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## Research Article

# Schwarz-Pick Estimates for Holomorphic Mappings with Values in Homogeneous Ball

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Let  $B_X$  be the unit ball in a complex Banach space  $X$ . Assume  $B_X$  is homogeneous. The generalization of the Schwarz-Pick estimates of partial derivatives of arbitrary order is established for holomorphic mappings from the unit ball  $B^n$  to  $B_X$  associated with the Carathéodory metric, which extend the corresponding Chen and Liu, Dai et al. results.

## 1. Introduction

By the classical Pick's invariant form of Schwarz's lemma, a holomorphic function  $f(z)$  which is bounded by one in the unit disk  $D \subset \mathbb{C}$  satisfies the following inequality

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2} \quad (1.1)$$

at each point  $z$  of  $D$ . Ruscheweyh in [1] firstly obtained best-possible estimates of higher order derivatives of bounded holomorphic functions on the unit disk in 1985. Recently, a lot of attention (see Ghatage et al. [2], MacCluer et al. [3], Avkhadiev and Wirths [4], Ghatage and Zheng [5], Dai and Pan [6]) has been paid to the Schwarz-Pick estimates of high-order derivative estimates in one complex variable. The best result is given as follows:

$$|f^{(k)}(z)| \leq \frac{k!(1 - |f(z)|^2)}{(1 - |z|^2)^k} (1 + |z|)^{k-1}, \quad z \in D, \quad k \geq 1. \quad (1.2)$$

It is natural to consider an extension of the above Schwarz-Pick estimates to higher dimensions. Anderson et al. [7] gave Schwarz-Pick estimates of derivatives of arbitrary order of functions in the Schur-Agler class on the unit polydisk and the unit ball of  $\mathbb{C}^n$ , respectively. Recently, Chen and Liu in [8] obtained estimates of high-order derivatives for all the bounded holomorphic functions on the unit ball of  $\mathbb{C}^n$ . Later, Dai et al. in [9, 10] generalized the high order Schwarz-Pick estimates for holomorphic mappings between unit balls in complex Hilbert space. Their main result is expressed as follows.

**Theorem A.** *Suppose  $f(z)$  is holomorphic mapping from  $B^n$  to  $B^m$ . Then for any multiindex  $k \geq 1$  and  $\beta \in \mathbb{C}^n \setminus \{0\}$ ,*

$$H_{f(z)}\left(D^k(f, z, \beta), D^k(f, z, \beta)\right) \leq k! \left(1 + \frac{(|\langle \beta, z \rangle|)}{((1 - |z|^2)|\beta|^2 + |\langle \beta, z \rangle|^2)^{1/2}}\right)^{k-1} (H_z(\beta, \beta))^k, \quad (1.3)$$

where  $D^k(f, z, \beta) = \sum_{|\alpha|=k} (k!/\alpha!) (\partial f^k(z) / \partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \cdots \partial z_n^{\alpha_n}) \beta^\alpha$  and  $H_z(\beta, \beta)$  is the Bergman metric on  $B^n$ .

In this paper, we will extend Theorem A to holomorphic mappings from the unit ball  $B^n$  to  $B_X$  associated with the Carathéodory metric. In particular, when  $B_X = B^m$ , our result coincides with Theorem A. Furthermore, our result shows that the high-order Schwarz-Pick estimates on the unit ball do depend on the geometric property of the image domain  $B_X$ .

Throughout this paper, the symbol  $X$  is used to denote a complex Banach space with norm  $\|\cdot\|$ , and  $B_X = \{z \in X : \|z\| < 1\}$  is the unit ball in  $X$ . Let  $\mathbb{C}^n$  be the space of  $n$  complex variables  $z = (z_1, \dots, z_n)'$  with the Euclidean inner product  $\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j$ , where the symbol  $'$  stands for the transpose of vector or matrix. The unit ball of  $\mathbb{C}^n$  is always written by  $B^n$ . It is well known that if  $f$  is a holomorphic mapping from  $B_X$  into  $X$ , then the following well-known expansion

