Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 938910, 12 pages
doi:10.1155/2012/938910

*Research Article*

# Generalizations of $\mathcal{N}$-Subalgebras in BCK/BCI-Algebras Based on Point $\mathcal{N}$-Structures

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Received 13 June 2012; Accepted 31 July 2012

Academic Editor: Nicola Mastronardi



The aim of this article is to obtain more general forms than the papers of (Jun et al. (2010); Jun et al. (in press)). The notions of $\mathcal{N}$-subalgebras of types $(\in, q_k)$, $(\in, \in \vee q_k)$, and $(q, \in \vee q_k)$ are introduced, and the concepts of $q_k$-support and $\in \vee q_k$-support are also introduced. Several related properties are investigated. Characterizations of $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ are discussed, and conditions for an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ to be an $\mathcal{N}$-subalgebra of type $(\in, \in)$ are considered.

## 1. Introduction

A (crisp) set $A$ in a universe $X$ can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set $A$ and the value 0 for elements excluded from the set $A$. So far most of the generalizations of the crisp set have been conducted on the unit interval $[0, 1]$, and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fits the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [1] introduced a new function which is called negative-valued function and constructed $\mathcal{N}$-structures. They applied $\mathcal{N}$-structures to BCK/BCI-algebras and discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. Jun et al. [2] considered closed ideals in BCH-algebras based on $\mathcal{N}$-structures. To obtain more general form of an $\mathcal{N}$-subalgebra in BCK/BCI-algebras, Jun et al. [3] defined the notions of $\mathcal{N}$-subalgebras of types $(\in, \in)$, $(\in, q)$, $(\in, \in \vee q)$, $(q, \in)$, $(q, q)$, and $(q, \in \vee q)$ and investigated related properties. They also gave conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-subalgebra of type $(q, \in \vee q)$. Jun et al. provided a characterization of an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q)$ (see [3, 4]).

In this paper, we try to have more general form of the papers [3, 4]. We introduce the notions of $\mathcal{N}$-subalgebras of types $(\in, q_k)$, $(\in, \in \vee q_k)$, and $(q, \in \vee q_k)$. We also introduce the concepts of $q_k$-support and $\in \vee q_k$-support and investigate several properties. We discuss characterizations of $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$. We consider conditions for an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ to be an $\mathcal{N}$-subalgebra of type $(\in, \in)$. The important achievement of the study of $\mathcal{N}$-subalgebras of types $(\in, q_k)$, $(\in, \in \vee q_k)$, and $(q, \in \vee q_k)$ is that the notions of $\mathcal{N}$-subalgebras of types $(\in, q)$, $(\in, \in \vee q)$, and $(q, \in \vee q)$ are a special case of $\mathcal{N}$-subalgebras of types $(\in, q_k)$, $(\in, \in \vee q_k)$, and $(q, \in \vee q_k)$, and thus so many results in the papers [3, 4] are corollaries of our results obtained in this paper.

## 2. Preliminaries

Let $K(\tau)$ be the class of all algebras with type $\tau = (2, 0)$. By a BCI-algebra, we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(i) $((x * y) * (x * z)) * (z * y) = 0$,

(ii) $(x * (x * y)) * y = 0$,

(iii) $x * x = 0$,

(iv) $x * y = y * x = 0 \Rightarrow x = y$,

for all $x, y, z \in X$. If a BCI-algebra $X$ satisfies $0 * x = 0$ for all $x \in X$, then we say that $X$ is a *BCK*-algebra. We can define a partial ordering $\leq$ by

$$(\forall x, y \in X) \quad (x \leq y \Longleftrightarrow x * y = 0). \tag{2.1}$$

In a BCK/BCI-algebra $X$, the following hold:

(a1) $(\forall x \in X)(x * 0 = x)$,

(a2) $(\forall x, y, z \in X)((x * y) * z = (x * z) * y)$,

for all $x, y, z \in X$.

A nonempty subset $S$ of a BCK/BCI-algebras $X$ is called a subalgebra of $X$ if $x * y \in S$ for all $x, y \in S$. For our convenience, the empty set $\emptyset$ is regarded as a subalgebra of $X$.

We refer the reader to the books [5, 6] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\begin{gathered} \bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite}, \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise}, \end{cases} \\ \bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite}, \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise}. \end{cases} \end{gathered} \tag{2.2}$$

Denote by $\mathcal{F}(X,[-1,0])$ the collection of functions from a set $X$ to $[-1,0]$. We say that an element of $\mathcal{F}(X,[-1,0])$ is a negative-valued function from $X$ to $[-1,0]$ (briefly, $\mathcal{N}$-function on $X$). By an $\mathcal{N}$-structure, we mean an ordered pair $(X,f)$ of $X$ and an $\mathcal{N}$-function $f$ on $X$. In what follows, let $X$ denote a BCK/BCI-algebras and $f$ an $\mathcal{N}$-function on $X$ unless otherwise specified.

*Definition 2.1* (see [1]). By a subalgebra of $X$ based on $\mathcal{N}$-function $f$ (briefly, $\mathcal{N}$-subalgebra of $X$), we mean an $\mathcal{N}$-structure $(X,f)$ in which $f$ satisfies the following assertion:

$$(\forall x,y\in X)\quad \left(f(x*y)\leq \bigvee\{f(x),f(y)\}\right). \tag{2.3}$$

For any $\mathcal{N}$-structure $(X,f)$ and $t\in[-1,0)$, the set

$$C(f;t):=\{x\in X\mid f(x)\leq t\} \tag{2.4}$$

is called a *closed t-support* of $(X,f)$, and the set

$$O(f;t):=\{x\in X\mid f(x)<t\} \tag{2.5}$$

is called an *open t-support* of $(X,f)$.

