Hindawi Publishing Corporation
International Journal of Combinatorics
Volume 2012, Article ID 406250, 11 pages
doi:10.1155/2012/406250

Research Article

Variations of the Game 3-Euclid

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Received 30 November 2011; Revised 27 December 2011; Accepted 5 January 2012

Academic Editor: Toufik Mansour

We present two variations of the game 3-Euclid. The games involve a triplet of positive integers. Two players move alternately. In the first game, each move is to subtract a positive integer multiple of the smallest integer from one of the other integers as long as the result remains positive. In the second game, each move is to subtract a positive integer multiple of the smallest integer from the largest integer as long as the result remains positive. The player who makes the last move wins. We show that the two games have the same $\mathcal{P}$-positions and positions of Sprague-Grundy value 1. We present three theorems on the periodicity of $\mathcal{P}$-positions and positions of Sprague-Grundy value 1. We also obtain a theorem on the partition of Sprague-Grundy values for each game. In addition, we examine the misère versions of the two games and show that the Sprague-Grundy functions of each game and its misère version differ slightly.

1. Introduction

The game Euclid, introduced by Cole and Davie [1], is a two-player game based on the Euclidean algorithm. A position in Euclid is a pair of positive integers. The two players move alternately. Each move is to subtract from one of the entries a positive integer multiple of the other without making the result negative. The game stops when one of the entries is reduced to zero. The player who makes the last move wins. In the literature, the term Euclid has been also used for a variation presented by Grossman [2] in which the game stops when the two entries are equal. More details and discussions on Euclid and Grossman's game can be found in [3–9]. Some restrictions of Grossman's game can be found in [10–12]. The misère version of Grossman's game was studied in [13].

Collins and Lengyel [11] presented an extension of Grossman's game to three dimensions that they called *3-Euclid*. In 3-Euclid, a position is a triplet of positive integers. Each move is to subtract from one of the integers a positive integer multiple of one of the others as long as the result remains positive. Generally, from a position (a, b, c), where $a \leq b \leq c$, there are three types of moves in 3-Euclid: (i) 1-2 moves: subtracting a multiple of a from b; (ii) 1–3 moves: subtracting a multiple of a from c; (iii) 2-3 moves: subtracting a multiple of b from c.

In this paper, we present two restrictions of Collins and Lengyel's game. In the first restriction that we call G_1, a move is to subtract a positive integer multiple of the smallest integer from one of the other integers as long as the result remains positive. In the second restriction that we call G_2, each move is to subtract a positive integer multiple of the smallest integer from the largest integer as long as the result remains positive. The two games therefore stop when the three integers are equal. Thus, the game G_1 is a restriction of the game 3-Euclid allowing the 1-2 and 1–3 moves while the game G_2 is a restriction of 3-Euclid allowing only the 1–3 move. We investigate $\mathcal{P}$-positions and positions of Sprague-Grundy value 1 of the two games G_1 and G_2.

Recall that a position p is said to be an $\mathcal{N}$*-position* (we write $p \in \mathcal{N}$) if the next player to move from p has a strategy to win. Otherwise, p is said to be a $\mathcal{P}$*-position* (we write $p \in \mathcal{P}$). Note that every move from a $\mathcal{P}$-position terminates at some $\mathcal{N}$-position while from an $\mathcal{N}$-position, there exists a move terminating at some $\mathcal{P}$-position.

Throughout this paper, the Sprague-Grundy value of the position p is denoted by $\mathcal{G}(p)$. Not surprisingly, the Sprague-Grundy functions for 3-Euclid, G_1, and G_2 are pairwise distinct. For example, calculations show that for the position $(2,3,7)$ the Sprague-Grundy value is 1 for 3-Euclid, 2 for G_1, and 3 for G_2. The $\mathcal{P}$-positions of 3-Euclid also differ to those of G_1 and G_2. For example, $(3,6,7)$ is a $\mathcal{P}$-position in 3-Euclid, but it has Sprague-Grundy value 1 for G_1 and G_2. Curiously, G_1 and G_2 have the same $\mathcal{P}$-positions and the same positions of Sprague-Grundy value 1. This is the main result in this paper.

This paper is organized as follows. In Section 2, we show that the two games G_1 and G_2 have the same $\mathcal{P}$-positions before proving a periodicity result for the $\mathcal{P}$-positions. We also give some classes of $\mathcal{P}$-positions of the two games. In Section 3, we show that the two games G_1 and G_2 also have the same positions of Sprague-Grundy value 1 and then present some connections between positions of Sprague-Grundy value 1 and $\mathcal{P}$-positions. We give two theorems on the periodicity of the positions of Sprague-Grundy value 1. Some special cases of positions of Sprague-Grundy value 1 are also discussed. Section 4 discusses the existence of values c satisfying the condition $\mathcal{G}(a,b,c) = s$ for some given a,b,s. This is analogous to a result of Collins and Lengyel [11]. Section 5 examines the misère versions of the two games G_1 and G_2. It will be shown that these two games and their misère versions differ slightly only on a subset of positions of Sprague-Grundy values 0 and 1. This result also shows that, as established in [14], the miserability is quite common in impartial games.

This paper continues our investigations of variants of Euclid and related questions; see [3, 15, 16].

