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Research Article

Integrally Small Perturbations of Semigroups and Stability of Partial Differential Equations

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Let A be a generator of an exponentially stable operator semigroup in a Banach space, and let $C(t)\,(t \geq 0)$ be a linear bounded variable operator. Assuming that $\int_0^t C(s)ds$ is sufficiently small in a certain sense for the equation $dx/dt = Ax + C(t)x$, we derive exponential stability conditions. Besides, we do not require that for each $t_0 \geq 0$, the "frozen" autonomous equation $dx/dt = Ax + C(t_0)x$ is stable. In particular, we consider evolution equations with periodic operator coefficients. These results are applied to partial differential equations.

1. Introduction and Statement of the Main Result

In this paper, we investigate stability of linear nonautonomous equations in a Banach space, which can be considered as integrally small perturbations of autonomous equations. The stability theory of evolution equations in a Banach space is well developed, compare and confare with [1] and references therein, but the problem of stability analysis of evolution equations continues to attract the attention of many specialists despite its long history. It is still one of the most burning problems, because of the absence of its complete solution. One of the basic methods for the stability analysis is the direct Lyapunov method. By that method, many strong results were established, compare and confare with [2, 3]. But finding the Lyapunov functionals is usually a difficult mathematical problem. A fundamental approach to the stability of diffusion parabolic equations is the method of upper and lower solutions. A systematical treatment of that approach is given in [4]. In [5], stability conditions are established by a normalizing mapping. Note that a normalizing mapping enables us to use more complete information about the equation than a usual (number) norm. In [6], the "freezing" method for ordinary differential equations is extended to equations in a Banach space. About the recent results, see the interesting papers [7–11]. In particular, in [7] the Perron-Bellman theorem for evolutionary processes with exponential growth in Banach spaces is investigated. In the paper [8], a Rolewicz's type theorem of in-solid function spaces is proved. Dragan and Morozan [9] established criteria for exponential stability of linear differential equations on ordered Banach spaces. Paper [10] deals with the stability and controllability of hyperbolic type abstract evolution equations. Pucci and Serrin [11] investigated the asymptotic stability for nonautonomous wave equations.

Certainly, we could not survey the whole subject here and refer the reader to the previously listed publications and references given therein.

Let X be a complex Banach space with a norm $\|\cdot\|_X$ and the unit operator I. For a bounded operator K, $\|K\|$ is the operator norm.

Everywhere below A is a linear operator in X with a domain $\mathrm{Dom}(A)$, generating a strongly continuous semigroup $T(t)$; that is, $A = \lim_{h\downarrow 0}(1/h)(T(h) - I)$ in the strong topology, and $C(t)$ $(t \geq 0)$ is a linear bounded variable operator mapping $\mathrm{Dom}(A)$ into itself. Put $B(t) = A + C(t)$. In the present paper, we establish stability conditions for the equation as follows:

$$\frac{du}{dt} = B(t)\,u, \quad t \geq 0. \tag{1}$$

It should be noted that in the previously pointed papers it is assumed that for each $t_0 \geq 0$, the "frozen" autonomous equation $dx/dt = B(t_0)x$ is stable. We do not require that condition. The aim of this paper is to generalize the main result from the paper [12], which deals with finite dimensional equations.

A solution of (1), for a given $u_0 \in \mathrm{Dom}(A)$ is a function $u : [0.\infty) \to \mathrm{Dom}(A)$ having a strong derivative, satisfying (1) and $u(0) = u_0$. We will investigate (1) as a perturbation of the following equation:

$$\frac{dv}{dt} = Av, \quad t \geq 0. \tag{2}$$

Put

$$J(t) := \int_0^t C(s)\,ds,$$
$$m(t) := \|AJ(t) - J(t)B(t)\|, \quad (t \geq 0). \tag{3}$$

We say that (1) is exponentially stable if there is an α = const > 0, such that $\|u(t)\|_X \leq e^{-\alpha t}\|u(0)\|_X$ $(t \geq 0)$ for any solution $u(t)$ with $u(0) \in \mathrm{Dom}(A)$. Now we are in a position to formulate the main result of the paper.