Using the similar method to the transfer principle in fuzzy theory (see [7, 8]), Jun et al. [2] considered transfer principle in $\mathcal{N}$-structures as follows.

**Theorem 2.2** (see [2]; $\mathcal{N}$-transfer principle). *An $\mathcal{N}$-structure $(X,f)$ satisfies the property $\overline{\mathcal{P}}$ if and only if for all $\alpha\in[-1,0]$,*

$$C(f;\alpha)\neq\emptyset \Longrightarrow C(f;\alpha)\ \text{satisfies the property}\ \mathcal{P}. \tag{2.6}$$

**Lemma 2.3** (see [1]). *An $\mathcal{N}$-structure $(X,f)$ is an $\mathcal{N}$-subalgebra of $X$ if and only if every open t-support of $(X,f)$ is a subalgebra of $X$ for all $t\in[-1,0)$.*

## 3. General Form of $\mathcal{N}$-Subalgebras with Type $(\in,\in\vee q)$

In what follows, let $t$ and $k$ denote arbitrary elements of $[-1,0)$ and $(-1,0]$, respectively, unless otherwise specified.

Let $(X,f)$ be an $\mathcal{N}$-structure in which $f$ is given by

$$f(y)=\begin{cases}0 & \text{if } y\neq x,\\ t & \text{if } y=x.\end{cases} \tag{3.1}$$

In this case, $f$ is denoted by $x_t$, and we call $(X,x_t)$ a point $\mathcal{N}$-structure. For any $\mathcal{N}$-structure $(X,g)$, we say that a point $\mathcal{N}$-structure $(X,x_t)$ is an $\mathcal{N}_\in$-subset (resp., $\mathcal{N}_q$-subset) of $(X,g)$ if $g(x)\leq t$ (resp., $g(x)+t+1<0$). If a point $\mathcal{N}$-structure $(X,x_t)$ is an $\mathcal{N}_\in$-subset of $(X,g)$ or an $\mathcal{N}_q$-subset of $(X,g)$, we say $(X,x_t)$ is an $\mathcal{N}_{\in\vee q}$-subset of $(X,g)$. We say that a point $\mathcal{N}$-structure

$(X, x_t)$ is an $\mathcal{N}_{q_k}$-subset of $(X, g)$ if $g(x) + t - k + 1 < 0$. Clearly, every $\mathcal{N}_{q_k}$-subset with $k = 0$ is an $\mathcal{N}_q$-subset. Note that if $k, r \in (-1, 0]$ with $k < r$, then every $\mathcal{N}_{q_k}$-subset is an $\mathcal{N}_{q_r}$-subset.

*Definition 3.1.* An $\mathcal{N}$-structure $(X, f)$ is called an $\mathcal{N}$-subalgebra of type

(i) $(\in, \in)$ (resp., $(\in, q)$ and $(\in, \in \vee q)$) if whenever two point $\mathcal{N}$-structures $(X, x_{t_1})$ and $(X, y_{t_2})$ are $\mathcal{N}_\in$-subsets of $(X, f)$ then the point $\mathcal{N}$-structure $(X, (x * y)_{\vee\{t_1, t_2\}})$ is an $\mathcal{N}_\in$-subset (resp., $\mathcal{N}_q$-subset and $\mathcal{N}_{\in\vee q}$-subset) of $(X, f)$.

(ii) $(q, \in)$ (resp., $(q, q)$ and $(q, \in \vee q)$) if whenever two point $\mathcal{N}$-structures $(X, x_{t_1})$ and $(X, y_{t_2})$ are $\mathcal{N}_q$-subsets of $(X, f)$ then the point $\mathcal{N}$-structure $(X, (x * y)_{\vee\{t_1, t_2\}})$ is an $\mathcal{N}_\in$-subset (resp., $\mathcal{N}_q$-subset and $\mathcal{N}_{\in\vee q}$-subset) of $(X, f)$.

*Definition 3.2.* An $\mathcal{N}$-structure $(X, f)$ is called an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ (resp., $(q, \in \vee q_k)$) if whenever two point $\mathcal{N}$-structures $(X, x_{t_1})$ and $(X, y_{t_2})$ are $\mathcal{N}_\in$-subsets (resp., $\mathcal{N}_q$-subsets) of $(X, f)$ then the point $\mathcal{N}$-structure $(X, (x * y)_{\vee\{t_1, t_2\}})$ is an $\mathcal{N}_{\in\vee q_k}$-subset of $(X, f)$.

*Example 3.3.* Consider a $BCI$-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

| $*$ | 0 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 0 | 0 | $a$ | $b$ | $c$ |
| $a$ | $a$ | 0 | $c$ | $b$ |
| $b$ | $b$ | $c$ | 0 | $a$ |
| $c$ | $c$ | $b$ | $a$ | 0 |

(3.2)

Let $(X, f)$ be an $\mathcal{N}$-structure in which $f$ is defined by

$$f = \begin{pmatrix} 0 & a & b & c \\ -0.6 & -0.7 & -0.3 & -0.3 \end{pmatrix}. \tag{3.3}$$

It is routine to verify that $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_{-0.2})$.

Note that if $k, r \in (-1, 0]$ with $k < r$, then every $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_r)$, but the converse is not true as seen in the following example.