2. On the $\mathcal{P}$-Positions

We show in this section that the two games G_1 and G_2 have the same $\mathcal{P}$-positions. We then present a periodicity property of $\mathcal{P}$-positions and some special classes of $\mathcal{P}$-positions.

Lemma 2.1. *Let $a \le b \le c$. In the game G_2, if $(a,b,c) \in \mathcal{P}$ then $c < a + b$.*

Proof. Assume by contradiction that there exists a $\mathcal{P}$-position (a,b,c) with $a \le b \le c$ and $c \ge a + b$. Then $(a,b,c-a) \in \mathcal{N}$. There must be a move from $(a,b,c-a)$ to some $\mathcal{P}$-position p. As $c - a \ge b$, the position p must be of the form $(a,b,c-a-ia)$ for some $i \ge 1$. This is a contradiction as there is then a move from (a,b,c) to p. □

Theorem 2.2. *The $\mathcal{P}$-positions in the game G_2 are identical to those in the game G_1.*

Proof. Let A, B be the sets of $\mathcal{P}$-positions and $\mathcal{N}$-positions, respectively, of the game G_2. We prove that the following two properties hold for the game G_1:

(i) every move from a position in A terminates in B;

(ii) from every position in B, there exists a move that terminates in A.

Since every move in G_2 is legal in G_1, (ii) holds. Assume by contradiction that (i) does not hold for G_1. Then there exists a position $(a,b,c) \in A$ with $a \le b \le c$ and a move in G_1 leading (a,b,c) to some position in A. This move cannot be a 1–3 move as (i) holds for G_2, and so this move must be a 1-2 move. Therefore, there exists $i \ge 1$ such that $(a, b - ia, c) \in A$. By Lemma 2.1, $c < a + b - ia \le b$ giving a contradiction. Therefore, (i) holds for G_1. □

Lemma 2.3. *Let $a \le b$. For the two games G_1 and G_2, there exists exactly one integer $c \ge 1$ in each residue class $r(\operatorname{mod} a)$ such that $(a,b,c) \in \mathcal{P}$.*

Proof. We will prove the existence for the game G_2 and the uniqueness for the game G_1. The lemma then follows Theorem 2.2. For the existence, let d be an integer such that $a \le b \le d$ and $d \equiv r \bmod a$. Consider the position (a,b,d) in the game G_2. If $\mathcal{G}(a,b,d) = 0$ then we are done. If $\mathcal{G}(a,b,d) > 0$ then there exists a move from (a,b,d) to some $\mathcal{P}$-position p. This move must be a 1–3 move, and so p is of the form $(a,b,d-ia)$ as required. For the uniqueness, assume by contradiction that there are two positive integers $c_1 < c_2$ in the residue class $r(\operatorname{mod}\ a)$ such that both (a,b,c_1) and (a,b,c_2) are $\mathcal{P}$-positions in the game G_1. Since c_1, c_2 are in the same residue class $r \bmod a$, we have $c_2 > a$. If $c_2 < b$ then there exists a 1-2 move from (a, c_2, b) to (a, c_1, b). This is a contradiction. If $c_2 \ge b$ then there exists a 1–3 move from (a,b,c_2) to (a,b,c_1). This is a contradiction. Therefore, the uniqueness holds. □

We now present a periodicity result for the $\mathcal{P}$-positions. Note that the following theorem holds for both games by Theorem 2.2.

Theorem 2.4. *Let $a \le b \le c$. Then $(a,b,c) \in \mathcal{P}$ if and only if $(a, b+a, c+a) \in \mathcal{P}$.*

Proof. By Theorem 2.2, it is enough to prove that the theorem holds for G_2. For the necessary condition, let $(a,b,c) \in \mathcal{P}$, let $m = \lfloor b/a \rfloor$, the integer part of b/a, and let r denote the remainder, $r = b - ma$. Note that $c < a + b$ by Lemma 2.1. By Lemma 2.3, there is exactly one integer n such that $p = (a, na + r, c + a) \in \mathcal{P}$. We show that $n = m + 1$. First assume that $n \le m$. We have $na + r + a \le ma + r + a = b + a \le c + a$. However, as $p \in \mathcal{P}$, by Lemma 2.1, $a + na + r > c + a$ giving a contradiction. Next, assume that $n \ge m+2$ and so $na + r \ge b + 2a$. Note that $a + b > c$ by Lemma 2.1 and so $na + r > c + a$. Consider the position $q = (a, b+a, c+a)$. Since there exists a move from $p = (a, c+a, na+r)$ to q, we have $q \in \mathcal{N}$ and so $(a, b+a, c+a-ja) \in \mathcal{P}$ for some positive integer j. By Lemma 2.1, we have $b + a < a + c + a - ja$ which implies $j = 1$, as $b + a > c$, and so $(a, b+a, c) \in \mathcal{P}$. Note that $b + a > c$ and so there is a move from $(a, b+a, c)$ to (a,b,c) which is also a $\mathcal{P}$-position. This is a contradiction.

