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Research Article

Existence of Solutions for Nonlinear Impulsive Fractional Differential Equations of Order $\alpha \in (2, 3]$ with Nonlocal Boundary Conditions

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We investigate the existence and uniqueness of solutions to the nonlocal boundary value problem for nonlinear impulsive fractional differential equations of order $\alpha \in (2, 3]$. By using some well-known fixed point theorems, sufficient conditions for the existence of solutions are established. Some examples are presented to illustrate the main results.

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines such as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, control theory, signal and image processing, biophysics, electrodynamics of complex medium, polymer rheology, and fitting of experimental data [1–6]. For example, one could mention the problem of anomalous diffusion [7–9], the nonlinear oscillation of earthquake can be modeled with fractional derivative [10], and fluid-dynamic traffic model with fractional derivatives [11] can eliminate the deficiency arising from the assumption to continuum traffic flow and many other [12, 13] recent developments in the description of anomalous transport by fractional dynamics. For some recent development on nonlinear fractional differential equations, see [14–29] and the references therein.

In this paper, we investigate a three-point boundary value problems for nonlinear impulsive fractional differential equations of order $\alpha \in (2, 3]$:

$$^{C}D^{\alpha}u(t) = f(t, u(t)), \quad t \in J',$$

$$\Delta u'(t_k) = I_k(u(t_k)), \quad k = 1, 2, \ldots, p,$$

$$\Delta u''(t_k) = I_k^*(u(t_k)), \quad k = 1,2,\dots,p,$$
$$u(0) = u'(0) = 0, \quad \beta u(\eta) = u(1), \tag{1.1}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative, $2 < \alpha \leq 3$, $f \in C(J \times R, R)$, I_k, $I_k^* \in C(R,R)$, $J = [0,1]$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_p < t_{p+1} = 1$, $J' = J \setminus \{t_1, t_2, \dots, t_p\}$, $0 < \eta < 1$, $\eta \neq t_k (k = 1,2,\dots,p)$, $0 < \beta < 1/\eta^2$, $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, where $u'(t_k^+)$ and $u'(t_k^-)$ denote the right and the left limits of $u'(t)$ at $t = t_k$ $(k = 1,2,\dots,p)$, respectively. $\Delta u''(t_k)$ has a similar meaning for $u''(t)$.

Impulsive differential equations arise in many engineering and scientific disciplines as the important mathematical modeling of systems and processes in the fields of biology, physics, engineering, and so forth. Due to their significance, it is important to study the solvability of impulsive differential equations. The theory of impulsive differential equations of integer order has emerged as an important area of investigation. Recently, the impulsive differential equations of fractional order have also attracted a considerable attention, and a variety of results can be found in the papers [30–42].

The study of multipoint boundary-value problems was initiated by Bicadze and Samarskiĭ in [43]. Many authors since then considered nonlinear multipoint boundary-value problems, see [44–51] and the references therein. The multipoint boundary conditions are important in various physical problems of applied science when the controllers at the end points of the interval (under consideration) dissipate or add energy according to the sensors located at intermediate points. For example, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multipoint boundary-value problem.

To our knowledge, no paper has considered nonlinear impulsive fractional differential equations of order $\alpha \in (2,3]$ with nonlocal boundary conditions, that is, problem (1.1). This paper fills this gap in the literature. Our purpose here is to give the existence and uniqueness of solutions for nonlinear impulsive fractional differential equations (1.1). Our results are based on some well-known fixed point theorems.

2. Preliminaries

Let $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], \dots,$ $J_{p-1} = (t_{p-1}, t_p]$, $J_p = (t_p, 1]$.

We introduce the space:

$$PC^2(J,R) = \left\{u : J \longrightarrow R \mid u \in C^2(J_k),\ k = 0,1,\dots,p,\ u'(t_k^+), u''(t_k^+) \text{ exist},\ k = 1,2,\dots,p, \right\}, \tag{2.1}$$

with the norm:

$$\|u\| = \sup_{t \in J} |u(t)|, \qquad \|u\|_{PC^2} = \max\left\{\|u\|, \left\|u'\right\|, \left\|u''\right\|\right\}. \tag{2.2}$$

Obviously, $PC^2(J,R)$ is a Banach space.

Definition 2.1. A function $u \in PC^2(J, R)$ with its Caputo derivative of order α existing on J is a solution of (1.1) if it satisfies (1.1).

Theorem 2.2 (see [52]). *Let E be a Banach space. Assume that $A : E \to E$ is a completely continuous operator, and the set $V = \{u \in E \mid u = \mu Au,\ 0 < \mu < 1\}$ is bounded. Then A has a fixed point in E.*

Theorem 2.3 (see [52]). *Let E be a Banach space. Assume that Ω is an open bounded subset of E with $\theta \in \Omega$, and let $A : \overline{\Omega} \to E$ be a completely continuous operator such that*

$$\|Au\| \leq \|u\|, \quad \forall u \in \partial\Omega. \tag{2.3}$$

Then, A has a fixed point in $\overline{\Omega}$.

Lemma 2.4. *Let $2 < \alpha \leq 3$, $1 \neq \beta\eta^2$, $\eta \in (t_m, t_{m+1})$, m is a nonnegative integer, $0 \leq m \leq p$. For a given $y \in C[0,1]$, a function u is a solution of the following impulsive boundary value problem:*

$$\begin{gathered}
{}^{C}D^{\alpha}u(t) = y(t), \quad t \in J', \\
\Delta u'(t_k) = I_k(u(t_k)), \quad k = 1, 2, \ldots, p, \\
\Delta u''(t_k) = I_k^*(u(t_k)), \quad k = 1, 2, \ldots, p, \\
u(0) = u'(0) = 0, \quad \beta u(\eta) = u(1),
\end{gathered} \tag{2.4}$$

if and only if u is a solution of the impulsive fractional integral equation:

$$u(t) = \begin{cases}
\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} y(s)\,ds + Ct^2, & t \in J_0; \\
\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_k}^t (t-s)^{\alpha-1} y(s)\,ds + \dfrac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} y(s)\,ds & \\
\quad + \displaystyle\sum_{i=1}^{k-1} \dfrac{(t_k - t_i)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} y(s)\,ds & \\
\quad + \displaystyle\sum_{i=1}^{k-1} \dfrac{(t_k - t_i)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} y(s)\,ds & \\
\quad + \displaystyle\sum_{i=1}^{k} \dfrac{(t - t_k)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} y(s)\,ds & \\
\quad + \displaystyle\sum_{i=1}^{k-1} \dfrac{(t - t_k)(t_k - t_i)}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} y(s)\,ds & \\
\quad + \displaystyle\sum_{i=1}^{k} \dfrac{(t - t_k)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} y(s)\,ds + \sum_{i=1}^{k-1} (t_k - t_i) I_i(u(t_i)) & \\
\quad + \displaystyle\sum_{i=1}^{k-1} \dfrac{(t_k - t_i)^2}{2} I_i^*(u(t_i)) + \sum_{i=1}^{k} (t - t_k) I_i(u(t_i)) & \\
\quad + \displaystyle\sum_{i=1}^{k-1} (t - t_k)(t_k - t_i) I_i^*(u(t_i)) & \\
\quad + \displaystyle\sum_{i=1}^{k} \dfrac{(t - t_k)^2}{2} I_i^*(u(t_i)) + Ct^2, & t \in J_k,\ k = 1, 2, \ldots, p,
\end{cases} \tag{2.5}$$

