omega(s',t,x))\, ds'\right) ds, \\
\Psi(t,x) &= \int_{e(t,x)}^t f_2(s, \omega(s,t,x)) \\
&\qquad \times \exp\left(\int_r^t b_2(s', \omega(s',t,x))\, ds'\right) ds, \\
\Upsilon(t,x) &= \int_{e(t,x)}^t f_3(s, \omega(s,t,x)) \\
&\qquad \times \exp\left(\int_r^t b_3(s', \omega(s',t,x))\, ds'\right) ds.
\end{aligned} \tag{34}$$

For $(t,x) \in R$, we have

$$\begin{aligned}
\Phi(t,x) &= \int_{e(t,x)}^t f_1(s, \omega(s,t,x)) \\
&\qquad \times \exp\left(\int_r^t b_1(s', \omega(s',t,x))\, ds'\right) ds, \\
\Psi(t,x) &= \int_{e(t,x)}^t f_2(s, \omega(s,t,x)) \\
&\qquad \times \exp\left(\int_r^t b_2(s', \omega(s',t,x))\, ds'\right) ds, \\
\Upsilon(t,x) &= \int_{e(t,x)}^t f_3(s, \omega(s,t,x)) \\
&\qquad \times \exp\left(\int_r^t b_3(s', \omega(s',t,x))\, ds'\right) ds.
\end{aligned} \tag{35}$$

Now since $\|P(t,\cdot)\|_{L^\infty(0,1)} \le 5\|\Phi(t,\cdot)\|_{L^\infty(0,1)}$, $\|Q(t,\cdot)\|_{L^\infty(0,1)} \le 5\|\Psi(t,\cdot)\|_{L^\infty(0,1)}$, $\|\mathrm{H}(t,\cdot)\|_{L^\infty(0,1)} \le 5\|\Upsilon(t,\cdot)\|_{L^\infty(0,1)}$ and $\widetilde{m}, \partial_x\widetilde{m}, \widetilde{n}, \partial_x\widetilde{n}, \widetilde{l}, \partial_x\widetilde{l}$ bounded, we see that for some $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$,

$$\begin{aligned}
&\|f_1(t,\cdot)\|_{L^\infty(0,1)} \\
&\quad \le \lambda_1\left(\|\Phi(t,\cdot)\|_{L^\infty(0,1)} + \|\Psi(t,\cdot)\|_{L^\infty(0,1)} + \|\Upsilon(t,\cdot)\|_{L^\infty(0,1)}\right), \\
&\|f_2(t,\cdot)\|_{L^\infty(0,1)} \\
&\quad \le \lambda_2\left(\|\Phi(t,\cdot)\|_{L^\infty(0,1)} + \|\Psi(t,\cdot)\|_{L^\infty(0,1)} + \|\Upsilon(t,\cdot)\|_{L^\infty(0,1)}\right), \\
&\|f_3(t,\cdot)\|_{L^\infty(0,1)} \\
&\quad \le \lambda_3\left(\|\Phi(t,\cdot)\|_{L^\infty(0,1)} + \|\Psi(t,\cdot)\|_{L^\infty(0,1)+\|\Upsilon(t,\cdot)\|_{L^\infty(0,1)}}\right),
\end{aligned} \tag{36}$$

since p, q, r, and B_i, $(i = 1, 2, 3)$ are bounded, we get that for some $\lambda_1' > 0$, $\lambda_2' > 0$, $\lambda_3' > 0$,

$$\begin{aligned}
&\|\Phi(t,\cdot)\|_{L^\infty(0,1)} \\
&\quad \le \lambda_1' \int_0^t \left(\|\Phi(s,\cdot)\|_{L^\infty(0,1)} + \|\Psi(s,\cdot)\|_{L^\infty(0,1)} \right. \\
&\qquad \left. + \|\Upsilon(s,\cdot)\|_{L^\infty(0,1)}\right) ds, \\
&\|\Psi(t,\cdot)\|_{L^\infty(0,1)} \\
&\quad \le \lambda_2' \int_0^t \left(\|\Phi(s,\cdot)\|_{L^\infty(0,1)} + \|\Psi(s,\cdot)\|_{L^\infty(0,1)} \right. \\
&\qquad \left. + \|\Upsilon(s,\cdot)\|_{L^\infty(0,1)}\right) ds, \\
&\|\Upsilon(t,\cdot)\|_{L^\infty(0,1)} \\
&\quad \le \lambda_3' \int_0^t \left(\|\Phi(s,\cdot)\|_{L^\infty(0,1)} + \|\Psi(s,\cdot)\|_{L^\infty(0,1)} \right. \\
&\qquad \left. + \|\Upsilon(s,\cdot)\|_{L^\infty(0,1)}\right) ds.
\end{aligned} \tag{37}$$

We can obtain that

$$\begin{aligned}
&\|\Phi(t,\cdot)\|_{L^\infty(0,1)} + \|\Psi(t,\cdot)\|_{L^\infty(0,1)} + \|\Upsilon(t,\cdot)\|_{L^\infty(0,1)} \\
&\le \left(\lambda_1' + \lambda_2' + \lambda_3'\right) \\
&\quad \times \int_0^t \left(\|\Phi(s,\cdot)\|_{L^\infty(0,1)} + \|\Psi(s,\cdot)\|_{L^\infty(0,1)} \right. \\
&\qquad \left. + \|\Upsilon(s,\cdot)\|_{L^\infty(0,1)}\right) ds.
\end{aligned} \tag{38}$$