Conversely, assume that $(a, b+a, c+a) \in \mathcal{P}$. By Lemma 2.1, we have $c + a < a + b + a$ or $c < a + b$. Since there exists a 1–3 move from $(a, b+a, c+a)$ to $(a, c, b+a)$, we have $(a, c, b+a) \in \mathcal{N}$. Then there exists a move from $(a, c, b+a)$ to some $\mathcal{P}$-position p'. Since $b + a > c$, p' must be of the form $(a, b + a - ja, c)$ for some $j \ge 1$. By Lemma 2.1, we have $a + b + a - ja > c$, and so $j = 1$. Therefore, $(a,b,c) \in \mathcal{P}$. □

Collins and Lengyel [11] suggested a similar result for the game 3-Euclid. They claimed that in the game 3-Euclid, if $b, c > a^2$ then $(a, b, c) \in \mathcal{P}$ if and only if $(a, b+a, c+a) \in \mathcal{P}$. To our knowledge, a proof for this claim has not appeared in the literature. In our opinion, a proof for this claim would be much more complicated.

We now solve some special cases for $\mathcal{P}$-positions.

Corollary 2.5. *Let $a \le b$. Then $(a, b, b) \in \mathcal{P}$.*

Proof. By Theorem 2.2, it is enough to prove that $(a, b, b) \in \mathcal{P}$ in G_2.

Assume by contradiction that $(a, b, b) \in \mathcal{N}$. Then there exists a positive integer i such that $(a, b - ia, b) \in \mathcal{P}$. By Lemma 2.1, $b < a + b - ia \le b$ giving a contradiction. Therefore, $(a, b, b) \in \mathcal{P}$. □

We state one further result, whose proof we postpone until the end of the next section.

Proposition 2.6. *(i) Let $2 \le a \le b$. Then $(a, b - 1, b) \in \mathcal{P}$ if and only if $\gcd(a, b) \neq 1$.*

(ii) Let $a < b$. Then $(a, b - a, b - 1) \in \mathcal{P}$ if and only if $\gcd(a, b) = 1$.

3. The Positions of Sprague-Grundy Value 1

In this section, we first give some basic results on the Sprague-Grundy function of the game G_2 before showing that the two games G_1 and G_2 have the same set of positions of Sprague-Grundy value 1. We then show that there is a bridge between the set of $\mathcal{P}$-positions and the set of positions of Sprague-Grundy value 1. Next, we present two theorems on the periodicity of positions of Sprague-Grundy value 1. Finally, we solve the Sprague-Grundy function for some special cases.

Lemma 3.1. *Let $a \le b \le c$. In the game G_2, if $\mathcal{G}(a, b, c) = 1$ then $(a, b, c - a) \in \mathcal{P}$.*

Proof. Assume by contradiction that $(a, b, c - a) \notin \mathcal{P}$. Then $\mathcal{G}(a, b, c - a) > 1$, and so there exists a move from $(a, b, c - a)$ to some position p of Sprague-Grundy value 1. If $c - a \ge b$, p must be of the form $(a, b, c - a - ia)$ for some $i \ge 1$. This is a contradiction as there exists a move from (a, b, c) to p. Therefore, $c - a < b$. Moreover, since $\mathcal{G}(a, b, c) = 1$, there exists a move from (a, b, c) to some $\mathcal{P}$-position q. So q must be of the form $(a, b, c - ia)$ for some $i \ge 1$. Since $(a, b, c - a) \notin \mathcal{P}$, we have $i \ge 2$, and so $a + c - ia \le c - a < b$. However, by Lemma 2.1, $a + c - ia > b$ giving a contradiction. Therefore, $(a, b, c - a) \in \mathcal{P}$. □

Lemma 3.2. *Let $a \le b \le c$. In the game G_2, if $\mathcal{G}(a, b, c) = 1$ then $c < 2a + b$.*

Proof. This is immediate from Lemmas 2.1 and 3.1. □

Lemma 3.3. *Let $a \le b \le c$. In the game G_2,*

(i) *if $c > a + b$ then $\mathcal{G}(a, b, c) \ge 2$;*

(ii) *if $c < a + b$ then $\mathcal{G}(a, b, c) \le 1$.*

Proof. The proof is by induction on c. One can check by hand that the lemma is true for $c \le 3$. Assume that the lemma is true for $c \le n$ for some $n \ge 3$, we will show that the lemma is true for $c = n + 1$. Note that if $b = c$ then the lemma is true by Corollary 2.5. Therefore, it is sufficient to prove the lemma for $a \le b < c = n + 1$.

For (i), assume by contradiction that there exist $a \leq b$ such that $a+b < c$ and $\mathcal{G}(a, b, c) \leq 1$. By Lemma 2.1, we have $\mathcal{G}(a, b, c) \neq 0$ and so $\mathcal{G}(a, b, c) = 1$. By Lemma 3.1, we have $(a, b, c - a) \in \mathcal{P}$. Since $c > a+b$, we have $c-2a > 0$. As $\mathcal{G}(a, b, c) = 1$, by Lemma 3.2, we have $c-2a < b$. Consider the position $(a, c-2a, b)$. We have $a+c-2a = c-a > b$. Note that $b \leq n$. By the inductive hypothesis on (ii), we have $\mathcal{G}(a, c-2a, b) \leq 1$. Note that there exists a move from (a, b, c) whose Sprague-Grundy value is 1 to $(a, c-2a, b)$ and so $\mathcal{G}(a, c-2a, b) \neq 1$. Also note that there exists a move from $(a, b, c-a) \in \mathcal{P}$ to $(a, c-2a, b)$ and so $\mathcal{G}(a, c-2a, b) \neq 0$, giving a contradiction. Thus, (i) is true for $c = n+1$.