*Example 3.4.* The $\mathcal{N}$-subalgebra $(X, f)$ of type $(\in, \in \vee q_{-0.2})$ in Example 3.3 is not of type $(\in, \in \vee q_{-0.4})$ since $(X, a_{-0.65})$ and $(X, a_{-0.68})$ are $\mathcal{N}_\in$-subsets of $(X, f)$, but

$$\left(X, (a * a)_{\vee\{-0.65, -0.68\}}\right) \tag{3.4}$$

is not an $\mathcal{N}_{\in\vee q_{-0.4}}$-subset of $(X, f)$.

**Theorem 3.5.** *Every $\mathcal{N}$-subalgebra of type $(\in, \in)$ is of type $(\in, \in \vee q_k)$.*

*Proof.* Straightforward. □

Taking $k = 0$ in Theorem 3.5 induces the following corollary.

**Corollary 3.6.** *Every $\mathcal{N}$-subalgebra of type $(\in, \in)$ is of type $(\in, \in \vee q)$.*

The converse of Theorem 3.5 is not true as seen in the following example.

*Example 3.7.* Consider the $\mathcal{N}$-subalgebra $(X, f)$ of type $(\in, \in \vee q_{-0.2})$ which is given in Example 3.3. Then $(X, f)$ is not an $\mathcal{N}$-subalgebra of type $(\in, \in)$ since $(X, a_{-0.65})$ and $(X, a_{-0.68})$ are $\mathcal{N}_{\in}$-subsets of $(X, f)$, but $(X, (a * a)_{\vee\{-0.65,-0.68\}})$ is not an $\mathcal{N}_{\in}$-subset of $(X, f)$.

*Definition 3.8.* An $\mathcal{N}$-structure $(X, f)$ is called an $\mathcal{N}$-subalgebra of type $(\in, q_k)$ if whenever two point $\mathcal{N}$-structure $(X, x_{t_1})$ and $(X, y_{t_2})$ are $\mathcal{N}_{\in}$-subsets of $(X, f)$ then the point $\mathcal{N}$-structure $(X, (x * y)_{\vee\{t_1,t_2\}})$ is an $\mathcal{N}_{q_k}$-subset of $(X, f)$.

**Theorem 3.9.** *Every $\mathcal{N}$-subalgebra of type $(\in, q_k)$ is of type $(\in, \in \vee q_k)$.*

*Proof.* Straightforward. □

Taking $k = 0$ in Theorem 3.9 induces the following corollary.

**Corollary 3.10.** *Every $\mathcal{N}$-subalgebra of type $(\in, q)$ is of type $(\in, \in \vee q)$.*

The converse of Theorem 3.9 is not true as seen in the following example.

*Example 3.11.* Consider the $\mathcal{N}$-subalgebra $(X, f)$ of type $(\in, \in \vee q_{-0.2})$ which is given in Example 3.3. Then $(X, a_{-0.65})$ and $(X, b_{-0.25})$ are $\mathcal{N}$-subsets of $(X, f)$, but

$$\left(X, (a * b)_{\vee\{-0.65,-0.25\}}\right) = (X, c_{-0.2}) \tag{3.5}$$

is not an $\mathcal{N}_{q_k}$-subset of $(X, f)$ for $k = -0.2$ since $f(c) - 0.25 - 0.2 + 1 > 0$.

We consider a characterization of an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$.

**Theorem 3.12.** *An $\mathcal{N}$-structure $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ if and only if it satisfies*

$$(\forall x, y \in X) \quad \left( f(x * y) \leq \bigvee\left\{ f(x), f(y), \frac{k-1}{2} \right\}\right). \tag{3.6}$$

*Proof.* Let $(X, f)$ be an $\mathcal{N}$-structure of type $(\in, \in \vee q_k)$. Assume that (3.6) is not valid. Then there exists $a, b \in X$ such that

$$f(a * b) > \bigvee\left\{ f(a), f(b), \frac{k-1}{2} \right\}. \tag{3.7}$$

If $\bigvee\{f(a), f(b)\} > (k-1)/2$, then $f(a * b) > \bigvee\{f(a), f(b)\}$. Hence

$$f(a * b) > t \geq \bigvee\{f(a), f(b)\} \tag{3.8}$$

for some $t \in [-1, 0)$. It follows that point $\mathcal{N}$-structures $(X, a_t)$ and $(X, b_t)$ are $\mathcal{N}_{\in}$-subsets of $(X, f)$, but the point $\mathcal{N}$-structure $(X, (a * b)_t)$ is not an $\mathcal{N}_{\in}$-subset of $(X, f)$. Moreover,

$$f(a * b) + t - k + 1 > 2t - k + 1 = 0, \tag{3.9}$$

and so $(X, (a * b)_t)$ is not an $\mathcal{N}_{q_k}$-subset of $(X, f)$. Consequently, $(X, (a * b)_t)$ is not an $\mathcal{N}_{\in \vee q_k}$-subset of $(X, f)$. This is a contradiction. If $\bigvee\{f(a), f(b)\} \leq (k-1)/2$, then $f(a) \leq (k-1)/2$, $f(b) \leq (k-1)/2$ and $f(a * b) > (k-1)/2$. Thus $(X, a_{(k-1)/2})$ and $(X, b_{(k-1)/2})$ are $\mathcal{N}_{\in}$-subsets of $(X, f)$, but $(X, (a * b)_{(k-1)/2})$ is not an $\mathcal{N}_{\in}$-subset of $(X, f)$. Also,

$$f(a * b) + \frac{k-1}{2} - k + 1 > \frac{k-1}{2} + \frac{k-1}{2} - k + 1 = 0, \tag{3.10}$$

that is, $(X, (a * b)_{(k-1)/2})$ is not an $\mathcal{N}_{q_k}$-subset of $(X, f)$. Hence $(X, (a * b)_{(k-1)/2})$ is not an $\mathcal{N}_{\in \vee q_k}$-subset of $(X, f)$, a contradiction. Therefore (3.6) is valid.