As a result, we get the result of the uniqueness by Gronwall's inequality when $(t, x) \in I$, $(t, x) \in L$, $(t, x) \in R$. Then, we complete the proof of the uniqueness results. □

3. Asymptotic Stabilization

3.1. Preliminary Results. The equilibrium state that we want to stabilize is $m = n = l = 0$, $p = q = r = G * B_i = 0$, and $i = 1, 2, 3$. A natural idea is using Lyapunov indirection method to investigate whether the linearized system around the equilibrium state is stabilizable or not. Its stabilization would provide a local stabilization result on the nonlinear system. However, there is a difficulty in the stabilization problem. We have to prescribe y_l, and we just need to make a continuous transition at $(t, x) = (0, 0)$, and that y_l asymptotically converge in time. For convenience, the system (6)–(8) can rewrite in the following

$$\begin{gathered}
\partial_t y - \breve{a}(t,x)\,\partial_x y = \breve{b}(t,x)\,y + \breve{f}(t,x), \\
y(0,\cdot) = y_0, \qquad y(\cdot,0) = y_l, \qquad y(1,\cdot) = y_r, \\
\breve{g}(t,x) = G * y(t,x), \qquad \breve{g}(t,0) = \breve{g}(t,1) = 0, \\
\breve{B}(t,x) = 0, \qquad G * \breve{B}(t,0) = v_l(t), \\
G * \breve{B}(t,1) = v_r(t),
\end{gathered} \tag{39}$$

where

$$\begin{gathered}
\breve{b}(t,x) = \begin{pmatrix} 2\left(p + G * \breve{B}\right)_x + v_x + w_x & 0 & 0 \\ 0 & 2\left(q + G * \breve{B}\right)_x + u_x + w_x & 0 \\ 0 & 0 & 2\left(r + G * \breve{B}\right)_x + u_x + v_x \end{pmatrix}, \\
\breve{a}(t,x) = \begin{pmatrix} \left(p + G * \breve{B} + v + w\right) & 0 & 0 \\ 0 & \left(u + q + G * \breve{B} + w\right) & 0 \\ 0 & 0 & \left(u + v + r + G * \breve{B}\right) \end{pmatrix}, \\
y = \begin{pmatrix} m \\ n \\ l \end{pmatrix}, \qquad y_0 = \begin{pmatrix} m_0 \\ n_0 \\ l_0 \end{pmatrix}, \\
\breve{f}(t,x) = \begin{pmatrix} n v_x + l w_x \\ m u_x + l w_x \\ m u_x + n v_x \end{pmatrix} = \begin{pmatrix} 0 & n & l \\ m & 0 & l \\ m & n & 0 \end{pmatrix} \begin{pmatrix} u_x \\ v_x \\ w_x \end{pmatrix}, \\
\breve{g}(t,x) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \qquad y_l = \begin{pmatrix} m_l \\ n_l \\ l_l \end{pmatrix}, \qquad y_r = \begin{pmatrix} m_r \\ n_r \\ l_r \end{pmatrix}.
\end{gathered} \tag{40}$$

Our feedback law for (3) reads

$$y \in C^0([0,1]) \longmapsto \begin{cases} v_l(y) = A_l \|y\|_{C^0([0,1])}, \\ v_r(y) = A_r \|y\|_{C^0([0,1])}, \\ \partial_t y_l = M y_l, \end{cases} \tag{41}$$

where $A_l > 2\sinh(1)$, $A_r > A_l\cosh(1) + \sinh(2)$, $M > 0$, $T > 0$, and $M = \left(\begin{smallmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{smallmatrix}\right)$, $\mu < 0$, a symmetric matrix, is the unique matrix solution to the matrix function:

$$PM + M^T P = -Z, \tag{42}$$

for some symmetric positive-definite matrices P and Z. Indeed, let $V(t, y_l) = y_l^T P y_l$ be the Lyapunov candidate, and that y_l asymptotically converges in time is equivalent to that the time derivative of the $V, \dot{V}(t, y_l) = -y_l^T Z y_l$ is strictly negative. A fixed-point strategy will be used again to prove the existence of a solution to the closed-loop system, we begin by defining the domain of the operator.

Definition 13. Let X be the space of $(g, N) \in C^0([0,T] \times [0,1]) \times C^0([0,1])$ satisfying

(1) for all $(t,x) \in [0,T] \times [0,1]$, $g(0,x) = y_0(x)$, $g(t,0) = y_0(0)e^{Mt}$,

(2) for all $t \in [0,T]$, $\|g(t,\cdot)\|_{C^0([0,1])} \le N(t)$,

(3) N is nonincreasing, and $N(0) \leq \|y_0\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}$.

Lemma 14. *The domain X is nonempty, convex, bounded, and closed with respect to the uniform topology.*

The proof is elementary and one notices that $(y_0(x)e^{Mt}, \|y_0\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}e^{Mt}) \in X$, so X is nonempty.