For (ii), assume by contradiction that there exist a, b such that $a \leq b < c$, $c < a+b$ and $\mathcal{G}(a, b, c) \geq 2$. Then there exist integers $i, j \geq 1$ such that $\mathcal{G}(a, b, c-ia) = 0$ and $\mathcal{G}(a, b, c-ja) = 1$. Since $\mathcal{G}(a, b, c-ia) = 0$, by Lemma 2.1, $a+c-ia > b$. Note that $c-a < b$ and so $i = 1$. This also implies that $j \geq 2$. We claim that $j = 2$. Assume by contradiction that $j \geq 3$. Then $2a+c-ja \leq c-a < b$. If $c-ja < a$, since $\mathcal{G}(c-ja, a, b) = 1$, by Lemma 3.2, $2(c-ja)+a > b$. However, $c-ja < a$ also implies $2(c-ja)+a < 2a+c-ja$ giving a contradiction. If $a \leq c-ja$, since $\mathcal{G}(a, c-ja, b) = 1$, by Lemma 3.2, $2a+c-ja > b$ giving a contradiction. Therefore, $j = 2$. Note that $c-2a < b$. Now consider the position $(a, c-2a, b)$. Since $a+c-2a = c-a < b$ and $b < n+1$, by the inductive hypothesis on (i), we have $\mathcal{G}(a, c-2a, b) \geq 2$ giving a contradiction. Therefore, (ii) is true for $c = n+1$.

Thus, by the inductive principle, the lemma is true. □

Question 1. Let $a \leq b$ be positive integers. What is the relationship between a, b so that $\mathcal{G}(a, b, a+b) = 1$?

By the part (i) of Lemma 3.3, we have a result stronger than Lemma 3.2 as follows.

Corollary 3.4. *Let $a \leq b \leq c$. In the game G_2, if $\mathcal{G}(a, b, c) = 1$ then $c \leq a+b$.*

We are now in the position to show that all results early in this section are also true for the game G_1.

Theorem 3.5. *The positions of Sprague-Grundy value 1 in the game G_2 are identical to those in the game G_1.*

Proof. Let A be the set of positions of Sprague-Grundy value 1 in the game G_2. We show that the following two properties hold for G_1:

(i) there is no move from a position in A to a position in A,

(ii) from every position that is not in $\mathcal{P} \cup A$, there is a move to a position in A.

Note that every move in G_2 is also legal in G_1 and so (ii) holds for G_1. Assume by contradiction that (i) does not hold for G_1. Then, there exist a position $p = (a, b, c) \in A$ with $a \leq b \leq c$ and a 1-2 move from p to some position $q \in A$. This implies that $q = (a, b-ia, c)$ for some $i \geq 1$. By Corollary 3.4, we have $c \leq a+b-ia \leq b$ implying $c = b$. But if $c = b$ then, by Corollary 2.5, $(a, b, c) \in \mathcal{P}$ giving a contradiction. Therefore (i) holds. □

In the next part, we find some connections between the $\mathcal{P}$-positions and the set of positions of Sprague-Grundy value 1.

Corollary 3.6. *Let $a \leq b \leq c$. For the two games G_1 and G_2, if $\mathcal{G}(a, b, c) = 1$ then $(a, b, c-a) \in \mathcal{P}$ and $(a, c, b+a) \in \mathcal{P}$.*

Proof. We work with the game G_2. By Lemma 3.1 and Theorem 3.5, $(a,b,c-a) \in \mathcal{P}$. So it remains to show that $(a,c,b+a) \in \mathcal{P}$. Note that $c \leq a+b$ by Corollary 3.4. Assume by contradiction that $(a,c,b+a) \notin \mathcal{P}$. Then there exists a move from $(a,c,b+a)$ to some $\mathcal{P}$-position p. So p must be of the form $(a,c,b+a-ia)$ for some $i \geq 1$. Since $\mathcal{G}(a,b,c) = 1$, we must have $i \geq 2$, and so $a+b+a-ia \leq b \leq c$. But as $(a,c,b+a-ia) \in \mathcal{P}$, by Lemma 2.1, we have $a+b+a-ia > c$ giving a contradiction. Therefore, $(a,c,b+a) \in \mathcal{P}$. □

The following corollary is the converse of Corollary 3.6 when $b < c$.

Corollary 3.7. *Let $a \leq b < c$. If $(a,b,c) \in \mathcal{P}$, then $\mathcal{G}(a,c-a,b) = 1$ and $\mathcal{G}(a,c,b+a) = 1$.*

Proof. By Lemma 2.1, we have $a+b > c$. We first prove that $\mathcal{G}(a,c-a,b) = 1$. Assume by contradiction that $\mathcal{G}(a,c-a,b) \neq 1$. Then $\mathcal{G}(a,c-a,b) > 1$. By Lemma 3.3, $b \geq a+c-a = c$ giving a contradiction. Therefore, $\mathcal{G}(a,c-a,b) = 1$.