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x) ((y-x)^n) \quad (1.4)$$

holds for all  $y$  in some neighborhood of  $x \in B_X$ , where  $D^n f(x)$  means the  $n$ th Fréchet derivative of  $f$  at the point  $x$ , and

$$D^n f(x) ((y-x)^n) = D^n f(x)(y-x, y-x, \dots, y-x). \quad (1.5)$$

Furthermore,  $D^n f(x)$  is a bounded symmetric  $n$ -linear mapping from  $\prod_{j=1}^n X$  into  $X$ . For a domain  $\Omega \in X$ , a mapping  $f : \Omega \rightarrow X$  is called to be biholomorphic if  $f(\Omega)$  is a domain; the inverse  $f^{-1}$  exists and is holomorphic on  $f(\Omega)$ . Let  $\text{Aut}(\Omega)$  denote the set of biholomorphic mappings of  $\Omega$  onto itself.  $\Omega$  is said to be homogeneous, if for each pair of points  $x, y \in \Omega$ , there is an  $f \in \text{Aut}(\Omega)$  such that  $f(x) = y$ .

In multiindex notation,  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$  is an  $n$ -tuple of nonnegative integers,  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ ,  $\alpha! = \alpha_1! \cdots \alpha_n!$ ,  $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ .

Let  $K(z, z)$  be the Bergman kernel function. Then the Bergman metric  $H_z(\beta, \beta)$  can be defined as

$$H_z(\beta, \beta) = \sum_{j,k=1}^n \frac{\partial^2 \log K(z, z)}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k, \tag{1.6}$$

where  $z \in \Omega$ ,  $u = (u_1, u_2, \dots, u_n) \in \mathbb{C}^n$ . It is well known that  $H_z(\beta, \beta) = (1 - |z|^2 + |\langle \beta, z \rangle|^2) / (1 - |z|^2)^2$  in [9].

Let  $F_c^{B_X}(z, \xi)$  be the infinitesimal form of Carathéodory metric of domain  $B_X$ . By the definition of the Carathéodory metric [11], we have for any  $\xi \in X$ ,

$$F_c^{B_X}(z, \xi) = \sup \{ |Df(z)\xi| : f \in H(B_X, B_X), f(z) = 0 \}, \tag{1.7}$$

where  $H(B_X, B_X)$  denotes the family of holomorphic mappings which map  $B_X$  into  $B_X$ .

## 2. Some Lemmas

In order to prove the main results, we need the following lemmas. Let  $B_X$  be the unit ball in a complex Banach space  $X$ , and  $B_X$  is homogeneous.

**Lemma 2.1** (see [11]). *If  $f \in H(B_X, B_X)$ , then*

$$F_c^{B_X}(f(z), Df(z)\xi) \leq F_c^{B_X}(z, \xi), \quad z \in B_X, \xi \in X. \tag{2.1}$$

*In particular, when  $f$  is biholomorphic mapping, then  $F_c^{B_X}(f(z), Df(z)\xi) = F_c^{B_X}(z, \xi)$ .*

**Lemma 2.2** (see [12]). *Consider the following:*

$$F_c^{B_X}(0, \xi) = \|\xi\|, \quad \xi \in X. \tag{2.2}$$

**Lemma 2.3.** *Let  $f \in H(D, B_X)$ . Then  $f$  can be written with the following  $n$ -variable power series given by*

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad z \in D. \tag{2.3}$$

*Then the following holds*

$$F_c^{B_X}(a_0, a_k) \leq 1 \tag{2.4}$$

*for any integer  $k \geq 0$ .*

*Proof.* For the fixed  $k$ , we define

$$f_k(z) = \sum_{j=1}^k \frac{f(e^{i(2\pi j/k)} z)}{k}. \quad (2.5)$$

Then  $f_k \in H(D, B_X)$ . It is clear that

$$\frac{1}{k} \sum_{j=1}^k e^{i(2\pi jl/k)} = \begin{cases} 1, & \text{if } l \equiv 0 \pmod{k}, \\ 0, & \text{otherwise.} \end{cases} \quad (2.6)$$