where

$$\begin{aligned}
C = {} & \frac{1}{1-\beta\eta^2} \\
& \times \left\{ \frac{\beta}{\Gamma(\alpha)} \int_{t_m}^{\eta} (\eta - s)^{\alpha-1} y(s) ds + \sum_{i=1}^{m} \frac{\beta}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} y(s) ds \right. \\
& + \sum_{i=1}^{m-1} \frac{\beta(t_m - t_i)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} y(s) ds + \sum_{i=1}^{m-1} \frac{\beta(t_m - t_i)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} y(s) ds \\
& + \sum_{i=1}^{m} \frac{\beta(\eta - t_m)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} y(s) ds + \sum_{i=1}^{m-1} \frac{\beta(\eta - t_m)(t_m - t_i)}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} y(s) ds \\
& + \sum_{i=1}^{m} \frac{\beta(\eta - t_m)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} y(s) ds - \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} y(s) ds \\
& - \sum_{i=1}^{p-1} \frac{(t_p - t_i)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} y(s) ds - \sum_{i=1}^{p-1} \frac{(t_p - t_i)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} y(s) ds \\
& - \sum_{i=1}^{p} \frac{(1 - t_p)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} y(s) ds - \sum_{i=1}^{p-1} \frac{(1 - t_p)(t_p - t_i)}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} y(s) ds \\
& - \sum_{i=1}^{p} \frac{(1 - t_p)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} y(s) ds + \sum_{i=1}^{m-1} \beta(t_m - t_i) I_i(u(t_i)) + \sum_{i=1}^{m-1} \frac{\beta(t_m - t_i)^2}{2} I_i^*(u(t_i)) \\
& + \sum_{i=1}^{m} \beta(\eta - t_m) I_i(u(t_i)) + \sum_{i=1}^{m-1} \beta(\eta - t_m)(t_m - t_i) I_i^*(u(t_i)) + \sum_{i=1}^{m} \frac{\beta(\eta - t_m)^2}{2} I_i^*(u(t_i)) \\
& - \sum_{i=1}^{p-1} (t_p - t_i) I_i(u(t_i)) - \sum_{i=1}^{p-1} \frac{(t_p - t_i)^2}{2} I_i^*(u(t_i)) - \sum_{i=1}^{p} (1 - t_p) I_i(u(t_i)) \\
& \left. - \sum_{i=1}^{p-1} (1 - t_p)(t_p - t_i) I_i^*(u(t_i)) - \sum_{i=1}^{p} \frac{(1 - t_p)^2}{2} I_i^*(u(t_i)) \right\}.
\end{aligned} \tag{2.6}$$

Proof. Let u is a solution of (2.4), it holds

$$u(t) = I^{\alpha} y(t) - c_1 - c_2 t - c_3 t^2 = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds - c_1 - c_2 t - c_3 t^2, \quad t \in J_0, \tag{2.7}$$

for some $c_1, c_2, c_3 \in R$. Then, we have

$$\begin{aligned}
u'(t) &= \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} y(s) ds - c_2 - 2c_3 t, \quad t \in J_0, \\
u''(t) &= \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} y(s) ds - 2c_3, \quad t \in J_0.
\end{aligned} \tag{2.8}$$

In view of $u(0) = u'(0) = 0$, it follows $c_1 = c_2 = 0$.

If $t \in J_1$, then

$$
\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} y(s)ds - d_1 - d_2(t-t_1) - d_3(t-t_1)^2, \\
u'(t) &= \frac{1}{\Gamma(\alpha-1)} \int_{t_1}^{t} (t-s)^{\alpha-2} y(s)ds - d_2 - 2d_3(t-t_1), \\
u''(t) &= \frac{1}{\Gamma(\alpha-2)} \int_{t_1}^{t} (t-s)^{\alpha-3} y(s)ds - 2d_3,
\end{aligned}
\tag{2.9}
$$

for some $d_1, d_2, d_3 \in R$.

Thus, we have

$$
\begin{aligned}
u'(t_1^-) &= \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} y(s)ds - 2c_3 t_1, \qquad u'(t_1^+) = -d_2, \\
u''(t_1^-) &= \frac{1}{\Gamma(\alpha-2)} \int_0^{t_1} (t_1-s)^{\alpha-3} y(s)ds - 2c_3, \qquad u''(t_1^+) = -2d_3.
\end{aligned}
\tag{2.10}
$$

In view of $u(t_1^-) = u(t_1^+)$, $\Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = I_1(u(t_1))$ and $\Delta u''(t_1) = u''(t_1^+) - u''(t_1^-) = I_1^*(u(t_1))$, we have

$$
\begin{aligned}
-d_1 &= \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} y(s)ds - c_3 t_1^2, \\
-d_2 &= \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} y(s)ds - 2c_3 t_1 + I_1(u(t_1)), \\
-2d_3 &= \frac{1}{\Gamma(\alpha-2)} \int_0^{t_1} (t_1-s)^{\alpha-3} y(s)ds - 2c_3 + I_1^*(u(t_1)).
\end{aligned}
\tag{2.11}
$$

Consequently,

$$
\begin{aligned}
u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} y(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} y(s)ds \\
& + \frac{t-t_1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} y(s)ds + \frac{(t-t_1)^2}{2\Gamma(\alpha-2)} \int_0^{t_1} (t_1-s)^{\alpha-3} y(s)ds \\
& + (t-t_1) I_1(u(t_1)) + \frac{1}{2}(t-t_1)^2 I_1^*(u(t_1)) - c_3 t^2, \quad t \in J_1.
\end{aligned}
\tag{2.12}
$$

Similarly, we can get

$$\begin{aligned}
u(t) = {} & \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}y(s)ds + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}y(s)ds \\
& + \sum_{i=1}^{k-1}\frac{(t_k-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}y(s)ds + \sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}y(s)ds \\
& + \sum_{i=1}^{k}\frac{(t-t_k)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}y(s)ds + \sum_{i=1}^{k-1}\frac{(t-t_k)(t_k-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}y(s)ds \\
& + \sum_{i=1}^{k}\frac{(t-t_k)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}y(s)ds + \sum_{i=1}^{k-1}(t_k-t_i)I_i(u(t_i)) \\
& + \sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2}I_i^*(u(t_i)) + \sum_{i=1}^{k}(t-t_k)I_i(u(t_i)) + \sum_{i=1}^{k-1}(t-t_k)(t_k-t_i)I_i^*(u(t_i)) \\
& + \sum_{i=1}^{k}\frac{(t-t_k)^2}{2}I_i^*(u(t_i)) - c_3t^2, \quad t\in J_k,\ k=1,2,\ldots,p.
\end{aligned} \tag{2.13}$$

By (2.13), it follows

$$\begin{aligned}
u(1) = {} & \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}y(s)ds + \sum_{i=1}^{p-1}\frac{(t_p-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}y(s)ds \\
& + \sum_{i=1}^{p-1}\frac{(t_p-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}y(s)ds + \sum_{i=1}^{p}\frac{(1-t_p)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}y(s)ds \\
& + \sum_{i=1}^{p-1}\frac{(1-t_p)(t_p-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}y(s)ds + \sum_{i=1}^{p}\frac{(1-t_p)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}y(s)ds \\
& + \sum_{i=1}^{p-1}(t_p-t_i)I_i(u(t_i)) + \sum_{i=1}^{p-1}\frac{(t_p-t_i)^2}{2}I_i^*(u(t_i)) + \sum_{i=1}^{p}(1-t_p)I_i(u(t_i)) \\
& + \sum_{i=1}^{p-1}(1-t_p)(t_p-t_i)I_i^*(u(t_i)) + \sum_{i=1}^{p}\frac{(1-t_p)^2}{2}I_i^*(u(t_i)) - c_3, \\
u(\eta) = {} & \frac{1}{\Gamma(\alpha)}\int_{t_m}^{\eta}(\eta-s)^{\alpha-1}y(s)ds + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}y(s)ds \\
& + \sum_{i=1}^{m-1}\frac{(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}y(s)ds + \sum_{i=1}^{m-1}\frac{(t_m-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}y(s)ds \\
& + \sum_{i=1}^{m}\frac{(\eta-t_m)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}y(s)ds + \sum_{i=1}^{m-1}\frac{(\eta-t_m)(t_m-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}y(s)ds
\end{aligned}$$

$$+\sum_{i=1}^{m}\frac{(\eta-t_m)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}y(s)ds+\sum_{i=1}^{m-1}(t_m-t_i)I_i(u(t_i))+\sum_{i=1}^{m-1}\frac{(t_m-t_i)^2}{2}I_i^*(u(t_i))$$
$$+\sum_{i=1}^{m}(\eta-t_m)I_i(u(t_i))+\sum_{i=1}^{m-1}(\eta-t_m)(t_m-t_i)I_i^*(u(t_i))+\sum_{i=1}^{m}\frac{(\eta-t_m)^2}{2}I_i^*(u(t_i))-c_3\eta^2. \tag{2.14}$$