We now prove that $\mathcal{G}(a,c,b+a) = 1$. Assume by contradiction that $\mathcal{G}(a,c,b+a) \neq 1$. Then $\mathcal{G}(a,c,b+a) > 1$. There exists a move from $(a,c,b+a)$ to some position p of Sprague-Grundy value 1. So, working in G_2, p must be of the form $(a,c,b+a-ia)$ for some $i \geq 1$. Since $p \neq (a,b,c)$, we have $i \geq 2$ and so $a+b+a-ia \leq b < c$. But by Lemma 3.3, $\mathcal{G}(a,c,b+a-ia) \geq 2$ giving a contradiction. Therefore, $\mathcal{G}(a,c,b+a) = 1$. □

Note that Corollary 3.7 does not hold for $b = c$. For example, for $a = 1, b = 3$, we have $\mathcal{G}(a,b,b) = 0$ for both games but $\mathcal{G}(a,b-a,b) = 3$ for the game G_1 and $\mathcal{G}(a,b-a,b) = 2$ for the game G_2. For $a = 1, b = 2$, we have $\mathcal{G}(a,b,b) = 0$ for both games but $\mathcal{G}(a,b,b+a) = 3$ for the game G_1 and $\mathcal{G}(a,b,b+a) = 2$ for the game G_2. An answer for Question 1 would also provide a criteria for the case $b = c$ to hold in Corollary 3.7.

We now present two theorems giving periodicity of positions of Sprague-Grundy value 1 of the forms (a,b,c) with $a+b > c$ and of the form $(a,b,a+b)$.

Theorem 3.8. *Let $a \leq b \leq c$ and $a+b > c$. Then $\mathcal{G}(a,b,c) = 1$ if and only if $\mathcal{G}(a,b+a,c+a) = 1$.*

Proof. Assume that $\mathcal{G}(a,b,c) = 1$. By Corollary 3.6, we have $(a,c,b+a) \in \mathcal{P}$. By Corollary 3.7, we have $\mathcal{G}(a,b+a,c+a) = 1$. Now assume that $\mathcal{G}(a,b+a,c+a) = 1$. By Corollary 3.6, we have $(a,c,b+a) \in \mathcal{P}$. By Corollary 3.7, we have $\mathcal{G}(a,b,c) = 1$. □

The previous theorem does not hold for $c = a+b$. For example, $\mathcal{G}(1,3,4) = 1$, but $\mathcal{G}(1,4,5) = 2$ for the game G_2. Nevertheless, one has the following.

Theorem 3.9. *Let $a \leq b$. Then $\mathcal{G}(a,b,a+b) = 1$ if and only if $\mathcal{G}(a,b+2a,b+3a) = 1$.*

Proof. It is sufficient to prove that the theorem holds for the game G_2. We first prove the necessary condition. Assume by contradiction that $\mathcal{G}(a,b+2a,b+3a) \neq 1$. By Lemma 2.1, $\mathcal{G}(a,b+2a,b+3a) \neq 0$. Then, $\mathcal{G}(a,b+2a,b+3a) \geq 2$ and so there exists a move from $(a,b+2a,b+3a)$ to some position p of Sprague-Grundy value 1. So p must be of the form $(a,b+2a,b+3a-ia)$ for some $i \geq 1$. We claim that $i = 2$. By Corollary 2.5, $(a,b+2a,b+2a) \in \mathcal{P}$, so $i \neq 1$. If $i \geq 3$, then $a+b+3a-ia \leq b+a < b+2a$ and so $\mathcal{G}(a,b+2a,b+3a-ia) \geq 2$ by Lemma 3.3. So $i = 2$. But this is impossible as there exists a move from p to the position $(a,b,a+b)$. Therefore, $\mathcal{G}(a,b+2a,b+3a) = 1$.

Conversely, assume that $\mathcal{G}(a,b+2a,b+3a) = 1$. Then $\mathcal{G}(a,b+a,b+2a) \neq 1$. Note that $(a,b+a,b+a) \in \mathcal{P}$ by Corollary 2.5. Since there exists a move from $(a,b+a,b+2a)$ to $(a,b+a,b+a)$, $\mathcal{G}(a,b+a,b+2a) \geq 2$. It follows that there exists a move from $(a,b+a,b+2a)$

to some position q of Sprague-Grundy value 1. So q must be of the form $(a, b+a, b+2a-ia)$ for some $i \geq 1$. We claim that $i = 2$. Note that $i \neq 1$ as $(a, b+a, b+a) \in \mathcal{P}$. If $i \geq 3$ then $a+b+2a-ia \leq b < b+a$, and so, by Lemma 3.3, we have $\mathcal{G}(q) \geq 2$ giving a contradiction. Hence $i = 2$ and $\mathcal{G}(a, b, b+a) = 1$. □

We now present some special classes of positions of Sprague-Grundy value 1. The following corollary follows from the above theorem by induction on a.

Corollary 3.10. $\mathcal{G}(1, a, a+1) = 1$ *if and only if a is odd.*

Proposition 3.11. *(i) Let $2 \leq a \leq b$. Then $\mathcal{G}(a, b-1, b) = 1$ if and only if $\gcd(a, b) = 1$.*
(ii) Let $2 \leq a < b$. Then $\mathcal{G}(a, b-a, b-1) = 1$ if and only if $\gcd(a, b) \neq 1$.