From the power series expansion of the holomorphic function  $f$ , we get

$$\begin{aligned} f_k(z) &= \frac{1}{k} \left( \sum_{j=1}^k \left( a_0 + \sum_{l=1}^{\infty} e^{i(2\pi jl/k)} \sum_{|\alpha|=l} a_{\alpha} z^{\alpha} \right) \right) \\ &= a_0 + \sum_{l=1}^{\infty} a_{lk} z^{lk}. \end{aligned} \quad (2.7)$$

In terms of the homogeneity of  $B_X$ , we can take  $\Psi \in \text{Aut}(B_X)$  and  $\Psi(a_0) = 0$ , then  $\Psi \circ f_k \in H(D, B_X)$ . This implies that

$$\begin{aligned} \Psi \circ f_k(z) &= \Psi \left( a_0 + \sum_{l=1}^{\infty} a_{lk} z^{lk} \right) \\ &= \Psi(a_0) + D\Psi(a_0) \left( \sum_{l=1}^{\infty} a_{lk} z^{lk} \right) + D^2\Psi(a_0) \left( \sum_{l=1}^{\infty} a_{lk} z^{lk} \right) + \dots \\ &= D\Psi(a_0)(a_k) z^k + D\Psi(a_0)(a_{2k}) z^{2k} + D\Psi(a_0)(a_{3k}) z^{3k} + \dots \end{aligned} \quad (2.8)$$

By making use of the orthogonality, we obtain

$$D\Psi(a_0)(a_{\alpha}) z^{\alpha} = \frac{1}{2\pi} \int_0^{2\pi} (\Psi \circ f_k)(ze^{i\theta}) e^{-i\alpha\theta} d\theta. \quad (2.9)$$

Hence,

$$\|D\Psi(a_0)(a_{\alpha}) z^{\alpha}\| \leq \frac{1}{2\pi} \int_0^{2\pi} \|(\Psi \circ f_k)(ze^{i\theta}) e^{-i\alpha\theta}\| d\theta \leq 1. \quad (2.10)$$

This implies the following inequality

$$\|D\Psi(a_0)(a_{\alpha})\| |z|^{\alpha} \leq 1 \quad (2.11)$$

holds for any  $z \in D$ . Thus,

$$\|D\Psi(a_0)(a_\alpha z^\alpha)\| \leq 1 \tag{2.12}$$

holds for any  $z \in \overline{D}$ . It means that  $\|D\Psi(a_0)(a_\alpha)\| \leq 1$ .

By Lemmas 2.1 and 2.2, we obtain

$$F_c^{B_X}(a_0, a_\alpha) = F_c^{B_X}(0, D\Psi(a_0)(a_\alpha)) = \|D\Psi(a_0)(a_\alpha)\| \leq 1, \tag{2.13}$$

which is the desired result. □

### 3. Main Results

**Theorem 3.1.** *Let  $f : D \rightarrow B_X$  be a holomorphic mapping. Then the following inequality*

$$F_c^{B_X}(f(z), f^{(k)}(z)) \leq k! \frac{(1 + |z|)^{k-1}}{(1 - |z|^2)^k} \tag{3.1}$$

holds for  $k \geq 1$  and  $z \in D$ .

*Proof.* Let  $g(\xi)$  be a holomorphic function on  $D$  defined by

$$g(\xi) = f\left(\frac{z + \xi}{1 + \overline{z}\xi}\right), \quad \xi \in D. \tag{3.2}$$

Then  $g$  can be written as a power series as follows:

$$g(\xi) = \sum_{j=0}^{\infty} a_j \xi^j. \tag{3.3}$$

In order to obtain Theorem 3.1, we need to prove the following equality:

$$f^{(k)}(z) = \frac{k!}{1 - |z|^2} \sum_{j=0}^k \binom{k-1}{j} a_{k-j} \overline{z}^{|j|}. \tag{3.4}$$

Let  $0 < r < 1$  such that  $D(z, r) \subset D$ , the Cauchy integral formula shows that

$$f(z) = \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w - z} dw. \tag{3.5}$$

Thus,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{|w|=r} \frac{f(w)}{(w-z)^{k+1}} dw. \quad (3.6)$$