In view of the condition $\beta u(\eta)=u(1)$, we have

$$\begin{aligned}
c_3=-\frac{1}{1-\beta\eta^2}\Bigg\{&\frac{\beta}{\Gamma(\alpha)}\int_{t_m}^{\eta}(\eta-s)^{\alpha-1}y(s)ds+\sum_{i=1}^{m}\frac{\beta}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}y(s)ds\\
&+\sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}y(s)ds+\sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}y(s)ds\\
&+\sum_{i=1}^{m}\frac{\beta(\eta-t_m)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}y(s)ds\\
&+\sum_{i=1}^{m-1}\frac{\beta(\eta-t_m)(t_m-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}y(s)ds\\
&+\sum_{i=1}^{m}\frac{\beta(\eta-t_m)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}y(s)ds-\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}y(s)ds\\
&-\sum_{i=1}^{p-1}\frac{(t_p-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}y(s)ds-\sum_{i=1}^{p-1}\frac{(t_p-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}y(s)ds\\
&-\sum_{i=1}^{p}\frac{(1-t_p)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}y(s)ds-\sum_{i=1}^{p-1}\frac{(1-t_p)(t_p-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}y(s)ds\\
&-\sum_{i=1}^{p}\frac{(1-t_p)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}y(s)ds+\sum_{i=1}^{m-1}\beta(t_m-t_i)I_i(u(t_i))\\
&+\sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)^2}{2}I_i^*(u(t_i))\\
&+\sum_{i=1}^{m}\beta(\eta-t_m)I_i(u(t_i))+\sum_{i=1}^{m-1}\beta(\eta-t_m)(t_m-t_i)I_i^*(u(t_i))+\sum_{i=1}^{m}\frac{\beta(\eta-t_m)^2}{2}I_i^*(u(t_i))\\
&-\sum_{i=1}^{p-1}(t_p-t_i)I_i(u(t_i))-\sum_{i=1}^{p-1}\frac{(t_p-t_i)^2}{2}I_i^*(u(t_i))-\sum_{i=1}^{p}(1-t_p)I_i(u(t_i))\\
&-\sum_{i=1}^{p-1}(1-t_p)(t_p-t_i)I_i^*(u(t_i))-\sum_{i=1}^{p}\frac{(1-t_p)^2}{2}I_i^*(u(t_i))\Bigg\}.
\end{aligned} \tag{2.15}$$

Substituting the value of c_i $(i = 1, 2, 3)$ in (2.7) and (2.13) and letting $C = -c_3$, we can get (2.5). Conversely, assume that u is a solution of the impulsive fractional integral equation (2.5), then by a direct computation, it follows that the solution given by (2.5) satisfies (2.4).

This completes the proof. □

3. Main Results

Let $2 < \alpha \leq 3$, $1 \neq \beta\eta^2$, $\eta \in (t_m, t_{m+1})$, m is a nonnegative integer, and $0 \leq m \leq p$. Define the operator $A : C(J) \to C(J)$ as follows:

$$\begin{aligned}
Au(t) = {}& \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s,u(s))ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} f(s,u(s))ds \\
&+ \sum_{i=1}^{k-1} \frac{(t_k - t_i)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f(s,u(s))ds + \sum_{i=1}^{k-1} \frac{(t_k - t_i)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f(s,u(s))ds \\
&+ \sum_{i=1}^{k} \frac{(t - t_k)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f(s,u(s))ds + \sum_{i=1}^{k-1} \frac{(t - t_k)(t_k - t_i)}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f(s,u(s))ds \\
&+ \sum_{i=1}^{k} \frac{(t - t_k)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f(s,u(s))ds + \sum_{i=1}^{k-1} (t_k - t_i) I_i(u(t_i)) + \sum_{i=1}^{k-1} \frac{(t_k - t_i)^2}{2} I_i^*(u(t_i)) \\
&+ \sum_{i=1}^{k} (t - t_k) I_i(u(t_i)) + \sum_{i=1}^{k-1} (t - t_k)(t_k - t_i) I_i^*(u(t_i)) + \sum_{i=1}^{k} \frac{(t - t_k)^2}{2} I_i^*(u(t_i)) + \frac{t^2}{1 - \beta\eta^2} \\
&\times \Bigg\{ \frac{\beta}{\Gamma(\alpha)} \int_{t_m}^{\eta} (\eta - s)^{\alpha-1} f(s,u(s))ds + \sum_{i=1}^{m} \frac{\beta}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} f(s,u(s))ds \\
&\quad + \sum_{i=1}^{m-1} \frac{\beta(t_m - t_i)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f(s,u(s))ds \\
&\quad + \sum_{i=1}^{m-1} \frac{\beta(t_m - t_i)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f(s,u(s))ds \\
&\quad + \sum_{i=1}^{m} \frac{\beta(\eta - t_m)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f(s,u(s))ds \\
&\quad + \sum_{i=1}^{m-1} \frac{\beta(\eta - t_m)(t_m - t_i)}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f(s,u(s))ds \\
&\quad + \sum_{i=1}^{m} \frac{\beta(\eta - t_m)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f(s,u(s))ds - \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} f(s,u(s))ds \\
&\quad - \sum_{i=1}^{p-1} \frac{(t_p - t_i)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f(s,u(s))ds - \sum_{i=1}^{p-1} \frac{(t_p - t_i)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f(s,u(s))ds
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{p}\frac{(1-t_p)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f(s,u(s))ds\\
&-\sum_{i=1}^{p-1}\frac{(1-t_p)(t_p-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f(s,u(s))ds\\
&-\sum_{i=1}^{p}\frac{(1-t_p)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f(s,u(s))ds+\sum_{i=1}^{m-1}\beta(t_m-t_i)I_i(u(t_i))\\
&+\sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)^2}{2}I_i^*(u(t_i))+\sum_{i=1}^{m}\beta(\eta-t_m)I_i(u(t_i))+\sum_{i=1}^{m-1}\beta(\eta-t_m)(t_m-t_i)I_i^*(u(t_i))\\
&+\sum_{i=1}^{m}\frac{\beta(\eta-t_m)^2}{2}I_i^*(u(t_i))-\sum_{i=1}^{p-1}(t_p-t_i)I_i(u(t_i))-\sum_{i=1}^{p-1}\frac{(t_p-t_i)^2}{2}I_i^*(u(t_i))\\
&-\sum_{i=1}^{p}(1-t_p)I_i(u(t_i))-\sum_{i=1}^{p-1}(1-t_p)(t_p-t_i)I_i^*(u(t_i))-\sum_{i=1}^{p}\frac{(1-t_p)^2}{2}I_i^*(u(t_i))\Bigg\},
\end{aligned}
\tag{3.1}
$$

then (1.1) has a solution if and only if the operator A has a fixed point.

Lemma 3.1. *The operator $A : C(J) \to C(J)$ is completely continuous.*

Proof. Obviously, A is continuous in view of continuity of f, I_k, and I_k^*.

Let $\Omega \subset C(J)$ be bounded. Then, there exist positive constants $L_i > 0$ $(i = 1,2,3)$ such that $|f(t,u)| \le L_1$, $|I_k(u)| \le L_2$ and $|I_k^*(u)| \le L_3$, for all $u \in \Omega$. Thus, for all $u \in \Omega$, we have