Now for $(y,N) \in X$, we define $\overset{\vee}{g}$ and $G* \overset{\vee}{B}$ as the solutions of

$$\forall (t,x) \in \Omega_T, \quad \overset{\vee}{g}(t,x) = G * y(t,x),$$
$$\overset{\vee}{g}(t,0) = \overset{\vee}{g}(t,1) = 0, \quad \overset{\vee}{B}(t,x) = 0, \tag{43}$$
$$G* \overset{\vee}{B}(t,0) = A_l N(t), \quad G* \overset{\vee}{B}(t,1) = A_r N(t).$$

One has the following exact formulas:

$$\begin{aligned}&\forall (t,x) \in \Omega_T,\\ &\overset{\vee}{g}(t,x) = -\int_0^x \sinh(x-\breve{x})\, y(t,\breve{x})\, d\breve{x} - \frac{\sinh(x)}{\sinh(1)}\int_0^1 \sinh(\breve{x}-1)\, y(t,\breve{x})\, d\breve{x},\\ &G* \overset{\vee}{B}(t,x) = \frac{N(t)}{\sinh(1)}\left(A_r \sinh(x) + A_l \sinh(1-x)\right).\end{aligned} \tag{44}$$

Therefore, we have the following inequalities:

$$\begin{aligned}&\forall (t,x) \in [0,T]\times[0,1],\\ &\left|\overset{\vee}{g}(t,x)\right| \leq 2(1+\cosh(1))\,\|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])},\\ &\left|\partial_x \overset{\vee}{g}(t,x)\right| \leq 2\cosh(1)\,\|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])},\\ &\left|\partial_{xx}^2 \overset{\vee}{g}(t,x)\right| \leq [2(\cosh(1)+1)+1] \times \|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])},\\ &\left|\partial_x\left(G* \overset{\vee}{B}\right)(t,x)\right| \geq \frac{A_r - 2\cosh(1)A_l}{\sinh(1)}N(t),\\ &\left|G* \overset{\vee}{B}(t,x)\right| \geq A_l N(t).\end{aligned} \tag{45}$$

Let $\overset{\vee}{c}(t,x) = G* \overset{\vee}{B}(t,x) + \overset{\vee}{g}(t,x)$, where $\overset{\vee}{c}(t,x) = \begin{pmatrix} u\\ v\\ w\end{pmatrix}(t,x)$, and in turn those provide

$$\begin{aligned}\overset{\vee}{c}(t,x) &\leq \left[2(1+\cosh(1)) + \frac{\cosh(1)}{\sinh(1)}(A_r + A_l)\right] \times \|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])},\\ \partial_x \overset{\vee}{c}(t,x) &\leq \frac{\sinh(2) + 2A_l\cosh(1) - A_r}{\sinh(1)} \times \|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])},\\ \partial_{xx}^2 \overset{\vee}{c}(t,x) &\leq \left[2(1+\cosh(1)) + 1 + \frac{\cosh(1)}{\sinh(1)}(A_r + A_l)\right] \times \|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}.\end{aligned} \tag{46}$$

Now, if ω is the flow of $\overset{\vee}{c}$, ω is C^1, and since $\overset{\vee}{c} \geq 0$, $\omega(\cdot,t,x)$ is nondecreasing. This allows us to define the entrance time and then the operator S as follows. Let $e(t,x) = \min\{s \in [0,t] \mid \omega(s,t,x) = 0\}$.

Now, for for all $(t,x) \in [0,T]\times[0,1]$, $S(y,N) = (\widetilde{y}, \widetilde{N})$ with the following:

(1) *if $x \geq \omega(t,0,0)$,*

$$\begin{aligned}&y(t,x)\\ &\quad = y_0(\omega(0,t,x))\exp\left(\int_0^t \overset{\vee}{b}(r,\omega(r,t,x))\,dr\right) + \int_0^t \overset{\vee}{f}(r,\omega(r,t,x)) \times \exp\left(\int_r^t \overset{\vee}{b}\left(r',\omega\left(r',t,x\right)\right)dr'\right)dr,\end{aligned} \tag{47}$$

(2) *if $x \leq \omega(t,0,0)$,*

$$\begin{aligned}&y(t,x)\\ &\quad = y_0(0)\,e^{Me(t,x)}\exp\left(\int_0^t \overset{\vee}{b}(r,\omega(r,t,x))\,dr\right) + \int_{e(t,x)}^t \overset{\vee}{f}(r,\omega(r,t,x)) \times \exp\left(\int_r^t \overset{\vee}{b}\left(r',\omega\left(r',t,x\right)\right)dr'\right)dr,\end{aligned} \tag{48}$$

(3) $N(t) = \|\widetilde{y}(t,\cdot)\|_{C^0([0,1])}$.

Lemma 15. *(1) The operator S maps X to X.*

(2) The family $S(X)$ is uniformly bounded and equicontinuous.

(3) S is continuous w.r.t. the uniform topology.

The proof is very similar to [10], except for the state y here is a three-component vector and the proof is omitted.

Now, we can apply Schauder's fixed point theorem to S and get (y, N) fixed point of S.

3.2. Stabilization and Global Existence

Theorem 16. *For any $y_0 \in C^0([0,1]) \times C^0([0,1]) \times C^0([0,1])$, there exists $y \in C^0(\Omega_T) \times C^0([0,T], C^2([0,1]))$ a weak solution of (39) satisfying*

$$\forall x \in [0,1] \quad y(0,x) = y_0(x). \tag{49}$$

Furthermore, any maximal solution of (39) and (41) is global, and if we let

$$k = \max\left(2(1+\cosh(1)) + 1 + \frac{\cosh(1)}{\sinh(1)}(A_r + A_l), \frac{\sinh(2) + 2A_l\cosh(1) - A_r}{\sinh(1)}\right),$$
$$\tau = \frac{1}{\|M\|_3}\ln\left(\frac{\|M\|_3^2}{8k^3\|y_0\|^2_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}}\right), \tag{50}$$

then we have

$$\forall t \geq \tau \quad \|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \leq \frac{\|M\|_3}{2k}(1 + 2k\|M\|_3)\, e^{\|M\|_3(t-x)}. \tag{51}$$

To finish the proof of Theorem (39), we have to prove the global existence of a maximal solution and the estimate (51).