Theorem 1. *Let*

$$\|T\|_{L^1} := \int_0^\infty \|T(t)\|\,dt < \infty, \tag{4}$$

$$\sup_{t\geq 0}\left(\|J(t)\| + \int_0^t \|T(t-s)\|\,m(s)\,ds\right) < 1. \tag{5}$$

Then, (1) *is exponentially stable.*

The proof of this theorem is divided into a series of lemmas which are presented in the next section. To the best of our knowledge, Theorem 1 is new even in the case of bounded operators. In Section 3 we consider particular cases of Theorem 1. In Section 4, the previously pointed results are applied to a partial differential equation. For the brevity, we restrict ourselves by a scalar equation with the periodic boundary condition, but our results enable us to consider coupled systems of equations and other boundary conditions, for example, the Dirichlet condition.

2. Proofs

We need the following simple result.

Lemma 2. *Let $w(t)$, $f(t)$, and $v(t)$ $(0 \leq t \leq a \leq \infty)$ be functions whose values are bounded linear operators. Assume that $w(t)$ is integrable and $f(t)$ and $v(t)$ have integrable derivatives on $[0, a]$. Then, with the notation $j_w(t) = \int_0^t w(s)ds$, one has*

$$\int_0^t f(s)\,w(s)\,v(s)\,ds$$
$$= f(t)\,j_w(t)\,v(t)$$
$$- \int_0^t \left[f'(s)\,j_w(s)\,v(s) + f(s)\,j_w(s)\,v'(s)\right]ds, \quad (t \leq a). \tag{6}$$

Proof. Clearly,

$$\frac{d}{dt} f(t)\,j_w(t)\,v(t)$$
$$= f'(t)\,j_w(t)\,v(t) + f(t)\,w(t)\,v(t) + f(t)\,j_w(t)\,v'(t). \tag{7}$$

Integrating this equality and taking into account that $j_w(0) = 0$, we arrive at the required result. □

Let $V(t)$ be the Cauchy operator to (1); that is, $V(t)u(0) = u(t)$ for a solution $u(t)$ of (1).

Lemma 3. *One has*

$$(I - J(t))\,V(t)$$
$$= T(t) + \int_0^t T(t-s)\,[AJ(s) - J(s)B(s)]\,V(s)\,ds. \tag{8}$$

Proof. As it is well known,

$$V(t) - T(t) = \int_0^t T(t-s)\,C(s)\,V(s)\,ds, \tag{9}$$

compare and confare with [13]. Thanks to the previous lemma, one has

$$\int_0^t T(t-s)\,C(s)\,V(s)\,ds$$
$$= T(0)\,J(t)\,V(t) - \int_0^t \left[\left(\frac{dT(t-s)}{ds}\right) J(s)\,V(s) + T(t-s)\,J(s)\,V'(s)\right] ds. \tag{10}$$

But $dT(t-s)/ds = -AT(t-s)$. In addition, $V'(s) = B(s)V(s)$. Thus,

$$\int_0^t T(t-s)\,C(s)\,V(s)\,ds$$
$$= J(t)\,V(t) + \int_0^t T(t-s)\,[AJ(s) - J(s)B(s)]\,V(s)\,ds. \tag{11}$$

Now, (9) implies the required result. □

Let,

$$\zeta(t) := \inf_{h\in X;\|h\|=1} \|(J(t) - I)\,h\|. \tag{12}$$

Lemma 4. *Let condition*

$$\inf_{t\geq 0}\zeta(t) > 0 \tag{13}$$

hold. Then, $\|V(t)\| \leq z(t)$, $t \geq 0$, where $z(t)$ is a solution of the following equation:

$$z(t) = \frac{1}{\zeta(t)} \times \left[\|T(t)\| + \int_0^t \|T(t-s)\|\,m(s)\,z(s)\,ds\right], \quad t \geq 0. \tag{14}$$

Proof. Thanks to the previous lemma,

$$\begin{aligned} &\|(I - J(t)) V(t)\| \\ &\quad \leq \|T(t)\| + \int_0^t \|T(t-s)\| m(s) \|V(s)\| ds. \end{aligned} \tag{15}$$

Hence

$$\begin{aligned} &\zeta(t) \|V(t)\| \\ &\quad \leq \|T(t)\| + \int_0^t \|T(t-s)\| m(s) \|V(s)\| ds. \end{aligned} \tag{16}$$

Then by the well-known (comparison) Lemma 3.2.1 from [14] we have the required result. □

Let

$$\eta_0 := \sup_{t\geq 0} \frac{1}{\zeta(t)} \int_0^t \|T(t-s)\| m(s)\, ds < 1. \tag{17}$$

Then (14) implies

$$\sup_t z(t) \leq \sup_{t\geq 0} \frac{\|T(t)\|}{\zeta(t)} + \sup_t z(t)\, \eta_0. \tag{18}$$

Due to the previous lemma we get the following.