$$
\begin{aligned}
&|(Au)(t)|\\
&\le\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left|f(s,u(s))\right|ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u(s))\right|ds\\
&\quad+\sum_{i=1}^{k-1}\frac{(t_k-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds+\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds\\
&\quad+\sum_{i=1}^{k}\frac{(t-t_k)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds+\sum_{i=1}^{k-1}\frac{(t-t_k)(t_k-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds\\
&\quad+\sum_{i=1}^{k}\frac{(t-t_k)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds+\sum_{i=1}^{k-1}(t_k-t_i)|I_i(u(t_i))|+\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2}\left|I_i^*(u(t_i))\right|\\
&\quad+\sum_{i=1}^{k}(t-t_k)|I_i(u(t_i))|+\sum_{i=1}^{k-1}(t-t_k)(t_k-t_i)\left|I_i^*(u(t_i))\right|+\sum_{i=1}^{k}\frac{(t-t_k)^2}{2}\left|I_i^*(u(t_i))\right|+\frac{t^2}{1-\beta\eta^2}\\
&\quad\times\left\{\frac{\beta}{\Gamma(\alpha)}\int_{t_m}^{\eta}(\eta-s)^{\alpha-1}\left|f(s,u(s))\right|ds+\sum_{i=1}^{m}\frac{\beta}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u(s))\right|ds\right.\\
&\quad\quad+\sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds\\
&+\sum_{i=1}^{m}\frac{\beta(\eta-t_m)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&+\sum_{i=1}^{m-1}\frac{\beta(\eta-t_m)(t_m-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds\\
&+\sum_{i=1}^{m}\frac{\beta(\eta-t_m)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u(s))\right|ds\\
&+\sum_{i=1}^{p-1}\frac{(t_p-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds+\sum_{i=1}^{p-1}\frac{(t_p-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds\\
&+\sum_{i=1}^{p}\frac{(1-t_p)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&+\sum_{i=1}^{p-1}\frac{(1-t_p)(t_p-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds\\
&+\sum_{i=1}^{p}\frac{(1-t_p)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds+\sum_{i=1}^{m-1}\beta(t_m-t_i)|I_i(u(t_i))|\\
&+\sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)^2}{2}\left|I_i^*(u(t_i))\right|+\sum_{i=1}^{m}\beta(\eta-t_m)|I_i(u(t_i))|+\sum_{i=1}^{m-1}\beta(\eta-t_m)(t_m-t_i)\left|I_i^*(u(t_i))\right|\\
&+\sum_{i=1}^{m}\frac{\beta(\eta-t_m)^2}{2}\left|I_i^*(u(t_i))\right|+\sum_{i=1}^{p-1}(t_p-t_i)|I_i(u(t_i))|+\sum_{i=1}^{p-1}\frac{(t_p-t_i)^2}{2}\left|I_i^*(u(t_i))\right|\\
&+\left.\sum_{i=1}^{p}(1-t_p)|I_i(u(t_i))|+\sum_{i=1}^{p-1}(1-t_p)(t_p-t_i)\left|I_i^*(u(t_i))\right|+\sum_{i=1}^{p}\frac{(1-t_p)^2}{2}\left|I_i^*(u(t_i))\right|\right\}\\
\le{}&\frac{L_1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}ds+\frac{L_1}{\Gamma(\alpha)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}ds+\sum_{i=1}^{p-1}\frac{L_1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}ds\\
&+\sum_{i=1}^{p-1}\frac{L_1}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}ds+\sum_{i=1}^{p}\frac{L_1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}ds\\
&+\sum_{i=1}^{p-1}\frac{L_1}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}ds+\sum_{i=1}^{p}\frac{L_1}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}ds+\sum_{i=1}^{p-1}L_2+\sum_{i=1}^{p-1}\frac{1}{2}L_3\\
&+\sum_{i=1}^{p}L_2+\sum_{i=1}^{p-1}L_3+\sum_{i=1}^{p}\frac{1}{2}L_3+\frac{1}{1-\beta\eta^2}\\
&\times\left\{\frac{L_1\beta}{\Gamma(\alpha)}\int_{t_m}^{\eta}(\eta-s)^{\alpha-1}ds+\sum_{i=1}^{p}\frac{L_1\beta}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{i=1}^{p-1} \frac{L_1 \beta}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} ds + \sum_{i=1}^{p-1} \frac{L_1 \beta}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} ds \\
&+ \sum_{i=1}^{p} \frac{L_1 \beta}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} ds + \sum_{i=1}^{p-1} \frac{L_1 \beta}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} ds \\
&+ \sum_{i=1}^{p} \frac{L_1 \beta}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} ds + \sum_{i=1}^{p+1} \frac{L_1}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} ds \\
&+ \sum_{i=1}^{p-1} \frac{L_1}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} ds + \sum_{i=1}^{p-1} \frac{L_1}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} ds \\
&+ \sum_{i=1}^{p} \frac{L_1}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} ds + \sum_{i=1}^{p-1} \frac{L_1}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} ds \\
&+ \sum_{i=1}^{p} \frac{L_1}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} ds + \sum_{i=1}^{p-1} \beta L_2 + \sum_{i=1}^{p-1} \frac{\beta}{2} L_3 + \sum_{i=1}^{p} \beta L_2 + \sum_{i=1}^{p-1} \beta L_3 \\
&+ \sum_{i=1}^{p} \frac{\beta}{2} L_3 + \sum_{i=1}^{p-1} L_2 + \sum_{i=1}^{p-1} \frac{1}{2} L_3 + \sum_{i=1}^{p} L_2 + \sum_{i=1}^{p-1} L_3 + \sum_{i=1}^{p} \frac{1}{2} L_3 \Bigg\} \\
\le{}& \frac{L_1}{\Gamma(\alpha+1)} + \frac{pL_1}{\Gamma(\alpha+1)} + \frac{(p-1)L_1}{\Gamma(\alpha)} + \frac{(p-1)L_1}{2\Gamma(\alpha-1)} + \frac{pL_1}{\Gamma(\alpha)} + \frac{(p-1)L_1}{\Gamma(\alpha-1)} + \frac{pL_1}{2\Gamma(\alpha-1)} \\
&+ (p-1)L_2 + \frac{p-1}{2} L_3 + pL_2 + (p-1)L_3 + \frac{p}{2} L_3 + \frac{1}{1-\beta\eta^2} \\
&\times \Bigg\{ \frac{\beta L_1}{\Gamma(\alpha+1)} + \frac{\beta p L_1}{\Gamma(\alpha+1)} + \frac{\beta(p-1)L_1}{\Gamma(\alpha)} + \frac{\beta(p-1)L_1}{2\Gamma(\alpha-1)} + \frac{\beta p L_1}{\Gamma(\alpha)} + \frac{\beta(p-1)L_1}{\Gamma(\alpha-1)} + \frac{\beta p L_1}{2\Gamma(\alpha-1)} \\
&+ \frac{(p+1)L_1}{\Gamma(\alpha+1)} + \frac{(p-1)L_1}{\Gamma(\alpha)} + \frac{(p-1)L_1}{2\Gamma(\alpha-1)} + \frac{pL_1}{\Gamma(\alpha)} \\
&+ \frac{(p-1)L_1}{\Gamma(\alpha-1)} + \frac{pL_1}{2\Gamma(\alpha-1)} + \beta(p-1)L_2 + \frac{\beta(p-1)}{2} L_3 \\
&+ \beta p L_2 + \beta(p-1)L_3 + \frac{\beta p}{2} L_3 + (p-1)L_2 + \frac{p-1}{2} L_3 + pL_2 + (p-1)L_3 + \frac{p}{2} L_3 \Bigg\} \\
={}& \frac{2+\beta(1-\eta^2)}{1-\beta\eta^2} \left\{ \frac{(p+1)L_1}{\Gamma(\alpha+1)} + \frac{(2p-1)L_1}{\Gamma(\alpha)} + \frac{(4p-3)L_1}{2\Gamma(\alpha-1)} + (2p-1)L_2 + \left(2p - \frac{3}{2}\right) L_3 \right\},
\end{aligned}
\tag{3.2}
$$

which implies

$$
\begin{aligned}
\|Au\| \le{}& \frac{2+\beta(1-\eta^2)}{1-\beta\eta^2} \\
&\times \left\{ \frac{(p+1)L_1}{\Gamma(\alpha+1)} + \frac{(2p-1)L_1}{\Gamma(\alpha)} + \frac{(4p-3)L_1}{2\Gamma(\alpha-1)} + (2p-1)L_2 + \left(2p - \frac{3}{2}\right) L_3 \right\} := L.
\end{aligned}
\tag{3.3}
$$