Conversely, suppose that (3.6) is valid. Let $x, y \in X$ and $t_1, t_2 \in [-1, 0)$ be such that two point $\mathcal{N}$-structures $(X, x_{t_1})$ and $(X, y_{t_2})$ are $\mathcal{N}_{\in}$-subsets of $(X, f)$. Then

$$f(x * y) \leq \bigvee\left\{f(x), f(y), \frac{k-1}{2}\right\} \leq \bigvee\left\{t_1, t_2, \frac{k-1}{2}\right\}. \tag{3.11}$$

Assume that $t_1 \geq (k-1)/2$ or $t_2 \geq (k-1)/2$. Then $f(x * y) \leq \bigvee\{t_1, t_2\}$, and so $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_{\in}$-subset of $(X, f)$. Now suppose that $t_1 < (k-1)/2$ and $t_2 < (k-1)/2$. Then $f(x * y) \leq (k-1)/2$, and thus

$$f(x * y) + \bigvee\{t_1, t_2\} - k + 1 < \frac{k-1}{2} + \frac{k-1}{2} - k + 1 = 0, \tag{3.12}$$

that is, $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_{q_k}$-subset of $(X, f)$. Therefore $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_{\in \vee q_k}$-subset of $(X, f)$ and consequently $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$. □

**Corollary 3.13** (see [3]). *An $\mathcal{N}$-structure $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q)$ if and only if it satisfies*

$$(\forall x, y \in X) \quad \left(f(x * y) \leq \bigvee\{f(x), f(y), -0.5\}\right). \tag{3.13}$$

*Proof.* It follows from taking $k = 0$ in Theorem 3.12. □

We provide conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-subalgebra of type $(q, \in \vee q_k)$.

**Theorem 3.14.** *Let $S$ be a subalgebra of $X$ and let $(X, f)$ be an $\mathcal{N}$-structure such that*

(a) $(\forall x \in X)(x \in S \Rightarrow f(x) \leq (k-1)/2)$,

(b) $(\forall x \in X)(x \notin S \Rightarrow f(x) = 0)$.

*Then $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(q, \in \vee q_k)$.*

*Proof.* Let $x, y \in X$ and $t_1, t_2 \in [-1, 0)$ be such that two point $\mathcal{N}$-structures $(X, x_{t_1})$ and $(X, y_{t_2})$ are $\mathcal{N}_q$-subsets of $(X, f)$. Then $f(x) + t_1 + 1 < 0$ and $f(y) + t_2 + 1 < 0$. Thus $x * y \in S$ because if it is impossible, then $x \notin S$ or $y \notin S$. Thus $f(x) = 0$ or $f(y) = 0$, and so $t_1 < -1$ or $t_2 < -1$. This is a contradiction. Hence $f(x * y) \leq (k-1)/2$. If $\bigvee\{t_1, t_2\} < (k-1)/2$, then $f(x * y) + \bigvee\{t_1, t_2\} - k + 1 < ((k-1)/2) + ((k-1)/2) - k + 1 = 0$ and so the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_{q_k}$-subset of $(X, f)$. If $\bigvee\{t_1, t_2\} \geq (k-1)/2$, then $f(x * y) \leq (k-1)/2 \leq \bigvee\{t_1, t_2\}$ and so the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_{\in}$-subset of $(X, f)$. Therefore the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_{\in \vee q_k}$-subset of $(X, f)$. This shows that $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(q, \in \vee q_k)$. □

Taking $k = 0$ in Theorem 3.14, we have the following corollary.

**Corollary 3.15** (see [3]). *Let $S$ be a subalgebra of $X$ and let $(X, f)$ be an $\mathcal{N}$-structure such that*

(a) $(\forall x \in X)(x \in S \Rightarrow f(x) \leq -0.5)$,

(b) $(\forall x \in X)(x \notin S \Rightarrow f(x) = 0)$.

*Then $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(q, \in \vee_q)$.*

**Theorem 3.16.** *Let $(X, f)$ be an $\mathcal{N}$-subalgebra of type $(q_k, \in \vee q_k)$. If $f$ is not constant on the open 0-support of $(X, f)$, then $f(x) \leq (k-1)/2$ for some $x \in X$. In particular, $f(0) \leq (k-1)/2$.*

*Proof.* Assume that $f(x) > (k-1)/2$ for all $x \in X$. Since $f$ is not constant on the open 0-support of $(X, f)$, there exists $x \in O(f; 0)$ such that $t_x = f(x) \neq f(0) = t_0$. Then either $t_0 < t_x$ or $t_0 > t_x$. For the case $t_0 < t_x$, choose $r < (k-1)/2$ such that $t_0 + r - k + 1 < 0 < t_x + r - k + 1$. Then the point $\mathcal{N}$-structure $(X, 0_r)$ is an $\mathcal{N}_{q_k}$-subset of $(X, f)$. Since $(X, x_{-1})$ is an $\mathcal{N}_{q_k}$-subset of $(X, f)$. It follows from (a1) that the point $\mathcal{N}$-structure $(X, (x * 0)_{\bigvee\{r, -1\}}) = (X, x_r)$ is an $\mathcal{N}_{\in \vee q_k}$-subset of $(X, f)$. But, $f(x) > (k-1)/2 > r$ implies that the point $\mathcal{N}$-structure $(X, x_r)$ is not an $\mathcal{N}_{\in}$-subset of $(X, f)$. Also, $f(x) + r - k + 1 = t_x + r - k + 1 > 0$ implies that the point $\mathcal{N}$-structure $(X, x_r)$ is not an $\mathcal{N}_{q_k}$-subset of $(X, f)$. This is a contradiction. Assume that $t_0 > t_x$ and take $r < (k-1)/2$ such that $t_x + r - k + 1 < 0 < t_0 + r - k + 1$. Then $(X, x_r)$ is an $\mathcal{N}_{q_k}$-subset of $(X, f)$. Since

$$f(x * x) = f(0) = t_0 > -r + k - 1 > -\frac{k-1}{2} + k - 1 = \frac{k-1}{2} > r, \tag{3.14}$$