Lemma 5. *Let conditions* (13) *and* (17) *hold. Then*

$$\sup_{t\geq 0} \|V(t)\| \leq \sup_{t\geq 0} \frac{\|T(t)\|}{(1-\eta_0)\zeta(t)}. \tag{19}$$

Proof of Theorem 1. Assume that

$$j(t) := \|J(t)\| \leq q < 1, \quad (q = \text{const};\ t \geq 0), \tag{20}$$

then $\zeta(t) \geq 1 - j(t)$. If

$$\eta_1 := \sup_{t\geq 0} \frac{1}{1-j(t)} \int_0^t \|T(t-s)\| m(s)\, ds < 1, \tag{21}$$

then $\eta_0 \leq \eta_1 < 1$ and thanks to the previous lemma, (1) is stable. But condition (5) implies that

$$j(t) + \int_0^t \|T(t-s)\| m(s)\, ds < 1 \tag{22}$$

or

$$\frac{1}{1-j(t)} \int_0^t \|T(t-s)\| m(s)\, ds < 1, \quad (t \geq 0). \tag{23}$$

Thus (5) implies the inequality $\eta_1 < 1$, and therefore, from (5), condition (21) follows. This proves the stability. To prove the exponential stability we use the well-known Theorem 4.1 [13, p. 116] (see also Theorem 2.44 [1, p. 49]). It asserts that the finiteness of the L^1-norm of T implies the inequality

$$\|T(t)\| \leq M e^{-\alpha t}, \quad t \geq 0, \tag{24}$$

where $M = \text{const} \geq 1$, $\alpha = \text{const} > 0$. So for $0 < \epsilon < \alpha$, the semigroup $T_\epsilon(t)$ generated by $A + I\epsilon$ satisfies the inequality $\|T_\epsilon(t)\| \leq M e^{-(\alpha-\epsilon)t}$, $t \geq 0$, and therefore it also has a finite L^1-norm. Substitute the equality

$$u(t) = y(t) e^{-\epsilon t} \tag{25}$$

into (1). Then we obtain the equation

$$\dot{y} = (B(t) + I\epsilon)\, y. \tag{26}$$

Denote the Cauchy operator of (26) by $V_\epsilon(t)$. Repeating our above arguments with $V_\epsilon(t)$ instead of $V(t)$ and the equation $\dot{x} = (A + I\epsilon)x$ instead of (2), due to Lemma 5 we can assert that $V_\epsilon(t)$ is bounded. Now (25) implies

$$\|V(t)\| \leq e^{-\epsilon t} \sup_{t\geq 0} \left\|V_\epsilon(t)\right\|, \quad t \geq 0. \tag{27}$$

This proves the theorem. □

3. A particular Case of Theorem 1

To illustrate Theorem 1, consider the following equation:

$$\frac{du}{dt} = Au + c(t) C_0 u, \tag{28}$$

where C_0 is a constant operator and $c(t)$ is a scalar real piecewise continuous function bounded on $[0, \infty)$. So, $C(t) = c(t)C_0$. Without any loss of generality, assume that

$$\sup_t |c(t)| = 1, \tag{29}$$

and with the notation

$$i_c(t) = \left| \int_0^t c(s)\, ds \right|, \tag{30}$$

we obtain

$$\begin{aligned} m(t) &= \|AJ(t) - J(t)B(t)\| \\ &\leq i_c(t) \left\|AC_0 - C_0(A + c(t)C_0)\right\| \\ &\leq i_c(t) \left(\left\|AC_0 - C_0A\right\| + |c(t)| \left\|C_0^2\right\|\right) \\ &\leq i_c(t) \left(\left\|AC_0 - C_0A\right\| + \left\|C_0^2\right\|\right). \end{aligned} \tag{31}$$

Due to (24),

$$\int_0^t \|T(t-s)\|\, ds \leq M \int_0^t e^{-\alpha s} ds \leq \frac{M}{\alpha}, \quad (t \geq 0). \tag{32}$$

Thus, denoting

$$\theta_0 = \sup_t \left| \int_0^t c(s)\, ds \right|, \tag{33}$$

due to Theorem 1, we arrive at the following result.