Let  $w = (z + \xi)/(1 + \bar{z}\xi)$ . Then

$$\frac{dw}{d\xi} = \frac{1 - |z|^2}{(1 + \bar{z}\xi)^2}, \quad w - z = \xi \frac{1 - |z|^2}{(1 + \bar{z}\xi)^2}. \quad (3.7)$$

Substituting (3.7) into (3.6), we get

$$\begin{aligned} f^{(k)}(z) &= \frac{k!}{2\pi i (1 - |z|^2)^k} \int_{|(z+\xi)/(1+\bar{z}\xi)|=r} \frac{g(\xi)(1 + \bar{z}\xi)^{k-1}}{\xi^{k+1}} d\xi \\ &= \frac{k!}{(1 - |z|^2)^k} \sum_{j=0}^{k-1} \binom{k-1}{j} a_{k-j} \bar{z}^j, \end{aligned} \quad (3.8)$$

which prove the equality (3.4).

From Lemma 2.3, we have for any integer  $k \geq 1$ ,

$$F_c^{B_X}(a_0, a_k) \leq 1. \quad (3.9)$$

This implies that

$$\begin{aligned} F_c^{B_X}(f(z), f^{(k)}(z)) &\leq F_c^{B_X}\left(a_0, \frac{k!}{(1 - |z|^2)^k} \sum_{j=0}^{k-1} \binom{k-1}{j} a_{k-j} |z|^j\right) \\ &\leq \frac{k!}{(1 - |z|^2)^k} (1 + |z|)^{k-1} \end{aligned} \quad (3.10)$$

which completes the desired result.  $\square$

*Remark 3.2.* If  $B_X = D$ , then the inequality (3.1) reduces to

$$\left| f^{(k)}(z) \right| \leq k! \frac{1 - |f(z)|^2}{(1 - |z|^2)^k} (1 + |z|)^{k-1} \quad (3.11)$$

which coincides with the Theorem 1.1 of Dai and Pan [6] in one complex variable.

**Theorem 3.3.** *Let  $f : B^n \rightarrow B_X$  be a holomorphic mapping. Then the following inequality*

$$F_c^{B_X}(f(z), D^k(f, z, \beta)) \leq k! \left( 1 + \frac{|\langle \beta, z \rangle|}{((1 - |z|^2)|\beta|^2 + |\langle \beta, z \rangle|^2)^{1/2}} \right)^{k-1} \left[ F_c^{B^n}(z, \beta) \right]^k \quad (3.12)$$

holds for  $k \geq 1$ ,  $\beta \in \mathbb{C}^n \setminus \{0\}$  and  $z \in B^n$ .

*Proof.* For any fixed  $k \geq 1$ ,  $\beta \in \partial B^n$ , and  $\xi \in B^n$ . Define the following disk:

$$\Delta = \left\{ \lambda \in \mathbb{C} : |\xi + \lambda\beta|^2 < 1 \right\}. \quad (3.13)$$

Notice that  $\langle \beta, \xi - \langle \xi, \beta \rangle \beta \rangle = 0$ . Hence,

$$\begin{aligned} |\xi + \lambda\beta|^2 &= |(\lambda + \langle \xi, \beta \rangle)\beta + \xi - \langle \xi, \beta \rangle \beta|^2 \\ &= |\lambda + \langle \xi, \beta \rangle|^2 + |\xi - \langle \xi, \beta \rangle \beta|^2 < 1. \end{aligned} \quad (3.14)$$

That is,

$$|\lambda + \langle \xi, \beta \rangle| < \sqrt{1 - |\xi - \langle \xi, \beta \rangle \beta|^2} = \sqrt{1 - |\xi|^2 + |\langle \xi, \beta \rangle|^2}. \quad (3.15)$$

Set  $\sigma = \sqrt{1 - |\xi|^2 + |\langle \xi, \beta \rangle|^2}$ . For the fixed  $\xi$  and  $\beta$ , we define

$$g(\omega) = f(\xi + (\omega\sigma - \langle \xi, \beta \rangle)\beta), \quad \omega \in D. \quad (3.16)$$

Then  $g(\omega)$  is holomorphic mapping from the unit disk  $D$  to the homogeneous domain  $\Omega$ .