Proof. First, we rewrite (46) as the following:

$$\forall (t,x) \in \Omega_T \quad \check{c}(t,x) \leq k\|y(t,\cdot)\|_{C^0([0,1])},$$
$$\partial_x \check{c}(t,x) \leq k\|y(t,\cdot)\|_{C^0([0,1])}, \tag{52}$$
$$\partial_{xx} \check{c}(t,x) \leq k\|y(t,\cdot)\|_{C^0([0,1])},$$

where $k = \max(2(1+\cosh(1)) + 1 + (\cosh(1)/\sinh(1))(\mathrm{A}_r + \mathrm{A}_l), (\sinh(2) + 2A_l\cosh(1) - A_r)/\sinh(1))$.

For y is the solution of the transport (39) and it satisfies

$$y(t,x) = y(s,\omega(s,t,x))\exp\left(\int_0^t \check{b}(r,\omega(r,t,x))\,dr\right) + \int_0^t \check{f}(r,\omega(r,t,x)) \times \exp\left(\int_r^t \check{b}\left(r',\omega\left(r',t,x\right)\right)dr'\right)dr. \tag{53}$$

Combining those facts, we get for $t \geq s$ the following:

$$|y(t,x)| \leq |y(s,\omega(s,t,x))|\left(1 + 4k^2|y(s,\omega(s,t,x))|\,t\right) \times \exp\left(2\int_s^t k\|y(r,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}dr\right). \tag{54}$$

We have also imposed $y(t,0) = y(s,0)e^{M(t-s)}$ and thanks to the existence theorem that a maximal solution of the closed loop system is global. To get a more precise statement, we consider all the between time t and s, and we obtain.

For $0 \leq s \leq t$,

$$\|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \leq \|y(s,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \times \max\left[e^{\|M\|_3(r-x)}\left(1 + 4k^2\|y(s,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}r\right)\right] \times \exp\left(2k\int_r^t \|y(\alpha,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}d\alpha\right). \tag{55}$$

We define

$$g(r) = \left[e^{\|M\|_3(r-s)}\left(1 + 4k^2\|y(r,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}r\right)\right] \times \exp\left(2k\int_s^t \|y(\alpha,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}d\alpha\right), \tag{56}$$

and we set $g(r) = g_1(r) + g_2(r)$, where

$$g_1(r) = e^{\|M\|_3(r-s)} \times \exp\left(2k\int_r^t \|y(\alpha,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}\right),$$
$$g_2(r) = e^{\|M\|_3(r-s)}4k^2\|y(r,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}r \times \exp\left(2k\int_r^t \|y(\alpha,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}\right). \tag{57}$$

Then, we have

$$g'(r) = \left(\|M\|_3 - 2k\|y(r,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}\right)g_1(r) + \left(\frac{1}{r} + \|M\|_3 - 2k\|y(r,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}\right) \times g_2(r), \tag{58}$$

as long as the quantity $\|y(r,\cdot)\|_{C^0([0,1])}$ is not equal to zero, it strictly decreases, so if $\|y_0\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} >$

$\|M\|_3/2k$, for t small enough $\|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \geq \|M\|_3/2k$, and we have the following.

$$\begin{aligned}\|y(t,\cdot)\|&_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}\\&\leq \|y_0\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}\\&\quad\times\left(1+4k^2\|y_0\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}t\right)e^{\|M\|_3 t}.\end{aligned} \tag{59}$$

If we define $\tau = (1/\|M\|_3)\ \ln(\|M\|_3^2/8k^3\|y_0\|^2_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])})$, we get that

$$\|y(\tau,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \leq \frac{\|M\|_3}{2k}. \tag{60}$$

This provides $\tau \leq s \leq t$, the inequality (which was clear when $\|y_0\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \leq \|M\|_3/2k$)

$$\begin{aligned}\|y(t,\cdot)\|&_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}\\&\leq \|y(\tau,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}\\&\quad\times\left(1+4k^2\|y(\tau,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}t\right)\\&\quad\times\exp\left(2k\int_x^t\|y(r,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}dr\right)\\&\leq\frac{\|M\|_3}{2k}\left(1+2k\|M\|_3\right)e^{\|M\|_3(t-x)}.\end{aligned} \tag{61}$$

□

4. Blow-Up Phenomena

In this section, we present a result with the initial data and boundary profiles under a special condition that ensure strong solutions to following system blow-up in finite time as follows:

$$\begin{aligned}&\partial_t m-(u+v+w)(t,x)\,\partial_x m\\&\quad=\left(2\left(p+G*B\right)+v_x+w_x\right)(t,x)\,m\\&\qquad+\left(nv_x+lw_x\right)(t,x),\\&\partial_t n-(u+v+w)(t,x)\,\partial_x n\\&\quad=\left(2\left(q+G*B\right)+u_x+w_x\right)(t,x)\,n\\&\qquad+\left(mu_x+lw_x\right)(t,x),\\&\partial_t l-(u+v+w)(t,x)\,\partial_x l\\&\quad=\left(2\left(r+G*B\right)+u_x+v_x\right)(t,x)\,l\\&\qquad+\left(mu_x+nv_x\right)(t,x),\\&m(0,\cdot)=m_0,\quad n(0,\cdot)=n_0,\quad l(0,\cdot)=l_0,\quad x\in[0,1],\\&m(t,0)=m_l=m(t,1)=m_r,\quad t\in[0,T],\\&n(t,0)=n_l=n(t,1)=n_r,\quad t\in[0,T],\\&l(t,0)=l_l=l(t,1)=l_r,\quad t\in[0,T],\end{aligned} \tag{62}$$