$(X, (x * x)_{\bigvee\{r, r\}})$ is not an $\mathcal{N}_{\in}$-subset of $(X, f)$. Since

$$f(x * x) + \bigvee\{r, r\} - k + 1 = f(0) + r - k + 1 = t_0 + r - k + 1 > 0, \tag{3.15}$$

$(X, (x * x)_{\bigvee\{r, r\}})$ is not an $\mathcal{N}_{q_k}$-subset of $(X, f)$. Hence $(X, (x * x)_{\bigvee\{r, r\}})$ is not an $\mathcal{N}_{\in \vee q_k}$-subset of $(X, f)$, which is a contradiction. Therefore $f(x) \leq (k-1)/2$ for some $x \in X$. We now prove that $f(0) \leq (k-1)/2$. Assume that $f(0) = t_0 > (k-1)/2$. Note that there exists $x \in X$ such that $f(x) = t_x \leq (k-1)/2$ and so $t_x < t_0$. Choose $t_1 < t_0$ such that $t_x + t_1 - k + 1 < 0 < t_0 + t_1 - k + 1$. Then $f(x) + t_1 - k + 1 = t_x + t_1 - k + 1 < 0$, and thus the point $\mathcal{N}$-structure $(X, x_{t_1})$ is an $\mathcal{N}_{q_k}$-subset of $(X, f)$. Now we have

$$f(x * x) + \bigvee\{t_1, t_1\} - k + 1 = f(0) + t_1 - k + 1 = t_0 + t_1 - k + 1 > 0 \tag{3.16}$$

and $f(x*x) = f(0) = t_0 > t_1 = \bigvee\{t_1, t_1\}$. Hence $(X, (x*x)_{\bigvee\{t_1,t_1\}})$ is not an $\mathcal{N}_{\in\vee q_k}$-subset of $(X, f)$. This is a contradiction, and therefore $f(0) \leq (k-1)/2$. □

**Corollary 3.17** (see [3]). *Let $(X, f)$ be an $\mathcal{N}$-subalgebra of type $(q, \in\vee_q)$. If $f$ is not constant on the open 0-support of $(X, f)$, then $f(x) \leq -0.5$ for some $x \in X$. In particular, $f(0) \leq -0.5$.*

**Theorem 3.18.** *An $\mathcal{N}$-structure $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q_k)$ if and only if for every $t \in [(k-1)/2, 0]$ the nonempty closed $t$-support of $(X, f)$ is a subalgebra of $X$.*

*Proof.* Assume that $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q_k)$ and let $t \in [(k-1)/2, 0]$ be such that $C(f;t) \neq \emptyset$. Let $x, y \in C(f;t)$. Then $f(x) \leq t$ and $f(y) \leq t$. It follows from Theorem 3.12 that

$$f(x*y) \leq \bigvee\left\{f(x), f(y), \frac{k-1}{2}\right\} \leq \bigvee\left\{t, \frac{k-1}{2}\right\} = t \tag{3.17}$$

so that $x*y \in C(f;t)$. Therefore $C(f;t)$ is a subalgebra of $X$.

Conversely, let $(X, f)$ be an $\mathcal{N}$-structure such that the nonempty closed $t$-support of $(X, f)$ is a subalgebra of $X$ for all $t \in [(k-1)/2, 0]$. If there exist $a, b \in X$ such that $f(a*b) > \bigvee\{f(a), f(b), (k-1)/2\}$, then we can take $s \in [-1, 0]$ such that

$$f(a*b) > s \geq \bigvee\left\{f(a), f(b), \frac{k-1}{2}\right\}. \tag{3.18}$$

Thus $a, b \in C(f;s)$ and $s \geq (k-1)/2$. Since $C(f, s)$ is a subalgebra of $X$, it follows that $a*b \in C(f;s)$ so that $f(a*b) \leq s$. This is a contradiction, and therefore $f(x*y) \leq \bigvee\{f(x), f(y), (k-1)/2\}$ for all $x, y \in X$. Using Theorem 3.12, we conclude that $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q_k)$. □

Taking $k = 0$ in Theorem 3.18, we have the following corollary.