Proof. Note that (ii) follows from (i) by Corollary 3.6 and Lemma 3.3. Therefore, it is sufficient to prove (i). We prove (i) by induction on b. It can be checked that (i) holds for $b = 2, 3$. Assume that (i) holds for $b \leq n$ for some $n \geq 3$. We now show that (i) holds for $b = n+1$. If $a = b$ then $\mathcal{G}(a, b-1, b) = 0$ by Corollary 2.5. Therefore, (i) holds for $a = b$. We claim that (i) also holds for $a = b-1$. Note that $\mathcal{G}(1, a, a) = 0$ by Corollary 2.5 and the only move from $(a, a, a+1)$ is to $(1, a, a)$. Therefore, $\mathcal{G}(a, a, a+1) = 1$ by definition. Also note that $\gcd(a, a+1) = 1$. Therefore, (i) holds for $a = b-1$. Let $2 \leq a \leq b-2$. We show that $\mathcal{G}(a, b-1, b) = 1$ if and only if $\gcd(a, b) = 1$.

Suppose that $\mathcal{G}(a, b-1, b) = 1$. By Corollary 3.6, $\mathcal{G}(a, b-a, b-1) = 0$. We compare a with $b-a$. We claim that $a \neq b-a$. In fact, if $a = b-a$ then $\mathcal{G}(b-1-a, a, b-a) = 0$ by Corollary 2.5, but there exists a move from $(a, b-a, b-1)$ to $(b-1-a, a, b-a)$. This is impossible. Hence $a \neq b-a$. If $a < b-a$, then $\mathcal{G}(a, b-1-a, b-a) = 1$ by Corollary 3.7. Since $b-a < n$, the inductive hypothesis gives $\gcd(a, b-a) = 1$, and so $\gcd(a, b) = 1$. If $b-a < a$, then $\mathcal{G}(b-a, a, b-1-(b-a)) = 1$ by Corollary 3.7; that is, $\mathcal{G}(b-a, a-1, a) = 1$. Since $a < n$ and $b-a \geq 2$, the inductive hypothesis gives $\gcd(b-a, a) = 1$ and so $\gcd(a, b) = 1$.

Conversely, suppose that $\gcd(a, b) = 1$. Assume by contradiction that $\mathcal{G}(a, b-1, b) \neq 1$. Then $\mathcal{G}(a, b-1, b) = 0$ by Lemma 3.3. By Corollary 3.7, we have $\mathcal{G}(a, b-a, b-1) = 1$. We compare a with $b-a$. Note that $a \neq b-a$ as $\gcd(a, b) = 1$. If $a < b-a$, then $\mathcal{G}(a, b-1-a, b-a) = 0$ by Corollary 3.6. As $b-a < n$, by the inductive hypothesis, $\gcd(a, b-a) \neq 1$ giving a contradiction. If $b-a < a$, then $\mathcal{G}(b-a, a, b-1-(b-a)) = 0$ by Corollary 3.6; that is, $\mathcal{G}(b-a, a-1, a) = 0$. Since $a < n$ and $b-a \geq 2$, by the inductive hypothesis, $\gcd(b-a, a) \neq 1$, and so $\gcd(a, b) \neq 1$ giving a contradiction. Therefore, $\mathcal{G}(a, b-1, b) = 1$. □

As promised at the end of the previous section, we now give the following.

Proof of Proposition 2.6. For $a \geq 2$, Proposition 2.6 follows immediately from Proposition 3.11 and Lemma 3.3. For $a = 1$, Proposition 2.6(ii) follows from Corollary 2.5. □

4. On the Partition of Sprague-Grundy Values

This section extends a result from Collins and Lengyel's work on the game 3-Euclid. Let $a \leq b$, and let s be a nonnegative integer. We answer the question as to whether there exists a positive integer c such that $\mathcal{G}(a, b, c) = s$ and whether such an existence is unique.

Theorem 4.1. *Let $a \leq b$, and let s be a nonnegative integer. For the game G_1, there exists exactly one integer $c \geq 1$ in each residue class $r \pmod{a}$ such that $\mathcal{G}(a, b, c) = s$.*

Proof. The result holds for $s = 0$ by Lemma 2.3. So we may assume that $s > 0$.

For the uniqueness, assume by contradiction that there are two positive integers $c_1 < c_2$ in the residue class $r \pmod a$ such that $\mathcal{G}(a, b, c_1) = \mathcal{G}(a, b, c_2)$. Since c_1, c_2 are in the same residue class modulo a, we have $c_2 > a$. If $c_2 < b$ then there exists a 1-2 move from (a, c_2, b) to (a, c_1, b). This is a contradiction. If $c_2 \geq b$ then there exists a 1–3 move from (a, b, c_2) to (a, b, c_1). This is a contradiction. Therefore, the uniqueness holds.

For the existence, assume by contradiction that there is no integer c in the residue class $r \pmod a$ such that $\mathcal{G}(a, b, c) = s$. Let $m = \lfloor b/a \rfloor$ and let d be an integer such that $d \geq b$ and $d \equiv r \pmod a$. Let $b_0 = b - ma$. There are two cases for b_0.