where $m=u-u_{xx}$, $n=v-v_{xx}$, $l=w-w_{xx}$. This imply that

$$\begin{aligned}u(t,0)&=u(t,1),\\v(t,0)&=v(t,1),\\w(t,0)&=w(t,1).\end{aligned} \tag{63}$$

From Definition 3, we can also define

$$\begin{aligned}\omega_t&=-(u+v+w)\left(t,\omega(t,y)\right),\\&\qquad(t,x)\in[0,T]\times[0,1],\\\omega(0,x)\,1&=x,\quad x\in[0,1];\end{aligned} \tag{64}$$

where u, v, and w denote the solution to (62). Applying classical results in the theory of ordinary differential equations, one can obtain a result on which is crucial in studying blow-up phenomena.

From (62), we obtain that

$$\begin{aligned}m_t-\partial_x(mu+mv+mw)&=mu_x+nv_x+lw_x,\\n_t-\partial_x(mu+nv+mw)&=mu_x+nv_x+lw_x,\\l_t-\partial_x(lu+lv+lw)&=mu_x+nv_x+lw_x.\end{aligned} \tag{65}$$

Lemma 17. *Let $u, v, w \in C([0,T); H^s)\cap C^1([0,T); H^{s-1})$, ($s \geq 2$), then (64) has a unique solution $\omega \in C([0,T)\times[0,1])$. Moreover, the map $\omega(t,\cdot)$ is an increasing diffeomorphism with*

$$\begin{aligned}\omega_x(t,x)&=\exp\left\{-\int_0^t\left(u_x(s,\omega(s,x))\right.\right.\\&\qquad\left.\left.+v_x(s,\omega(s,x))+w_x(s,\omega(s,x))\,ds\right)\right\}\\&>0,\\\omega_x(0,x)&=1,\quad\forall x\in[0,1].\end{aligned} \tag{66}$$

Proof. The proof is omitted here, one can see a similar proof in [12]. □

Now, we have the following lemma that the potential $m-n$, $n-l$, $m-l$ with compactly supported initial datum m_0-n_0, n_0-l_0, m_0-l_0 also has compact x support as long as it exists.

Lemma 18. *Assume that $u_0(t,x), v_0(t,x), w_0(t,x) \in H^s\times H^s$ with $s > (3/2)$, (u,v,w) is the corresponding solution, if $m_0-n_0, n_0-l_0, m_0-l_0$ has compact support, then $m-n, n-l, m-l$ also has compact support, moreover, we can obtain that*

$$\begin{aligned}\|m(t,\cdot)-n(t,\cdot)\|&\leq e^{3KT}\left\|m_0(\cdot)-n_0(\cdot)\right\|,\\\|m(t,\cdot)-l(t,\cdot)\|&\leq e^{3KT}\left\|m_0(\cdot)-l_0(\cdot)\right\|,\\\|n(t,\cdot)-l(t,\cdot)\|&\leq e^{3KT}\left\|n_0(\cdot)-l_0(\cdot)\right\|.\end{aligned} \tag{67}$$

Proof. Since

$$\frac{d}{dt}\left(m\left(t,\omega\left(t,x\right)\right)\omega_x\right) = \left(m_t + m_x\omega_t\right)\omega_x + m\omega_{xt} = \left[m_t - \partial_x\left(mu + mv + mw\right)\right]\omega_x = \left(mu_x + nv_x + lw_x\right)\omega_x. \quad (68)$$

Similarly,

$$\frac{d}{dt}\left(n\left(t,\omega\left(t,x\right)\right)\omega_x\right) = \left(mu_x + nv_x + lw_x\right)\omega_x,$$
$$\frac{d}{dt}\left(l\left(t,\omega\left(t,x\right)\right)\omega_x\right) = \left(mu_x + nv_x + lw_x\right)\omega_x. \quad (69)$$

So it follows that

$$\frac{d}{dt}\left(\left(m\left(t,\omega\left(t,x\right)\right) - n\left(t,\omega\left(t,x\right)\right)\right)\omega_x\right) = 0,$$
$$\frac{d}{dt}\left(\left(m\left(t,\omega\left(t,x\right)\right) - l\left(t,\omega\left(t,x\right)\right)\right)\omega_x\right) = 0, \quad (70)$$
$$\frac{d}{dt}\left(\left(n\left(t,\omega\left(t,x\right)\right) - l\left(t,\omega\left(t,x\right)\right)\right)\omega_x\right) = 0.$$

We obtain

$$\left[m\left(t,\omega\left(t,x\right)\right) - n\left(t,\omega\left(t,x\right)\right)\right]\omega_x = m_0\left(x\right) - n_0\left(x\right),$$
$$\left[m\left(t,\omega\left(t,x\right)\right) - l\left(t,\omega\left(t,x\right)\right)\right]\omega_x = m_0\left(x\right) - l_0\left(x\right), \quad (71)$$
$$\left[n\left(t,\omega\left(t,x\right)\right) - l\left(t,\omega\left(t,x\right)\right)\right]\omega_x = n_0\left(x\right) - l_0\left(x\right).$$