**Corollary 3.19** (see [4]). *An $\mathcal{N}$-structure $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee_q)$ if and only if for every $t \in [-0.5, 0]$ the nonempty closed $t$-support of $(X, f)$ is a subalgebra of $X$.*

**Theorem 3.20.** *Let $S$ be a subalgebra of $X$. For any $t \in [(k-1)/2, 0)$, there exists an $\mathcal{N}$-subalgebra $(X, f)$ of type $(\in, \in\vee q_k)$ for which $S$ is represented by the closed $t$-support of $(X, f)$.*

*Proof.* Let $(X, f)$ be an $\mathcal{N}$-structure in which $f$ is given by

$$f(x) = \begin{cases} t & \text{if } x \in S, \\ 0 & \text{if } x \notin S, \end{cases} \tag{3.19}$$

for all $x \in X$ where $t \in [(k-1)/2, 0)$. Assume that $f(a*b) > \bigvee\{f(a), f(b), (k-1)/2\}$ for some $a, b \in X$. Since the cardinality of the image of $f$ is 2, we have $f(a*b) = 0$ and $\bigvee\{f(a), f(b), (k-1)/2\} = t$. Since $t \geq (k-1)/2$, it follows that $f(a) = t = f(b)$ so that $a, b \in S$. Since $S$ is a subalgebra of $X$, we obtain $a*b \in S$ and so $f(a*b) = t < 0$. This is a contradiction. Therefore $f(x*y) \leq \bigvee\{f(x), f(y), (k-1)/2\}$ for all $x, y \in X$. Using Theorem 3.12, we conclude that

$(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$. Obviously, $S$ is represented by the closed $t$-support of $(X, f)$. □

**Corollary 3.21** (see [4]). *Let $S$ be a subalgebra of $X$. For any $t \in [-0.5, 0)$, there exists an $\mathcal{N}$-subalgebra $(X, f)$ of type $(\in, \in \vee_q)$ for which $S$ is represented by the closed $t$-support of $(X, f)$.*

*Proof.* It follows from taking $k = 0$ in Theorem 3.20. □

Note that every $\mathcal{N}$-subalgebra of type $(\in, \in)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$, but the converse is not true in general (see Example 3.7). Now, we give a condition for an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ to be an $\mathcal{N}$-subalgebra of type $(\in, \in)$.

**Theorem 3.22.** *Let $(X, f)$ be an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ such that $f(x) > (k-1)/2$ for all $x \in X$. Then $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in)$.*

*Proof.* Let $x, y \in X$ and $t \in [-1, 0)$ be such that $(X, x_{t_1})$ and $(X, y_{t_2})$ are $\mathcal{N}_\in$-subsets of $(X, f)$. Then $f(x) \le t_1$ and $f(y) \le t_2$. It follows from Theorem 3.12 and the hypothesis that

$$f(x * y) \le \bigvee\left\{f(x), f(y), \frac{k-1}{2}\right\} = \bigvee\{f(x), f(y)\} \le \bigvee\{t_1, t_2\} \tag{3.20}$$

so that $(X, (x*y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_\in$-subset of $(X, f)$. Therefore $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in)$. □

**Corollary 3.23** (see [4]). *Let $(X, f)$ be an $\mathcal{N}$-structure of type $(\in, \in \vee_q)$ such that $f(x) > -0.5$ for all $x \in X$. Then $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in)$.*

*Proof.* It follows from taking $k = 0$ in Theorem 3.22. □

**Theorem 3.24.** *Let $\{(X, f_i) \mid i \in \Lambda\}$ be a family of $\mathcal{N}$-subalgebras of type $(\in, \in \vee q_k)$. Then $(X, \bigcup_{i\in\Lambda} f_i)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$, where $\bigcup_{i\in\Lambda} f_i$ is an $\mathcal{N}$-function on $X$ given by $(\bigcup_{i\in\Lambda} f_i)(x) = \bigvee_{i\in\Lambda} f_i(x)$ for all $x \in X$.*

*Proof.* Let $x, y \in X$ and $t_1, t_2 \in [-1, 0)$ be such that $(X, x_{t_1})$ and $(X, y_{t_2})$ are $\mathcal{N}_\in$-subsets of $(X, \bigcup_{i\in\Lambda} f_i)$. Assume that $(X, (x*y)_{\bigvee\{t_1,t_2\}})$ is not an $\mathcal{N}_{\in\vee q_k}$-subset of $(X, \bigcup_{i\in\Lambda} f_i)$. Then $(X, (x*y)_{\bigvee\{t_1,t_2\}})$ is neither an $\mathcal{N}_\in$-subset nor an $\mathcal{N}_{q_k}$-subset of $(X, \bigcup_{i\in\Lambda} f_i)$. Hence $(\bigcup_{i\in\Lambda} f_i)(x*y) > \bigvee\{t_1, t_2\}$ and

$$\left(\bigcup_{i\in\Lambda} f_i\right)(x*y) + \bigvee\{t_1, t_2\} - k + 1 \ge 0, \tag{3.21}$$

which imply that

$$\left(\bigcup_{i\in\Lambda} f_i\right)(x*y) > \frac{k-1}{2}. \tag{3.22}$$

Let $A_1 := \{i \in \Lambda \mid (X, (x*y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_\in$-subset of $(X, f_i)\}$ and $A_2 := \{i \in \Lambda \mid (X, (x*y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_{q_k}$-subset of $(X, f_i)\} \cap \{j \in \Lambda \mid (X, (x*y)_{\bigvee\{t_1,t_2\}})$ is not an $\mathcal{N}_\in$-subset of $(X, f_j)\}$. Then $\Lambda = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. If $A_2 = \emptyset$, then $(X, (x*y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_\in$-subset of $(X, f_i)$ for all $i \in \Lambda$, that is, $f_i(x*y) \le \bigvee\{t_1, t_2\}$ for all $i \in \Lambda$. Thus $(\bigcup_{i\in\Lambda} f_i)(x*y) \le \bigvee\{t_1, t_2\}$. This is a contradiction. Hence $A_2 \ne \emptyset$, and so for every $i \in A_2$, we have $f_i(x*y) > \bigvee\{t_1, t_2\}$ and $f_i(x*y) + \bigvee\{t_1, t_2\} - k + 1 < 0$. It follows that $\bigvee\{t_1, t_2\} < (k-1)/2$. Since $(X, x_{t_1})$ is an $\mathcal{N}_\in$-subset of $(X, \bigcup_{i\in\Lambda} f_i)$, we have