On the other hand, for any $t \in J_k,\ 0 \le k \le p$, we have

$$
\begin{aligned}
\left|(Au)'(t)\right| \le{}& \frac{1}{\Gamma(\alpha-1)}\int_{t_k}^{t}(t-s)^{\alpha-2}\left|f(s,u(s))\right|ds + \sum_{i=1}^{k}\frac{1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds \\
&+ \sum_{i=1}^{k-1}\frac{(t_k-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds \\
&+ \sum_{i=1}^{k}\frac{(t-t_k)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds \\
&+ \sum_{i=1}^{k}\left|I_i(u(t_i))\right| + \sum_{i=1}^{k-1}(t_k-t_i)\left|I_i^*(u(t_i))\right| + \sum_{i=1}^{k}(t-t_k)\left|I_i^*(u(t_i))\right| + \frac{2t}{1-\beta\eta^2} \\
&\times\left\{\frac{\beta}{\Gamma(\alpha)}\int_{t_m}^{\eta}(\eta-s)^{\alpha-1}\left|f(s,u(s))\right|ds + \sum_{i=1}^{m}\frac{\beta}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u(s))\right|ds\right. \\
&\quad+ \sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds \\
&\quad+ \sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds \\
&\quad+ \sum_{i=1}^{m}\frac{\beta(\eta-t_m)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds \\
&\quad+ \sum_{i=1}^{m-1}\frac{\beta(\eta-t_m)(t_m-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds \\
&\quad+ \sum_{i=1}^{m}\frac{\beta(\eta-t_m)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds \\
&\quad+ \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u(s))\right|ds \\
&\quad+ \sum_{i=1}^{p-1}\frac{(t_p-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds \\
&\quad+ \sum_{i=1}^{p-1}\frac{(t_p-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds \\
&\quad+ \sum_{i=1}^{p}\frac{(1-t_p)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds \\
&\quad+ \sum_{i=1}^{p-1}\frac{(1-t_p)(t_p-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{p}\frac{(1-t_p)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds+\sum_{i=1}^{m-1}\beta(t_m-t_i)|I_i(u(t_i))|\\
&+\sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)^2}{2}\left|I_i^*(u(t_i))\right|+\sum_{i=1}^{m}\beta(\eta-t_m)|I_i(u(t_i))|\\
&+\sum_{i=1}^{m-1}\beta(\eta-t_m)(t_m-t_i)\left|I_i^*(u(t_i))\right|\\
&+\sum_{i=1}^{m}\frac{\beta(\eta-t_m)^2}{2}\left|I_i^*(u(t_i))\right|+\sum_{i=1}^{p-1}(t_p-t_i)|I_i(u(t_i))|+\sum_{i=1}^{p-1}\frac{(t_p-t_i)^2}{2}\left|I_i^*(u(t_i))\right|\\
&+\sum_{i=1}^{p}(1-t_p)|I_i(u(t_i))|+\sum_{i=1}^{p-1}(1-t_p)(t_p-t_i)\left|I_i^*(u(t_i))\right|\\
&+\left.\sum_{i=1}^{p}\frac{(1-t_p)^2}{2}\left|I_i^*(u(t_i))\right|\right\}\\
\le{}&\frac{L_1}{\Gamma(\alpha-1)}\int_{t_k}^{t}(t-s)^{\alpha-2}ds+\sum_{i=1}^{p}\frac{L_1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}ds\\
&+\sum_{i=1}^{p-1}\frac{L_1}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}ds+\sum_{i=1}^{p}\frac{L_1}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}ds+\sum_{i=1}^{p}L_2+\sum_{i=1}^{p-1}L_3\\
&+\sum_{i=1}^{p}L_3+\frac{2}{1-\beta\eta^2}\\
&\times\left\{\frac{L_1\beta}{\Gamma(\alpha)}\int_{t_m}^{\eta}(\eta-s)^{\alpha-1}ds+\sum_{i=1}^{p}\frac{L_1\beta}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}ds\right.\\
&+\sum_{i=1}^{p-1}\frac{L_1\beta}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}ds+\sum_{i=1}^{p-1}\frac{L_1\beta}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}ds\\
&+\sum_{i=1}^{p}\frac{L_1\beta}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}ds+\sum_{i=1}^{p-1}\frac{L_1\beta}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}ds\\
&+\sum_{i=1}^{p}\frac{L_1\beta}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}ds+\sum_{i=1}^{p+1}\frac{L_1}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}ds\\
&+\sum_{i=1}^{p-1}\frac{L_1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}ds+\sum_{i=1}^{p-1}\frac{L_1}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}ds\\
&+\sum_{i=1}^{p}\frac{L_1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}ds+\sum_{i=1}^{p-1}\frac{L_1}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}ds\\
&+\sum_{i=1}^{p}\frac{L_1}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}ds+\sum_{i=1}^{p-1}\beta L_2+\sum_{i=1}^{p-1}\frac{\beta}{2}L_3+\sum_{i=1}^{p}\beta L_2+\sum_{i=1}^{p-1}\beta L_3\\
&+\left.\sum_{i=1}^{p}\frac{\beta}{2}L_3+\sum_{i=1}^{p-1}L_2+\sum_{i=1}^{p-1}\frac{1}{2}L_3+\sum_{i=1}^{p}L_2+\sum_{i=1}^{p-1}L_3+\sum_{i=1}^{p}\frac{1}{2}L_3\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{L_1}{\Gamma(\alpha)} + \frac{pL_1}{\Gamma(\alpha)} + \frac{(p-1)L_1}{\Gamma(\alpha-1)} + \frac{pL_1}{\Gamma(\alpha-1)} pL_2 + (p-1)L_3 + pL_3 + \frac{2}{1-\beta\eta^2} \\
&\quad \times \left\{ \frac{\beta L_1}{\Gamma(\alpha+1)} + \frac{\beta pL_1}{\Gamma(\alpha+1)} + \frac{\beta(p-1)L_1}{\Gamma(\alpha)} + \frac{\beta(p-1)L_1}{2\Gamma(\alpha-1)} + \frac{\beta pL_1}{\Gamma(\alpha)} + \frac{\beta(p-1)L_1}{\Gamma(\alpha-1)} \right. \\
&\qquad + \frac{\beta pL_1}{2\Gamma(\alpha-1)} + \frac{(p+1)L_1}{\Gamma(\alpha+1)} + \frac{(p-1)L_1}{\Gamma(\alpha)} + \frac{(p-1)L_1}{2\Gamma(\alpha-1)} + \frac{pL_1}{\Gamma(\alpha)} + \frac{(p-1)L_1}{\Gamma(\alpha-1)} \\
&\qquad + \frac{pL_1}{2\Gamma(\alpha-1)} + \beta(p-1)L_2 + \frac{\beta(p-1)}{2}L_3 + \beta pL_2 + \beta(p-1)L_3 + \frac{\beta p}{2}L_3 \\
&\qquad \left. + (p-1)L_2 + \frac{p-1}{2}L_3 + pL_2 + (p-1)L_3 + \frac{p}{2}L_3 \right\} \\
&= \frac{(p+1)L_1}{\Gamma(\alpha)} + \frac{(2p-1)L_1}{\Gamma(\alpha-1)} pL_2 + (2p-1)L_3 + \frac{2(\beta+1)}{1-\beta\eta^2} \\
&\quad \times \left\{ \frac{(p+1)L_1}{\Gamma(\alpha+1)} + \frac{(2p-1)L_1}{\Gamma(\alpha)} + \frac{(4p-3)L_1}{2\Gamma(\alpha-1)} + (2p-1)L_2 + \left(2p - \frac{3}{2}\right)L_3 \right\} := \overline{L}.
\end{aligned}
\tag{3.4}
$$

Hence, let $t_1, t_2 \in J_k,\ t_1 < t_2,\ 0 \leq k \leq p$, we have

$$
|(Au)(t_2) - (Au)(t_1)| \leq \int_{t_1}^{t_2} \left|(Au)'(s)\right| ds \leq \overline{L}(t_2 - t_1). \tag{3.5}
$$

This implies that A is equicontinuous on all $J_k,\ k = 0, 1, 2, \ldots, p$. The Arzela-Ascoli Theorem implies that $A : C(J) \to C(J)$ is completely continuous. □

Theorem 3.2. *Assume that the nonlinearity f is bounded and that the impulse functions $I_k,\ I_k^*,\ k = 1, 2, \ldots, p$ are bounded. Then, the nonlinear impulsive fractional three-point boundary value problems* (1.1) *have at least one solution.*

Proof. Firstly, by Lemma 3.1, we know that the operator $A : C(J) \to C(J)$ is completely continuous.

Let $L_i\ (i = 1, 2, 3)$ be nonnegative constants such that

$$
\left|f(t,u)\right| \leq L_1, \quad |I_k(u)| \leq L_2, \quad \left|I_k^*(u)\right| \leq L_3, \quad t \in J,\ u \in R. \tag{3.6}
$$

For $0 < \mu < 1$, consider the equation:

$$
u = \mu A u. \tag{3.7}
$$

If u is a solution of (3.7), then for every $t \in J$ we have that

$$
\begin{aligned}
|u(t)| &= \mu|Au(t)| \\
&\le \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left|f(s,u(s))\right|ds + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u(s))\right|ds \\
&\quad + \sum_{i=1}^{k-1}\frac{(t_k-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds \\
&\quad + \sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds \\
&\quad + \sum_{i=1}^{k}\frac{(t-t_k)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds \\
&\quad + \sum_{i=1}^{k-1}\frac{(t-t_k)(t_k-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds \\
&\quad + \sum_{i=1}^{k}\frac{(t-t_k)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds \\
&\quad + \sum_{i=1}^{k-1}(t_k-t_i)|I_i(u(t_i))| + \sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2}\left|I_i^*(u(t_i))\right| \\
&\quad + \sum_{i=1}^{k}(t-t_k)|I_i(u(t_i))| + \sum_{i=1}^{k-1}(t-t_k)(t_k-t_i)\left|I_i^*(u(t_i))\right| \\
&\quad + \sum_{i=1}^{k}\frac{(t-t_k)^2}{2}\left|I_i^*(u(t_i))\right| + \frac{t^2}{1-\beta\eta^2} \\
&\quad \times\Bigg\{\frac{\beta}{\Gamma(\alpha)}\int_{t_m}^{\eta}(\eta-s)^{\alpha-1}\left|f(s,u(s))\right|ds + \sum_{i=1}^{m}\frac{\beta}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u(s))\right|ds \\
&\qquad + \sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds \\
&\qquad + \sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds \\
&\qquad + \sum_{i=1}^{m}\frac{\beta(\eta-t_m)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds \\
&\qquad + \sum_{i=1}^{m-1}\frac{\beta(\eta-t_m)(t_m-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds \\
&\qquad + \sum_{i=1}^{m}\frac{\beta(\eta-t_m)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u(s))\right|ds\\
&+\sum_{i=1}^{p-1}\frac{(t_p-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&+\sum_{i=1}^{p-1}\frac{(t_p-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds\\
&+\sum_{i=1}^{p}\frac{(1-t_p)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&+\sum_{i=1}^{p-1}\frac{(1-t_p)(t_p-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds\\
&+\sum_{i=1}^{p}\frac{(1-t_p)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds+\sum_{i=1}^{m-1}\beta(t_m-t_i)|I_i(u(t_i))|\\
&+\sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)^2}{2}\left|I_i^*(u(t_i))\right|+\sum_{i=1}^{m}\beta(\eta-t_m)|I_i(u(t_i))|\\
&+\sum_{i=1}^{m-1}\beta(\eta-t_m)(t_m-t_i)\left|I_i^*(u(t_i))\right|\\
&+\sum_{i=1}^{m}\frac{\beta(\eta-t_m)^2}{2}\left|I_i^*(u(t_i))\right|+\sum_{i=1}^{p-1}(t_p-t_i)|I_i(u(t_i))|\\
&+\sum_{i=1}^{p-1}\frac{(t_p-t_i)^2}{2}\left|I_i^*(u(t_i))\right|\\
&+\sum_{i=1}^{p}(1-t_p)|I_i(u(t_i))|+\sum_{i=1}^{p-1}(1-t_p)(t_p-t_i)\left|I_i^*(u(t_i))\right|\\
&+\left.\sum_{i=1}^{p}\frac{(1-t_p)^2}{2}\left|I_i^*(u(t_i))\right|\right\}\\
\le{}&\frac{2+\beta(1-\eta^2)}{1-\beta\eta^2}\left\{\frac{(p+1)L_1}{\Gamma(\alpha+1)}+\frac{(2p-1)L_1}{\Gamma(\alpha)}+\frac{(4p-3)L_1}{2\Gamma(\alpha-1)}+(2p-1)L_2+\left(2p-\frac{3}{2}\right)L_3\right\},
\end{aligned}
\tag{3.8}
$$

which implies for any $t\in J$, it holds that

$$
\|u\|\le\frac{2+\beta(1-\eta^2)}{1-\beta\eta^2}\left\{\frac{(p+1)L_1}{\Gamma(\alpha+1)}+\frac{(2p-1)L_1}{\Gamma(\alpha)}+\frac{(4p-3)L_1}{2\Gamma(\alpha-1)}+(2p-1)L_2+\left(2p-\frac{3}{2}\right)L_3\right\}.
\tag{3.9}
$$

This shows that all the solutions of (3.7) are bounded independently of $0 < \mu < 1$. Using Schaeffer's theorem (see Theorem 2.2), we get that A has at least one fixed point, which implies that (1.1) has at least one solution. □

Now, we present some existence results when the nonlinearity and the impulse functions have sublinear growth.

Theorem 3.3. *Assume that*

(H_1) *there exist nonnegative constants a and b such that*

$$\left|f(t,u)\right| \leq a + b|u|^{\rho}, \quad 0 \leq \rho < 1,\ t \in J,\ u \in \mathbb{R}; \tag{3.10}$$

(H_2) *there exist nonnegative constants a_k, b_k, a_k^*, b_k^* such that*

$$|I_k(u)| \leq a_k + b_k|u|^{\rho_k}, \qquad \left|I_k^*(u)\right| \leq a_k^* + b_k^*|u|^{\rho_k^*}, \quad 0 \leq \rho_k,\ \rho_k^* < 1,\ u \in \mathbb{R},\ k = 1,2,\ldots,p. \tag{3.11}$$

Then, problem (1.1) *has at least one solution.*

Proof. If $u \in C(J)$, then we can write that

$$\begin{gathered}\left|f(s,u(s))\right| \leq a + b|u(s)|^{\rho}, \quad s \in J, \\ |I_k(u(t_k))| \leq a_k + b_k|u(t_k)|^{\rho_k}, \qquad \left|I_k^*(u(t_k))\right| \leq a_k^* + b_k^*|u(t_k)|^{\rho_k^*}, \quad k = 1,2,\ldots,p.\end{gathered} \tag{3.12}$$

So, if u is a solution of (3.7), then, by the similar process used to obtain (3.2), for every $t \in J$ we have that

$$\begin{aligned}|u(t)| \leq{}& \frac{\mu}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left|f(s,u(s))\right|ds + \frac{\mu}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u(s))\right|ds \\ &+ \sum_{i=1}^{k-1}\frac{\mu(t_k-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds \\ &+ \sum_{i=1}^{k-1}\frac{\mu(t_k-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds + \sum_{i=1}^{k}\frac{\mu(t-t_k)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds \\ &+ \sum_{i=1}^{k-1}\frac{\mu(t-t_k)(t_k-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds \\ &+ \sum_{i=1}^{k}\frac{\mu(t-t_k)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds + \sum_{i=1}^{k-1}\mu(t_k-t_i)|I_i(u(t_i))| \\ &+ \sum_{i=1}^{k-1}\frac{\mu(t_k-t_i)^2}{2}\left|I_i^*(u(t_i))\right| + \sum_{i=1}^{k}\mu(t-t_k)|I_i(u(t_i))| + \sum_{i=1}^{k-1}\mu(t-t_k)(t_k-t_i)\left|I_i^*(u(t_i))\right|\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k}\frac{\mu(t-t_k)^2}{2}\left|I_i^*(u(t_i))\right|+\frac{\mu t^2}{1-\beta\eta^2}\\
&\times\left\{\frac{\beta}{\Gamma(\alpha)}\int_{t_m}^{\eta}(\eta-s)^{\alpha-1}\left|f(s,u(s))\right|ds\right.\\
&\quad+\sum_{i=1}^{m}\frac{\beta}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u(s))\right|ds+\sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&\quad+\sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds\\
&\quad+\sum_{i=1}^{m}\frac{\beta(\eta-t_m)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&\quad+\sum_{i=1}^{m-1}\frac{\beta(\eta-t_m)(t_m-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds\\
&\quad+\sum_{i=1}^{m}\frac{\beta(\eta-t_m)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u(s))\right|ds\\
&\quad+\sum_{i=1}^{p-1}\frac{(t_p-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&\quad+\sum_{i=1}^{p-1}\frac{(t_p-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds\\
&\quad+\sum_{i=1}^{p}\frac{(1-t_p)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&\quad+\sum_{i=1}^{p-1}\frac{(1-t_p)(t_p-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds\\
&\quad+\sum_{i=1}^{p}\frac{(1-t_p)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))\right|ds+\sum_{i=1}^{m-1}\beta(t_m-t_i)|I_i(u(t_i))|\\
&\quad+\sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)^2}{2}\left|I_i^*(u(t_i))\right|+\sum_{i=1}^{m}\beta(\eta-t_m)|I_i(u(t_i))|\\
&\quad+\sum_{i=1}^{m-1}\beta(\eta-t_m)(t_m-t_i)\left|I_i^*(u(t_i))\right|\\
&\quad+\sum_{i=1}^{m}\frac{\beta(\eta-t_m)^2}{2}\left|I_i^*(u(t_i))\right|+\sum_{i=1}^{p-1}(t_p-t_i)|I_i(u(t_i))|+\sum_{i=1}^{p-1}\frac{(t_p-t_i)^2}{2}\left|I_i^*(u(t_i))\right|\\
&\quad\left.+\sum_{i=1}^{p}(1-t_p)|I_i(u(t_i))|+\sum_{i=1}^{p-1}(1-t_p)(t_p-t_i)\left|I_i^*(u(t_i))\right|+\sum_{i=1}^{p}\frac{(1-t_p)^2}{2}\left|I_i^*(u(t_i))\right|\right\}
\end{aligned}
$$

$$\leq M_1 a + M_2\left(\sum_{k=1}^{p} a_k + \sum_{k=1}^{p-1} a_k\right) + \frac{M_2}{2}\left(\sum_{k=1}^{p} a_k^* + 3\sum_{k=1}^{p-1} a_k^*\right) + M_1 b\|u\|^{\rho} + M_2\sum_{k=1}^{p} b_k\|u\|^{\rho_k}$$
$$+ M_2\sum_{k=1}^{p-1} b_k\|u\|^{\rho_k} + \frac{M_2}{2}\sum_{k=1}^{p} b_k^*\|u\|^{\rho_k^*} + \frac{3M_2}{2}\sum_{k=1}^{p-1} b_k^*\|u\|^{\rho_k^*}, \tag{3.13}$$

where

$$M_1 = \frac{2+\beta(1-\eta^2)}{1-\beta\eta^2}\left(\frac{(p+1)}{\Gamma(\alpha+1)} + \frac{(2p-1)}{\Gamma(\alpha)} + \frac{(4p-3)}{2\Gamma(\alpha-1)}\right), \qquad M_2 = \frac{2+\beta(1-\eta^2)}{1-\beta\eta^2}. \tag{3.14}$$