If $b_0 = 0$, we consider the sequence of positions

$$(a, b, d + a), (a, b, d + 2a), \ldots, (a, b, d + (s + m)a). \tag{4.1}$$

Note that each pair of these $s + m$ positions has distinct Sprague-Grundy values, and so there are at most s positions having Sprague-Grundy values from 0 to $s - 1$. Consequently, there are, in that sequence, at least m positions having Sprague-Grundy values more than s. Assume that these m positions are

$$(a, b, d + i_1 a), (a, b, d + i_2 a), \ldots, (a, b, d + i_m a). \tag{4.2}$$

From each position $(a, b, d + i_j a)$, there exists a move to some position p_j of Sprague-Grundy value s. This move must be a 1-2 move, and so p_j must be of the form $(a, b - k_j a, d + i_j a)$. Since there are at most $m - 1$ values for k_j (as $b - k_j a > 0$) while there are m positions p_j, there must be two distinct positions $p_{j_1} = (a, b - k_{j_1} a, d + i_{j_1} a)$ and $p_{j_2} = (a, b - k_{j_2} a, d + i_{j_2} a)$ having the same Sprague-Grundy value s in which $k_{j_1} = k_{j_2}$ and $i_{j_1} \neq i_{j_2}$. This is a contradiction as there is a 1–3 move (as $b - k_j a \geq a$) from one of the two positions p_{j_1}, p_{j_2} to the other.

If $b_0 > 0$, we consider the sequence of positions

$$(a, b, d + b_0 a), (a, b, d + 2b_0 a), \ldots, (a, b, d + (s + m + 1)b_0 a). \tag{4.3}$$

This sequence contains at least $m + 1$ positions having Sprague-Grundy values greater than s. Assume that these $m + 1$ positions are

$$(a, b, d + i_1 b_0 a), (a, b, d + i_2 b_0 a), \ldots, (a, b, d + i_{m+1} b_0 a). \tag{4.4}$$

From each position $(a, b, d + i_j b_0 a)$, there exists a move to some position q_j of Sprague-Grundy value s. This move must be a 1-2 move and so q_j must be of the form $(a, b - k_j a, d + i_j b_0 a)$. Since there are at most m values for k_j while there are $m + 1$ positions q_j, there must be two distinct positions $q_{j_1} = (a, b - k_{j_1} a, d + i_{j_1} b_0 a)$ and $q_{j_2} = (a, b - k_{j_2} a, d + i_{j_2} b_0 a)$ having the same Sprague-Grundy value s in which $k_{j_1} = k_{j_2}$ and $i_{j_1} \neq i_{j_2}$. If $b - k_{j_1} a \geq a$ then there exists a 1–3 move from one of two positions q_{j_1}, q_{j_2} to the other. This is a contradiction. Therefore, $b - k_{j_1} a < a$ and so $b - k_{j_1} a = b_0$, and the two integers $d + i_{j_1} b_0 a$ and $d + i_{j_2} b_0 a$ are in the same residue class $d \pmod{b_0}$. Thus, there are two distinct integers $d + i_{j_1} b_0 a, d + i_{j_2} b_0 a$ in the residue class $d \pmod{b_0}$ satisfying $\mathcal{G}(b_0, a, d + i_{j_1} b_0 a) = \mathcal{G}(b_0, a, d + i_{j_2} b_0 a)$. This contradicts the uniqueness part of the theorem. Therefore, the existence holds. □

Theorem 4.2. *Let $a \le b$, and let s be a nonnegative integer. For the game G_2, in each residue class $r(\operatorname{mod} a)$, there exists at least one integer $c \ge 1$ such that $\mathcal{G}(a, b, c) = s$.*

Proof. By Lemma 2.3, we may assume that $s > 0$. Let d be an integer such that $d \ge b$ and $d \equiv r \bmod a$. Consider the sequence of positions

$$(a, b, d), (a, b, d + a), (a, b, d + 2a), \ldots, (a, b, d + sa). \tag{4.5}$$

Note that each pair of these $s+1$ positions has distinct Sprague-Grundy values and so there are at most s positions having Sprague-Grundy values from 0 to $s - 1$. Therefore, if none of these positions has Sprague-Grundy value s then there exists at least one of these positions having Sprague-Grundy value more than s, say $(a, b, d + ia)$. Then there exists a 1–3 move from $(a, b, d + ia)$ to some position $\mathcal{G}(a, b, d + (i - j)a)$ of Sprague-Grundy value s, as required. □

In Theorem 4.2, the existence of c is not unique. For example, the three positions $(4, 4, 19), (4, 8, 19)$, and $(4, 12, 19)$ have the same Sprague-Grundy value 4. Whenever the existence of the value c is not unique, we have the following observation on the values c for the game G_2. We have checked this result as far as $b = 100$.

Observation 1. Let $a \le b$. In the game G_2, if there are two values c_1, c_2 in the same residue class modulo a such that $\mathcal{G}(a, b, c_1) = \mathcal{G}(a, b, c_2)$ then $a \le c_1, c_2 < b, b \ge 3a$, and c_1, c_2 are both multiples of a.

5. Miserability

In this section, we examine the misère versions of two games G_1 and G_2. Recall that an impartial game is under misère convention if the player making the last move loses. A game can be described by a finite directed acyclic graph Γ without multiple edges in which each vertex is a position, and there is a downward edge from p to q if and only if there is a move from the position p to the position q. Moreover, the graph can be assumed to have only one source. A source is a vertex with no incoming edges. The source of the graph is the original position of the game. The sinks are the vertices with no outgoing edges, so the sinks of the graph are the final (terminal) positions of the game. For convenience, a graph of a game is assumed to have precisely one sink. This is because when the graph has more than one sink, they can be coalesced together into one sink without changing the properties of the game.