According to Theorem 3.1 to the functions  $g$  and  $\omega' = (\langle \xi, \beta \rangle)/\sigma$ , we have

$$F_c^{B_X}(g(\omega'), g^{(k)}(\omega')) \leq k! \frac{(1 + |\omega'|)^{k-1}}{(1 - |\omega'|^2)^k}, \quad (3.17)$$

which holds for  $k \geq 1$ . Since  $g(\omega') = f(\xi)$ , and

$$|\omega'| = \frac{|\langle \beta, \xi \rangle|}{\sqrt{1 - |\xi|^2 + |\langle \xi, \beta \rangle|^2}}, \quad 1 - |\omega'|^2 = \frac{1 - |\xi|^2}{1 - |\xi|^2 + |\langle \xi, \beta \rangle|^2}. \quad (3.18)$$

In terms of the chain rule, we have

$$g^{(k)}(\omega') = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial f^k(\xi)}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \cdots \partial z_N^{\alpha_N}} (\sigma\beta)^\alpha = \sigma^k \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial f^k(\xi)}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \cdots \partial z_N^{\alpha_N}} \beta^\alpha = \sigma^k D^k(f, \xi, \beta). \quad (3.19)$$

Hence,

$$F_c^{B_X}(f(\xi), \sigma^k D^k(f, \xi, \beta)) \leq k! \left( 1 + \frac{|\langle \beta, \xi \rangle|}{(1 - |\xi|^2 + |\langle \beta, \xi \rangle|^2)^{1/2}} \right)^{k-1} \left[ \frac{(1 - |\xi|^2) + |\langle \beta, \xi \rangle|^2}{(1 - |\xi|^2)^2} \right]^k \sigma^k. \quad (3.20)$$

Note the definition of Carathéodory metric and  $F_c^{B^n}(z, \beta) = (1 - |z|^2 + |\langle \beta, z \rangle|^2)/(1 - |z|^2)^2$  in [11], we can get

$$F_c^{B_X}(f(z), D^k(f, z, \beta)) \leq k! \left( 1 + \frac{|\langle \beta, z \rangle|}{(1 - |z|^2 + |\langle \beta, z \rangle|^2)^{1/2}} \right)^{k-1} \left[ F_c^{B^n}(z, \beta) \right]^k. \quad (3.21)$$

This gives the proof of the case  $z = \xi$  and  $\beta \in \partial B_n$ . For general vector  $\beta \in \mathbb{C}^n \setminus \{0\}$ , we may substitute  $\beta/\|\beta\|$  for  $\beta$ . By the homogeneous of  $\beta$  from the above inequality, we can obtain the same result, which completes the proof of the Theorem 3.3.  $\square$

*Remark 3.4.* If  $B_X = B^m$ , then  $H_{f(z)}(D^k(f, z, \beta), D^k(f, z, \beta)) = F_c^{B^m}(f(z), D^k(f, z, \beta))$  and  $H_z(\beta, \beta) = F_c^{B^m}(z, \beta)$ . Thus, the Theorem 3.3 reduces to Theorem A established by Dai et al. [9].

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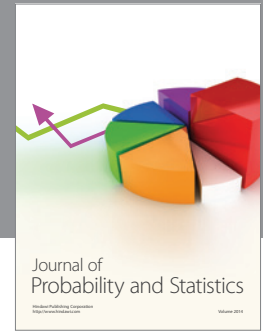
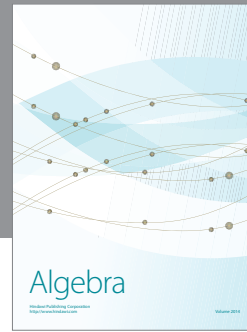
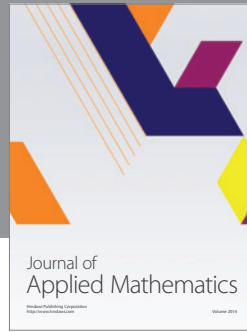
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