$$f_i(x) \le \left(\bigcup_{i\in\Lambda} f_i\right)(x) \le t_1 \le \bigvee\{t_1, t_2\} < \frac{k-1}{2} \tag{3.23}$$

for all $i \in \Lambda$. Similarly, $f_i(y) < (k-1)/2$ for all $i \in \Lambda$. Next suppose that $t := f_i(x*y) > (k-1)/2$. Taking $(k-1)/2 < r < t$, we know that $(X, x_r)$ and $(X, y_r)$ are $\mathcal{N}_\in$-subsets of $(X, f_i)$, but $(X, (x*y)_{\bigvee\{r,r\}}) = (X, (x*y)_r)$ is not an $\mathcal{N}_{\in\vee q_k}$-subset of $(X, f_i)$. This contradicts that $(X, f_i)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q_k)$. Hence $f_i(x*y) \le (k-1)/2$ for all $i \in \Lambda$, and so $(\bigcup_{i\in\Lambda} f_i)(x*y) \le (k-1)/2$ which contradicts (3.22). Therefore $(X, (x*y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_{\in\vee q_k}$-subset of $(X, \bigcup_{i\in\Lambda} f_i)$ and consequently $(X, \bigcup_{i\in\Lambda} f_i)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q_k)$. □

For any $\mathcal{N}$-structure $(X, f)$ and $t \in [-1, 0)$, the $q$-support and the $\in\vee_q$-support of $(X, f)$ related to $t$ are defined to be the sets (see [4])

$$\mathcal{N}_q(f;t) := \{x \in X \mid (X, x_t) \text{ is an } \mathcal{N}_q\text{-subset of } (X, f)\}, \tag{3.24}$$

$$\mathcal{N}_{\in\vee q}(f;t) := \{x \in X \mid (X, x_t) \text{ is an } \mathcal{N}_{\in\vee q}\text{-subset of } (X, f)\}, \tag{3.25}$$

respectively. Note that the $\in\vee q$-support is the union of the closed support and the $q$-support, that is,

$$\mathcal{N}_{\in\vee q}(f;t) = C(f;t) \cup \mathcal{N}_q(f;t), \quad t \in [-1, 0). \tag{3.26}$$

The $q_k$-support and the $\in\vee q_k$-support of $(X, f)$ related to $t$ are defined to be the sets

$$\mathcal{N}_{q_k}(f;t) := \{x \in X \mid (X, x_t) \text{ is an } \mathcal{N}_{q_k}\text{-subset of } (X, f)\}, \tag{3.27}$$

$$\mathcal{N}_{\in\vee q_k}(f;t) := \{x \in X \mid (X, x_t) \text{ is an } \mathcal{N}_{\in\vee q_k}\text{-subset of } (X, f)\}, \tag{3.28}$$

respectively. Clearly, $\mathcal{N}_{\in\vee q_k}(f;t) = C(f;t) \cup \mathcal{N}_{q_k}(f;t)$ for all $t \in [-1, 0)$.

**Theorem 3.25.** *An $\mathcal{N}$-structure $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q_k)$ if and only if the $\in\vee q_k$-support of $(X, f)$ related to $t$ is a subalgebra of $X$ for all $t \in [-1, 0)$.*

*Proof.* Suppose that $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q_k)$. Let $x, y \in \mathcal{N}_{\in\vee q_k}(f;t)$ for $t \in [-1, 0)$. Then $(X, x_t)$ and $(X, y_t)$ are $\mathcal{N}_{\in\vee q_k}$-subsets of $(X, f)$. Hence $f(x) \le t$ or $f(x) + t - k + 1 < 0$, and $f(y) \le t$ or $f(y) + t - k + 1 < 0$. Then we consider the following four cases:

(c1) $f(x) \le t$ and $f(y) \le t$,

(c2) $f(x) \le t$ and $f(y) + t - k + 1 < 0$,

(c3) $f(x) + t - k + 1 < 0$ and $f(y) \le t$,

(c4) $f(x) + t - k + 1 < 0$ and $f(y) + t - k + 1 < 0$.

Combining (3.6) and (c1), we have $f(x*y) \leq \bigvee\{t,(k-1)/2\}$. If $t \geq (k-1)/2$, then $f(x*y) \leq t$ and so $(X,(x*y)_t)$ is an $\mathcal{N}_{\in}$-subset of $(X,f)$. Hence $x*y \in C(f;t) \subseteq \mathcal{N}_{\in\vee q_k}(f;t)$. If $t < (k-1)/2$, then $f(x* \ y) \leq (k-1)/2$ and so $f(x* \ y)+t-k+1 < ((k-1)/2)+((k-1)/2)-k+1 = 0$, that is, $(X,(x*y)_t)$ is an $\mathcal{N}_{q_k}$-subset of $(X,f)$. Therefore $x*y \in \mathcal{N}_{q_k}(f;t) \subseteq \mathcal{N}_{\in\vee q_k}(f;t)$. For the case (c2), assume that $t < (k-1)/2$. Then

$$\begin{aligned} f(x*y) &\leq \bigvee\left\{f(x),f(y),\frac{k-1}{2}\right\} \\ &\leq \bigvee\left\{t,f(y),\frac{k-1}{2}\right\} = \bigvee\left\{f(y),\frac{k-1}{2}\right\} \\ &= \begin{cases} f(y) & \text{if } f(y) > \dfrac{k-1}{2}, \\ \dfrac{k-1}{2} & \text{if } f(y) \leq \dfrac{k-1}{2}, \end{cases} \\ &< k-1-t, \end{aligned} \tag{3.29}$$