From Lemma 17, we have

$$\omega_x = \exp\left(-\int_0^t u_x\left(s,\omega\left(s,x\right)\right) + v_x\left(s,\omega\left(s,x\right)\right) + w_x\left(s,\omega\left(s,x\right)\right)ds\right). \quad (72)$$

If there exist four constants K, K_1, K_2, and K_3 such that $u_x \le K_1, v_x \le K_2, w_x \le K_3$, and $K = \max(K_1, K_2, K_3)$, we can get that

$$\begin{aligned}&\left\|m\left(t,\cdot\right) - n\left(t,\cdot\right)\right\|_{L^\infty} \\ &= \left\|m\left(t,\omega\left(t,\cdot\right)\right) - n\left(t,\omega\left(t,\cdot\right)\right)\right\|_{L^\infty} \\ &= \left\|\exp\left(\int_0^t u_x\left(s,\omega\left(s,x\right)\right) + v_x\left(s,\omega\left(s,x\right)\right) + w_x\left(s,\omega\left(s,x\right)\right)ds\right)\times\left(m_0\left(x\right) - n_0\left(x\right)\right)ds\right\|_{L^\infty} \\ &\le e^{3KT}\left\|m_0\left(\cdot\right) - n_0\left(\cdot\right)\right\|.\end{aligned} \quad (73)$$

Similarly,

$$\left\|m\left(t,\cdot\right) - l\left(t,\cdot\right)\right\| \le e^{3KT}\left\|m_0\left(\cdot\right) - l_0\left(\cdot\right)\right\|,$$
$$\left\|n\left(t,\cdot\right) - l\left(t,\cdot\right)\right\| \le e^{3KT}\left\|n_0\left(\cdot\right) - l_0\left(\cdot\right)\right\|. \quad (74)$$

□

Theorem 19. *Let $u_0(x), v_0(x)$, and $w_0(x)$, $s > 3/2$ and $u(t,x), v(t,x), w(t,x)$ be the solution in (62) with time T. Then, T is finite if and only if*

$$\liminf_{t\to T}\left\{\inf_{x\in[0,1]}\left[u_x\left(x,t\right)\right]\right\} = -\infty \quad (75)$$

or

$$\liminf_{t\to T}\left\{\inf_{x\in[0,1]}\left[v_x\left(x,t\right)\right]\right\} = -\infty \quad (76)$$

or

$$\liminf_{t\to T}\left\{\inf_{x\in[0,1]}\left[w_x\left(x,t\right)\right]\right\} = -\infty. \quad (77)$$

Proof. Let $u_0(x), v_0(x)$, and $w_0(x)$, $s > 3/2$ and $u(t,x), v(t,x), w(t,x)$ be the solution (62) with time T. We know that $u(t,0) = u(t,1), v(t,0) = v(t,1)$, and $w(t,0) = w(t,1)$. By the definition of m, n, and l, we have

$$\|m\|_{L^2}^2 = \int_0^1\left(u - u_{xx}\right)^2dx = \int_0^1\left(u^2 + 2u_x^2 + u_{xx}^2\right)dx,$$
$$\|n\|_{L^2}^2 = \int_0^1\left(v - v_{xx}\right)^2dx = \int_0^1\left(v^2 + 2v_x^2 + v_{xx}^2\right)dx, \quad (78)$$
$$\|l\|_{L^2}^2 = \int_0^1\left(w - w_{xx}\right)^2dx = \int_0^1\left(w^2 + 2w_x^2 + w_{xx}^2\right)dx.$$

Hence, $\|u\|_{H^2}^2 \le \|m\|_{L^2}^2 \le 2\|u\|_{H^2}^2, \|v\|_{H^2}^2 \le \|n\|_{L^2}^2 \le 2\|v\|_{H^2}^2, \|w\|_{H^2}^2 \le \|l\|_{L^2}^2 \le 2\|w\|_{H^2}^2$.

Multiplying the first equation by m, the second one by n, and the third one by l, after integration by parts and adding up the results, we see that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\left(m^2 + n^2 + l^2\right)dx \\ &= \int_0^1\Big[\left(2u_x + v_x + w_x\right)m^2 + \left(2v_x + u_x + w_x\right)n^2 \\ &\quad + \left(2w_x + u_x + v_x\right)l^2 + \left(mm_x + nn_x + ll_x\right) \\ &\quad \times\left(u + v + w\right) + mn\left(u_x + v_x\right) \\ &\quad + lm\left(u_x + w_x\right) + \ln\left(v_x + w_x\right)\Big]dx \\ &= \int_0^1\left[\left(2u_x + v_x + w_x - \frac{1}{2}u_x - \frac{1}{2}v_x - \frac{1}{2}w_x\right)m^2\right.\end{aligned}$$