In consequence,

$$\|u\| \leq M + M_1 b\|u\|^{\rho} + M_2\sum_{k=1}^{p} b_k\|u\|^{\rho_k} + M_2\sum_{k=1}^{p-1} b_k\|u\|^{\rho_k} + \frac{M_2}{2}\sum_{k=1}^{p} b_k^*\|u\|^{\rho_k^*} + \frac{3M_2}{2}\sum_{k=1}^{p-1} b_k^*\|u\|^{\rho_k^*}, \tag{3.15}$$

where $M = M_1 a + M_2(\sum_{k=1}^{p} a_k + \sum_{k=1}^{p-1} a_k) + (M_2/2)(\sum_{k=1}^{p} a_k^* + 3\sum_{k=1}^{p-1} a_k^*)$.

Now, taking into account that $0 \leq \rho, \rho_k, \rho_k^* < 1$, $k = 1, 2, \ldots, p$, we can conclude that there exists a constant $C > 0$ such that $\|u\| \leq C$ for any solution of (3.7), $0 < \mu < 1$. Therefore, by Schaeffer's theorem (Theorem 2.2), we obtain the existence of a fixed point for A, which implies the problem (1.1) has at least one solution. □

Theorem 3.4. *Assume that*

(H_3) *there exist nonnegative constants a and b such that*

$$\left|f(t,u)\right| \leq a + b|u|, \quad t \in J, \; u \in \mathbb{R}; \tag{3.16}$$

(H_4) *there exist nonnegative constants a_k, b_k, a_k^*, b_k^* such that*

$$|I_k(u)| \leq a_k + b_k|u|, \qquad \left|I_k^*(u)\right| \leq a_k^* + b_k^*|u|, \quad u \in \mathbb{R}, \; k = 1, 2, \ldots, p. \tag{3.17}$$

If

$$M_1 b + M_2\left(\sum_{k=1}^{p}\left(b_k + \frac{1}{2}b_k^*\right) + \sum_{k=1}^{p-1}\left(b_k + \frac{3}{2}b_k^*\right)\right) < 1, \tag{3.18}$$

where

$$M_1 = \frac{2+\beta(1-\eta^2)}{1-\beta\eta^2}\left(\frac{(p+1)}{\Gamma(\alpha+1)} + \frac{(2p-1)}{\Gamma(\alpha)} + \frac{(4p-3)}{2\Gamma(\alpha-1)}\right), \qquad M_2 = \frac{2+\beta(1-\eta^2)}{1-\beta\eta^2}. \tag{3.19}$$

Then problem (1.1) *has at least one solution.*

Proof. The proof is similar to that of Theorem 3.3, so we omit it. □

Theorem 3.5. *Let* $\lim_{u\to 0} f(t,u)/u = 0$, $\lim_{u\to 0} I_k(u)/u = 0$, *and* $\lim_{u\to 0} I_k^*(u)/u = 0$, *then problem* (1.1) *has at least one solution.*

Proof. Now, in view of $\lim_{u\to 0} f(t,u)/u = 0$, $\lim_{u\to 0} I_k(u)/u = 0$, and $\lim_{u\to 0} I_k^*(u)/u = 0$, there exists a constant $r > 0$ such that $|f(t,u)| \le \delta_1|u|$, $|I_k(u)| \le \delta_2|u|$, and $|I_k^*(u)| \le \delta_3|u|$ for $0 < |u| < r$, where $\delta_i > 0$ $(i = 1,2,3)$ satisfy

$$\frac{2+\beta(1-\eta^2)}{1-\beta\eta^2}\left\{\frac{(p+1)\delta_1}{\Gamma(\alpha+1)}+\frac{(2p-1)\delta_1}{\Gamma(\alpha)}+\frac{(4p-3)\delta_1}{2\Gamma(\alpha-1)}+(2p-1)\delta_2+\left(2p-\frac{3}{2}\right)\delta_3\right\}\le 1. \tag{3.20}$$

Let $\Omega = \{u \in C(J) \mid \|u\| < r\}$. Take $u \in C(J)$ such that $\|u\| = r$, that is, $u \in \partial\Omega$. By Lemma 3.1, we know that A is completely continuous, and

$$\begin{aligned}|(Au)(t)| \le{}& \frac{2+\beta(1-\eta^2)}{1-\beta\eta^2}\\ &\times\left\{\frac{(p+1)\delta_1}{\Gamma(\alpha+1)}+\frac{(2p-1)\delta_1}{\Gamma(\alpha)}+\frac{(4p-3)\delta_1}{2\Gamma(\alpha-1)}+(2p-1)\delta_2+\left(2p-\frac{3}{2}\right)\delta_3\right\}\|u\|.\end{aligned} \tag{3.21}$$

Thus, in view of (3.21), we obtain $\|Au\| \le \|u\|$, $u \in \partial\Omega$. Therefore, by Theorem 2.3, the operator A has at least one fixed point, which implies that (1.1) has at least one solution $u \in \overline{\Omega}$. □

For the forthcoming analysis, we need the following assumption:

(H_5) there exist positive constants K_i $(i = 1,2,3)$ such that

$$\left|f(t,u)-f(t,v)\right| \le K_1|u-v|, \qquad |I_k(u)-I_k(v)| \le K_2|u-v|, \qquad \left|I_k^*(u)-I_k^*(v)\right| \le K_3|u-v|, \tag{3.22}$$

for $t \in J$, $u,v \in R$ and $k = 1,2,\dots,p$.

Define the constants:

$$\Lambda = \frac{2+\beta(1-\eta^2)}{1-\beta\eta^2}\left\{\frac{(p+1)K_1}{\Gamma(\alpha+1)}+\frac{(2p-1)K_1}{\Gamma(\alpha)}+\frac{(4p-3)K_1}{2\Gamma(\alpha-1)}+(2p-1)K_2+\left(2p-\frac{3}{2}\right)K_3\right\}. \tag{3.23}$$

Theorem 3.6. *If condition* (H_5) *holds. Then problem* (1.1) *has a unique solution provided* $\Lambda < 1$, *where* Λ *is given by* (3.23).