and so $f(x*y)+t-k+1 < 0$. Thus $(X,(x*y)_t)$ is an $\mathcal{N}_{q_k}$-subset of $(X,f)$. If $t \geq (k-1)/2$, then

$$\begin{aligned} f(x*y) &\leq \bigvee\left\{f(x),f(y),\frac{k-1}{2}\right\} \\ &\leq \bigvee\left\{t,f(y),\frac{k-1}{2}\right\} = \bigvee\{t,f(y)\} \\ &= \begin{cases} f(y) & \text{if } f(y) > t, \\ t & \text{if } f(y) \leq t, \end{cases} \end{aligned} \tag{3.30}$$

and thus $x*y \in \mathcal{N}_{q_k}(f;t)$ or $x*y \in C(f;t)$. Consequently, $x*y \in \mathcal{N}_{\in\vee q_k}(f;t)$. For the case (c3), it is similar to the case (c2). Finally, for the case (c4), if $t \geq (k-1)/2$, then $k-1-t \leq (k-1)/2 \leq t$. Hence

$$f(x*y) \leq \bigvee\left\{f(x),f(y),\frac{k-1}{2}\right\} \leq \bigvee\left\{k-1-t,\frac{k-1}{2}\right\} = \frac{k-1}{2} \leq t, \tag{3.31}$$

which implies that $x*y \in C(f;t)$. If $t < (k-1)/2$, then $t < (k-1)/2 < k-1-t$. Therefore

$$f(x*y) \leq \bigvee\left\{f(x),f(y),\frac{k-1}{2}\right\} \leq \bigvee\left\{k-1-t,\frac{k-1}{2}\right\} = k-1-t, \tag{3.32}$$

that is, $f(x*y)+t-k+1 < 0$, which means that $(X,(x*y)_t)$ is an $\mathcal{N}_{q_k}$-subset of $(X,f)$. Consequently, the $\in\vee q_k$-support of $(X,f)$ related to $t$ is a subalgebra of $X$ for all $t \in [-1,0)$.

Conversely, let $(X, f)$ be an $\mathcal{N}$-structure for which the $\in \vee q_k$-support of $(X, f)$ related to $t$ is a subalgebra of $X$ for all $t \in [-1, 0)$. Assume that there exist $a, b \in X$ such that $f(a * b) > \bigvee\{f(a), f(b), (k-1)/2\}$. Then

$$f(a * b) > s \geq \bigvee\left\{f(a), f(b), \frac{k-1}{2}\right\} \tag{3.33}$$

for some $s \in [(k-1)/2, 0)$. It follows that $a, b \in C(f; s) \subseteq \mathcal{N}_{\in \vee q_k}(f; s)$ but $a * b \notin C(f; s)$. Also, $f(a * b) + s - k + 1 > 2s - k + 1 \geq 0$, that is, $a * b \notin \mathcal{N}_{q_k}(f; s)$. Thus $a * b \notin \mathcal{N}_{\in \vee q_k}(f; s)$ which is a contradiction. Therefore

$$f(x * y) \leq \bigvee\left\{f(x), f(y), \frac{k-1}{2}\right\} \tag{3.34}$$

for all $x, y \in X$. Using Theorem 3.12, we conclude that $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$. □

If we take $k = 0$ in Theorem 3.25, we have the following corollary.

**Corollary 3.26** (see [4]). *An $\mathcal{N}$-structure $(X, f)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q)$ if and only if the $\in \vee q$-support of $(X, f)$ related to $t$ is a subalgebra of $X$ for all $t \in [-1, 0)$.*

## References

[1] Y. B. Jun, K. J. Lee, and S. Z. Song, "$\mathcal{N}$-ideals of BCK/BCI-algebras," *The Journal of Chungcheong Mathematical Society*, vol. 22, pp. 417–437, 2009.

[2] Y. B. Jun, M. A. Öztürk, and E. H. Roh, "$\mathcal{N}$-structures applied to closed ideals in BCH-algebras," *International Journal of Mathematics and Mathematical Sciences*, vol. 2010, Article ID 943565, 9 pages, 2010.

[3] Y. B. Jun, M. S. Kang, and C. H. Park, "$\mathcal{N}$-subalgebras in BCK/BCI-algebras based on point $\mathcal{N}$-structures," *International Journal of Mathematics and Mathematical Sciences*, vol. 2009, Article ID 303412, 9 pages, 2010.

[4] Y. B. Jun, K. J. Lee, and X. H. Zhang, "$\mathcal{N}$-subalgerbas of type($\in, \in$ Vq) based on point $\mathcal{N}$-structures in BCK/BCI-algebras," *Communications of the Korean Mathematical Society*. In press.

[5] Y. S. Huang, *BCI-Algebra*, Science Press, Beijing, China, 2006.

[6] J. Meng and Y. B. Jun, *BCK-Algebras*, Kyungmoon Sa, Seoul, South Korea, 1994.

[7] Y. B. Jun and M. Kondo, "On transfer principle of fuzzy BCK/BCI-algebras," *Scientiae Mathematicae Japonicae*, vol. 59, no. 1, pp. 35–40, 2004.

[8] M. Kondo and W. A. Dudek, "On the transfer principle in fuzzy theory," *Mathware & Soft Computing*, vol. 12, no. 1, pp. 41–55, 2005.


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