$$+\left(2v_x+u_x+w_x-\frac{1}{2}u_x-\frac{1}{2}v_x-\frac{1}{2}w_x\right)n^2$$
$$+\left(2w_x+u_x+v_x-\frac{1}{2}u_x-\frac{1}{2}v_x-\frac{1}{2}w_x\right)l^2$$
$$+mn\left(u_x+v_x\right)+lm\left(u_x+w_x\right)\ln\left(v_x+w_x\right)\Big]dx$$
$$=\int_0^1\Big[\left(\frac{3}{2}u_x+\frac{1}{2}v_x+\frac{1}{2}w_x\right)m^2+\left(\frac{3}{2}v_x+\frac{1}{2}u_x+\frac{1}{2}w_x\right)n^2$$
$$+\left(\frac{3}{2}w_x+\frac{1}{2}u_x+\frac{1}{2}v_x\right)l^2+mn\left(u_x+v_x\right)$$
$$+lm\left(u_x+w_x\right)+\ln\left(v_x+w_x\right)\Big]dx$$
$$=\int_0^1\Big[\frac{1}{2}\left(u_x+v_x+w_x\right)\left(m^2+n^2+l^2\right)$$
$$+m^2u_x+n^2v_x+l^2w_x+mn\left(u_x+v_x\right)$$
$$+lm\left(u_x+w_x\right)+\ln\left(v_x+w_x\right)\Big]dx$$
$$=\int_0^1\Big[\frac{1}{2}\left(u_x+v_x\right)(m+\mathrm{n})^2+\frac{1}{2}\left(u_x+w_x\right)(m+l)^2$$
$$+\frac{1}{2}\left(v_x+w_x\right)(l+n)^2+\frac{1}{2}u_xm^2$$
$$+\frac{1}{2}v_xn^2+\frac{1}{2}w_xl^2\Big]dx \tag{79}$$

So, we have

$$\frac{d}{dt}\int_0^1\left(m^2+n^2+l^2\right)dx$$
$$\leq\left(K_1+K_2+K_3\right)$$
$$\times\int_0^1\Big[(m+n)^2+(m+l)^2 \tag{80}$$
$$+(l+n)^2+m^2+n^2+l^2\Big]dx$$
$$\leq 5\left(K_1+K_2+K_3\right)\int_0^1\left(m^2+n^2+l^2\right)dx.$$

By Gronwall's inequality, we get

$$\|u\|_{H^2}^2+\|v\|_{H^2}^2+\|w\|_{H^2}^2$$
$$\leq\int_0^1\left(m^2+n^2+l^2\right)dx$$
$$\leq\exp\left[5T\left(K_1+K_2+K_3\right)\right]$$
$$\times\int_0^1\left(m_0^2+n_0^2+l_0^2\right)dx$$
$$\leq 2\exp\left[5T\left(K_1+K_2+K_3\right)\right]$$
$$\times\left(\left\|u_0\right\|_{H^2}^2+\left\|v_0\right\|_{H^2}^2+\left\|w_0\right\|_{H^2}^2\right), \tag{81}$$

The above inequality, Soblev's embedding theorem, ensure that the solution $(u(t,\mathrm{x}),v(t,x),w(t,x))$ cannot blow up in finite time.

On the other hand, if

$$\lim_{t\to T}\inf\left\{\inf_{x\in[0,1]}\left[u_x(x,t)\right]\right\}=-\infty \tag{82}$$

or

$$\lim_{t\to T}\inf\left\{\inf_{x\in[0,1]}\left[v_x(x,t)\right]\right\}=-\infty \tag{83}$$

or

$$\lim_{t\to T}\inf\left\{\inf_{x\in[0,1]}\left[w_x(x,t)\right]\right\}=-\infty, \tag{84}$$

then the solution will blow up in finite time. □

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References

[1] R. S. Johnson, "Camassa-Holm, Korteweg-de Vries and related models for water waves," *Journal of Fluid Mechanics*, vol. 455, pp. 63–82, 2002.

[2] K.-H. Kwek, H. Gao, W. Zhang, and C. Qu, "An initial boundary value problem of Camassa-Holm equation," *Journal of Mathematical Physics*, vol. 41, no. 12, pp. 8279–8285, 2000.

[3] S. Ma and S. Ding, "On the initial boundary value problem for a shallow water equation," *Journal of Mathematical Physics*, vol. 45, no. 9, pp. 3479–3497, 2004.

[4] L. Molinet, "On well-posedness results for Camassa-Holm equation on the line: a survey," *Journal of Nonlinear Mathematical Physics*, vol. 11, no. 4, pp. 521–533, 2004.

[5] G. Rodríguez-Blanco, "On the Cauchy problem for the Camassa-Holm equation," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 46, no. 3, pp. 309–327, 2001.

[6] E. Wahln, "Global existence of weak solutions to the Camassa-Holm equation," *Communications on Pure and Applied Mathematics*, vol. 21, pp. 883–906, 2006.

[7] Z. Xin and P. Zhang, "On the weak solutions to a shallow water equation," *Communications on Pure and Applied Mathematics*, vol. 53, no. 11, pp. 1411–1433, 2000.

[8] G. M. Coclite, H. Holden, and K. H. Karlsen, "Global weak solutions to a generalized hyperelastic-rod wave equation," *SIAM Journal on Mathematical Analysis*, vol. 37, no. 4, pp. 1044–1069, 2005.

[9] J. Escher and Z. Yin, "Initial boundary value problems for nonlinear dispersive wave equations," *Journal of Functional Analysis*, vol. 256, no. 2, pp. 479–508, 2009.

[10] G. Zhang, Z. Qiao, and F. Liu, "Cusp and smooth solitons of the Camassa-Holm equation under an inhomogeneous boundary condition," *Pacific Journal of Applied Mathematics*, vol. 1, no. 1, pp. 105–121, 2008.