Corollary 6. *If the inequality*

$$\theta_0 \left(\|C_0\| + \frac{M}{\alpha} \left(\|AC_0 - C_0A\| + \left\|C_0^2\right\| \right) \right) < 1 \tag{34}$$

holds, then (28) *is exponentially stable.*

For example, let $c(t) = \sin(\omega t)$ $(\omega > 0)$. Then, $i_c(t) \le 2/\omega$ and

$$m(t) \le \frac{2}{\omega}\left(\|AC_0 - C_0A\| + \|C_0^2\|\right). \tag{35}$$

Thus, (34) takes the following form:

$$\|C_0\| + \frac{M}{\alpha}\left(\|AC_0 - C_0A\| + \|C_0^2\|\right) < \frac{\omega}{2}. \tag{36}$$

4. Equations with Periodic Boundary Conditions

Consider the problem

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x} + (-b_0 + c(t)a(x))u(x,t), \quad (0 \le x \le 1), \tag{37}$$

$$u(0,t) = u(1,t), \quad (t \ge 0), \tag{38}$$

with a positive constant b_0 and a real differentiable function $a(x)$; $c(t)$ is the same as in the previous section.

Take $X = L^2(0,1)$, where $L^2 = L^2(0,1)$ is the Hilbert space of real functions f, h defined on $[0,1]$ with the scalar product

$$(f,h) = \int_0^1 f(x)h(x)\,dx, \tag{39}$$

and the norm $\|h\| = \sqrt{(h,h)}$. Set

$$\operatorname{Dom}(A) = \left\{f \in L^2 : f' \in L^2; f(0) = f(1)\right\}, \tag{40}$$

$$Au(x) = \frac{du(x)}{dx} - b_0u(x) \quad (u \in \operatorname{Dom}(A)), \tag{41}$$

then we have

$$\begin{aligned}(Au,u) &= \int_0^1 \left(\frac{du(x)}{dx} - b_0u(x)\right)u(x)\,dx \\ &= \int_0^1 \left(\frac{du(x)}{dx} - b_0u(x)\right)u(x)\,dx \\ &= \int_0^1 \left(\frac{1}{2}\frac{du^2(x)}{dx} - b_0u^2(x)\right)dx \\ &= \frac{1}{2}\left(u^2(1) - u^2(0)\right) \\ &\quad - b_0\int_0^1 u^2(x)\,dx = -b_0\int_0^1 u^2(x)\,dx.\end{aligned} \tag{42}$$

Let $v = v(t,x)$ be a solution of (2) with A defined by (41). Then, we obtain

$$\begin{aligned}\frac{d}{dt}(v,v) &= (\dot v, v) + (v, \dot v) = (Av,v) + (v,Av) = 2(Av,v) \\ &\le -2b_0(v,v).\end{aligned} \tag{43}$$

Hence, $(d/dt)\|v\| \le -b_0\|v\|$. Thus, $\|T(t)\| \le e^{-b_0t}$. In addition, $C_0u(x) = a(x)u(x)$,

$$\begin{aligned}(AC_0 - C_0A)u(x) &= \frac{d(a(x)u(x))}{dx} - a(x)\frac{du(x)}{dx} \\ &= a'(x)u(x), \quad (u \in \operatorname{Dom}(A)).\end{aligned} \tag{44}$$

Due to condition (34), we obtain the following.

Corollary 7. *If the inequality*

$$\theta_0\left(|a(x)| + \frac{1}{b_0}\left(|a'(x)| + |a(x)|^2\right)\right) < 1, \quad (0 \le x \le 1) \tag{45}$$

holds, then (37) *is exponentially stable.*

For example, let $c(t) = \sin(\omega t)(\omega > 0)$. Then $i_c(t) \le 2/\omega$ and (45) takes the form

$$|a(x)| + \frac{1}{b_0}\left(|a'(x)| + |a(x)|^2\right) < \frac{\omega}{2}, \quad (0 \le x \le 1). \tag{46}$$

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