Proof. Let $u, v \in C(J)$, by a simple computation, we can get

$$
\begin{aligned}
&|(Au)(t) - (Av)(t)| \\
&\quad\leq \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \left| f(s,u(s)) - f(s,v(s)) \right| ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \left| f(s,u(s)) - f(s,v(s)) \right| ds \\
&\qquad + \sum_{i=1}^{k-1} \frac{(t_k - t_i)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} \left| f(s,u(s)) - f(s,v(s)) \right| ds \\
&\qquad + \sum_{i=1}^{k-1} \frac{(t_k - t_i)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} \left| f(s,u(s)) - f(s,v(s)) \right| ds \\
&\qquad + \sum_{i=1}^{k} \frac{(t - t_k)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} \left| f(s,u(s)) - f(s,v(s)) \right| ds \\
&\qquad + \sum_{i=1}^{k-1} \frac{(t - t_k)(t_k - t_i)}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} \left| f(s,u(s)) - f(s,v(s)) \right| ds \\
&\qquad + \sum_{i=1}^{k} \frac{(t - t_k)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} \left| f(s,u(s)) - f(s,v(s)) \right| ds \\
&\qquad + \sum_{i=1}^{k-1} (t_k - t_i) |I_i(u(t_i)) - I_i(v(t_i))| + \sum_{i=1}^{k-1} \frac{(t_k - t_i)^2}{2} \left| I_i^*(u(t_i)) - I_i^*(v(t_i)) \right| \\
&\qquad + \sum_{i=1}^{k} (t - t_k) |I_i(u(t_i)) - I_i(v(t_i))| + \sum_{i=1}^{k-1} (t - t_k)(t_k - t_i) \left| I_i^*(u(t_i)) - I_i^*(v(t_i)) \right| \\
&\qquad + \sum_{i=1}^{k} \frac{(t - t_k)^2}{2} \left| I_i^*(u(t_i)) - I_i^*(v(t_i)) \right| + \frac{t^2}{1 - \beta\eta^2} \\
&\qquad \times \left\{ \frac{\beta}{\Gamma(\alpha)} \int_{t_m}^{\eta} (\eta - s)^{\alpha-1} \left| f(s,u(s)) - f(s,v(s)) \right| ds \right. \\
&\qquad\quad + \sum_{i=1}^{m} \frac{\beta}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \left| f(s,u(s)) - f(s,v(s)) \right| ds \\
&\qquad\quad + \sum_{i=1}^{m-1} \frac{\beta(t_m - t_i)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} \left| f(s,u(s)) - f(s,v(s)) \right| ds \\
&\qquad\quad + \sum_{i=1}^{m-1} \frac{\beta(t_m - t_i)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} \left| f(s,u(s)) - f(s,v(s)) \right| ds \\
&\qquad\quad + \sum_{i=1}^{m} \frac{\beta(\eta - t_m)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} \left| f(s,u(s)) - f(s,v(s)) \right| ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m-1}\frac{\beta(\eta-t_m)(t_m-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))-f(s,v(s))\right|ds\\
&+\sum_{i=1}^{m}\frac{\beta(\eta-t_m)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))-f(s,v(s))\right|ds\\
&+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u(s))-f(s,v(s))\right|ds\\
&+\sum_{i=1}^{p-1}\frac{(t_p-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))-f(s,v(s))\right|ds\\
&+\sum_{i=1}^{p-1}\frac{(t_p-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))-f(s,v(s))\right|ds\\
&+\sum_{i=1}^{p}\frac{(1-t_p)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))-f(s,v(s))\right|ds\\
&+\sum_{i=1}^{p-1}\frac{(1-t_p)(t_p-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))-f(s,v(s))\right|ds\\
&+\sum_{i=1}^{p}\frac{(1-t_p)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\left|f(s,u(s))-f(s,v(s))\right|ds\\
&+\sum_{i=1}^{m-1}\beta(t_m-t_i)|I_i(u(t_i))-I_i(v(t_i))|+\sum_{i=1}^{m-1}\frac{\beta(t_m-t_i)^2}{2}\left|I_i^*(u(t_i))-I_i^*(v(t_i))\right|\\
&+\sum_{i=1}^{m}\beta(\eta-t_m)|I_i(u(t_i))-I_i(v(t_i))|+\sum_{i=1}^{m-1}\beta(\eta-t_m)(t_m-t_i)\left|I_i^*(u(t_i))-I_i^*(v(t_i))\right|\\
&+\sum_{i=1}^{m}\frac{\beta(\eta-t_m)^2}{2}\left|I_i^*(u(t_i))-I_i^*(v(t_i))\right|+\sum_{i=1}^{p-1}(t_p-t_i)|I_i(u(t_i))-I_i(v(t_i))|\\
&+\sum_{i=1}^{p-1}\frac{(t_p-t_i)^2}{2}\left|I_i^*(u(t_i))-I_i^*(v(t_i))\right|+\sum_{i=1}^{p}(1-t_p)|I_i(u(t_i))-I_i(v(t_i))|\\
&+\left.\sum_{i=1}^{p-1}(1-t_p)(t_p-t_i)\left|I_i^*(u(t_i))-I_i^*(v(t_i))\right|+\sum_{i=1}^{p}\frac{(1-t_p)^2}{2}\left|I_i^*(u(t_i))-I_i^*(v(t_i))\right|\right\}\\
\le{}&\frac{2+\beta(1-\eta^2)}{1-\beta\eta^2}\left\{\frac{(p+1)K_1}{\Gamma(\alpha+1)}+\frac{(2p-1)K_1}{\Gamma(\alpha)}+\frac{(4p-3)K_1}{2\Gamma(\alpha-1)}+(2p-1)K_2+\left(2p-\frac{3}{2}\right)K_3\right\}\\
&\times\|u-v\|\\
\le{}&\Lambda\|u-v\|.
\end{aligned}
\tag{3.24}
$$

Thus, $\|Au - Av\| \le \Lambda\|u - v\|$.

As $\Lambda < 1$, therefore, A is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. □

4. Examples

Example 4.1. Consider the following nonlinear impulsive fractional differential equations:

$$\begin{gathered}
{}^{C}D^{\alpha}u(t) = \frac{e^{t}\cos^{3}u(t)}{1+u^{2}(t)}, \quad 0<t<1,\ t \neq \frac{1}{2}, \\
\Delta u'\left(\frac{1}{2}\right) = \frac{5+3u^{4}(1/2)}{1+u^{4}(1/2)}, \qquad \Delta u''\left(\frac{1}{2}\right) = 3\sin u\left(\frac{1}{2}\right), \\
u(0) = u'(0) = 0, \quad \beta u(\eta) = u(1),
\end{gathered} \tag{4.1}$$

where $2 < \alpha \le 3,\ 0 < \eta < 1,\ \eta \neq 1/2,\ 0 < \beta < 1/\eta^{2}$, and $p = 1$.

Obviously, for $L_1 = e,\ L_2 = 5$, and $L_3 = 3$, it is easy to verify condition of Theorem 3.2 holds. Hence, by Theorem 3.2, we can get that the above equation (4.1) has at least one solution.

Example 4.2. Consider the following fractional impulsive the three-point boundary value problem:

$$\begin{gathered}
{}^{C}D^{\alpha}u(t) = \frac{e^{5t}\cos^{5}\left[3u(t)+e^{8u(t)}\right]}{1+u^{4}(t)} + \frac{2\sin(2t+5)}{\sqrt{3+u^{2}(t)}}|u(t)|^{\rho}, \quad 0<t<1,\ t\neq\frac{1}{4}, \\
\Delta u'\left(\frac{1}{4}\right) = 3+2\cos^{2}\left[\ln\left(1+8u^{2}\left(\frac{1}{4}\right)\right)\right] + 3\cos u\left(\frac{1}{4}\right)\left|u\left(\frac{1}{4}\right)\right|^{\rho 1}, \\
\Delta u''\left(\frac{1}{4}\right) = \frac{1}{2} + \frac{5+2u^{2}(1/4)}{2+u^{2}(1/4)} + 3e^{-u^{2}(1/4)}\left|u\left(\frac{1}{4}\right)\right|^{\rho 2}, \\
u(0) = u'(0) = 0, \quad \beta u(\eta) = u(1),
\end{gathered} \tag{4.2}$$

where $2 < \alpha \le 3,\ 0 < \eta < 1,\ \eta \neq 1/4,\ 0 < \beta < 1/\eta^{2}$, and $p = 1$.

Obviously $a = e^{5},\ b = 2,\ a_1 = 5,\ b_1 = a_1^{*} = b_1^{*} = 3$. It is easy to verify that for $0 < \rho, \rho_k, \rho_k^{*} < 1$, the conditions of Theorem 3.3 hold. Therefore, by Theorem 3.3, the impulsive the three-point fractional boundary value problem (4.2) has at least one solution.

Example 4.3. Consider the following nonlinear impulsive fractional differential equations:

$$
\begin{gathered}
{}^{C}D^{\alpha}u(t) = t^2u^2(t) + u(t)\arctan u(t), \quad 0 < t < 1,\ t \neq \frac{1}{3}, \\
\Delta u'\left(\frac{1}{3}\right) = e^{u(t)}u^3\left(\frac{1}{3}\right), \qquad \Delta u''\left(\frac{1}{3}\right) = \frac{3u^2(1/3)}{1+2u^4(1/3)}, \\
u(0) = u'(0) = 0, \quad \beta u(\eta) = u(1),
\end{gathered}
\tag{4.3}
$$

where $2 < \alpha \leq 3,\ 0 < \eta < 1,\ \eta \neq 1/3,\ 0 < \beta < 1/\eta^2$, and $p = 1$.

Clearly, all the assumptions of Theorem 3.5 hold. Thus, by the conclusion of Theorem 3.5 we can get that (4.3) has at least one solution.

5. Conclusion

The existence and uniqueness of solutions to a three-point boundary value problem for fractional nonlinear differential equation with impulses have been discussed. We apply the concepts of fractional calculus together with fixed point theorems to establish the existence results. First of all, we investigate a linear three-point boundary value problem involving fractional derivatives and impulses, which in fact provides the platform to prove the existence of solutions for the associated nonlinear fractional equation with impulses. It is worth mentioning that the conditions of our theorems are easily to verify and that our approach is simple, so they are applicable to a variety of real world problems.

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