[11] V. Perrollaz, "Initial boundary value problem and asymptotic stabilization of the Camassa-Holm equation on an interval," *Journal of Functional Analysis*, vol. 259, no. 9, pp. 2333–2365, 2010.

[12] J. Escher and Z. Yin, "Initial boundary value problems of the Camassa-Holm equation," *Communications in Partial Differential Equations*, vol. 33, no. 1–3, pp. 377–395, 2008.

[13] G. MisioThlek, "Classical solutions of the periodic Camassa-Holm equation," *Geometric and Functional Analysis*, vol. 12, no. 5, pp. 1080–1104, 2002.

[14] J. Zhou and L. Tian, "Blow-up of solution of an initial boundary value problem for a generalized Camassa-Holm equation," *Physics Letters A*, vol. 372, no. 20, pp. 3659–3666, 2008.

[15] A. Bressan and A. Constantin, "Global conservative solutions of the Camassa-Holm equation," *Archive for Rational Mechanics and Analysis*, vol. 183, no. 2, pp. 215–239, 2007.

[16] Y. Fu and C. Qu, "Well posedness and blow-up solution for a new coupled Camassa-Holm equations with peakons," *Journal of Mathematical Physics*, vol. 50, no. 1, Article ID 012906, 25 pages, 2009.

[17] Q. Hu, L. Lin, and J. Jin, "Initial boundary value problem for a coupled Camassa-Holm system with peakons," *Journal of Applied Analysis*, pp. 1–17, 2012.

[18] A. Constantin and R. I. Ivanov, "On an integrable two-component Camassa-Holm shallow water system," *Physics Letters A*, vol. 372, no. 48, pp. 7129–7132, 2008.

[19] M. Chen, S.-Q. Liu, and Y. Zhang, "A two-component generalization of the Camassa-Holm equation and its solutions," *Letters in Mathematical Physics A*, vol. 75, no. 1, pp. 1–15, 2006.

[20] R. I. Ivanov, "Extended Camassa-Holm hierarchy and conserved quantities," *Zeitschrift Für Naturforschung*, vol. 61, pp. 133–138, 2006.

[21] J. Escher, O. Lechtenfeld, and Z. Yin, "Well-posedness and blow-up phenomena for the 2-component Camassa-Holm equation," *Discrete and Continuous Dynamical Systems A*, vol. 19, no. 3, pp. 493–513, 2007.

[22] P. Zhang and Y. Liu, "Stability of solitary waves and wave-breaking phenomena for the two-component Camassa-Holm system," *International Mathematics Research Notices*, no. 11, pp. 1981–2021, 2010.

[23] A. Constantin and R. I. Ivanov, "On an integrable two-component Camassa-Holm shallow water system," *Physics Letters A*, vol. 372, no. 48, pp. 7129–7132, 2008.

[24] Y. Fu, Y. Liu, and C. Qu, "Well-posedness and blow-up solution for a modified two-component periodic Camassa-Holm system with peakons," *Mathematische Annalen*, vol. 348, no. 2, pp. 415–448, 2010.

[25] M. Zhu, "Blow-up, global existence and persistence properties for the coupled Camassa-Holm equations," *Mathematical Physics*, vol. 14, no. 3, pp. 197–209, 2011.

[26] J. Yin, L. Tian, and X. Fan, "Stability of negative solitary waves for an integrable modified Camassa-Holm equation," *Journal of Mathematical Physics*, vol. 51, no. 5, Article ID 053515, 6 pages, 2010.

[27] W. Yan, L. Tian, and M. Zhu, "Local well-posedness and blow-up phenomenon for a modified two-component Camassa-Holm system in Besov spaces," *International Journal of Nonlinear Science*, vol. 13, no. 1, pp. 99–104, 2012.

[28] P. Zhang, "Global existence of solutions to the modified Camassa-Holm shallow water equation," *International Journal of Nonlinear Science*, vol. 9, no. 1, pp. 123–128, 2010.

[29] L. Tian, Y. Wang, and J. Zhou, "Global conservative and dissipative solutions of a coupled Camassa-Holm equations," *Journal of Mathematical Physics*, vol. 52, no. 6, Article ID 063702, 29 pages, 2011.

[30] L. Tian, W. Yan, and G. Gui, "On the local well posedness and blow-up solution of a coupled Camassa-Holm equations in Besov spaces," *Journal of Mathematical Physics*, vol. 53, no. 1, Article ID 013701, 10 pages, 2012.

[31] Q. Hu and Z. Yin, "Well-posedness and blow-up phenomena for a periodic two-component Camassa-Holm equation," *Proceedings of the Royal Society of Edinburgh A*, vol. 141, no. 1, pp. 93–107, 2011.

[32] C. Guan and Z. Yin, "Global existence and blow-up phenomena for an integrable two-component Camassa-Holm shallow water system," *Journal of Differential Equations*, vol. 248, no. 8, pp. 2003–2014, 2010.

[33] L. Tian and Y. Xu, "Attractor for a viscous coupled Camassa-Holm equation," *Advances in Difference Equations*, vol. 2010, Article ID 512812, 30 pages, 2010.

[34] Y. Fu and C. Qu, "On a new Three-Component Camassa-Holm equation with peakons," *Communications in Theoretical Physics*, vol. 53, no. 2, p. 223, 2010.

[35] Q. Hu, L. Lin, and J. Jin, "Well-posedness and blowup phenomena for a three-component Camassa-Holm system with peakons," *Journal of Hyperbolic Differential Equations*, vol. 9, no. 3, pp. 451–467, 2012.


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