Let G be an impartial game and Γ the corresponding digraph of the game G. The misère version of the game G can be considered as the graph obtained from Γ by adding an extra vertex v_0 and a move from the sink of Γ to v_0 [13]. In [14], a game is said to be *miserable* if its normal and misère versions are different only on some subset of positions of Sprague-Grundy values 0 and 1. More precisely, a game $\mathcal{G}$ is miserable if there exist subsets V_0 of $\mathcal{P}$-positions and V_1 of positions of Sprague-Grundy value 1 so that the two functions $\mathcal{G}$ and $\mathcal{G}^-$ swap on positions in V_0 and V_1 and are equal on other positions. Here $\mathcal{G}$ and $\mathcal{G}^-$ are the Sprague-Grundy functions for the game G and its misère version respectively. If V_0 is equal to the set of $\mathcal{P}$-positions and V_1 is equal to the set of positions of Sprague-Grundy value 1 then the game is said to be *strongly miserable*. We will show that the two games G_1 and G_2 are miserable but not strongly miserable. Before presenting this result, let us discuss some properties of positions of Sprague-Grundy values 0 and 1 of the misère versions. The proofs for the following results are similar to those in previous sections.

Lemma 5.1. *Let $a \le b \le c$. In the misère versions,*

(i) *if $c > a + b$ then $\mathcal{G}(a,b,c) \ge 2$;*

(ii) *if $c < a + b$ then $\mathcal{G}(a,b,c) \le 1$.*

Theorem 5.2. *The $\mathcal{P}$-positions of the misère version of G_2 are identical to those of the misère version of G_2.*

Theorem 5.3. *The positions of Sprague-Grundy value 1 in the misère version of G_2 are identical to those in the misère version of G_1.*

Some properties of periodicity of $\mathcal{P}$-positions and positions of Sprague-Grundy value 1 of the two games G_1 and G_2 are still true for their misère versions.

Theorem 5.4. *Let $a \le b \le c$ such that $(a,b,c) \ne (a,a,2a)$. In the misère versions of G_1 and G_2, if $(a,b,c) \in \mathcal{P}$ then $(a,b+a,c+a) \in \mathcal{P}$.*

Theorem 5.5. *Let $a \le b \le c$ such that $a+b > c$ and $(a,b,c) \ne (a,2a,2a)$. In the misère versions of G_1 and G_2, if (a,b,c) has Sprague-Grundy value 1 then $(a,b+a,c+a)$ has Sprague-Grundy value 1.*

Theorem 5.6. *Let $a \le b$. In the misère versions of G_1 and G_2, if $(a,b,a+b)$ has Sprague-Grundy value 1 then $(a,b+2a,b+3a)$ has Sprague-Grundy value 1.*

We now come back with the main result of this section. Note that the following theorem is also true for G_2 by Theorems 5.2 and 5.3.

Theorem 5.7. *The game G_1 is miserable.*

Proof. Consider the graph Γ_1^- with the sink v_0 of the misère version. For each vertex (position) v, the *height* $h(v)$ of v is the length of the longest directed path from v to the sink v_0. We will prove by induction on $h(v)$ that if either $\mathcal{G}_1(v) \ge 2$ or $\mathcal{G}_1^-(v) \ge 2$ then $\mathcal{G}_1^-(v) = \mathcal{G}_1(v)$. Note that the claim is true for $h(v) = 1$. Assume that the claim is true for $h(v) \le n$ for some $n \ge 1$. We show that the claim is true for $h(v) = n+1$.

We first assume that $\mathcal{G}_1(v) \ge 2$. For each $k < \mathcal{G}_1(v)$, there exists w_k such that $\mathcal{G}_1(w_k) = k$ and one can move from v to w_k. By the inductive hypothesis, $\mathcal{G}_1^-(w_0), \mathcal{G}_1^-(w_1) \le 1$, and if there is a move from v to some w with $\mathcal{G}_1(w) \ge 2$ then $\mathcal{G}_1^-(w) = \mathcal{G}_1(w)$. It remains to show that $\mathcal{G}_1^-(w_0) \ne \mathcal{G}_1^-(w_1)$. We show that there is a move between w_0, w_1 in the misère version of G_1. Assume that $v = (a,b,c)$. There are three following possibilities for moves from v to w_0, w_1:

(i) $w_0 = (a,b,c-ia), w_1 = (a,b,c-ja)$;

(ii) $w_0 = (a,b-ia,c), w_1 = (a,b-ja,c)$;

(iii) either $w_0 = (a,b,c-ia), w_1 = (a,b-ja,c)$ or $w_0 = (a,b-ia,c), w_1 = (a,b,c-ja)$.

One can check that there is a move between w_0, w_1 for the first two possibilities. We show that the third possibility does not occur. In fact, consider the assumption $w_0 = (a,b,c-ia), w_1 = (a,b-ja,c)$ (the other case can be treated similarly). By Lemma 5.1, we have $a+b-ja \le c$, and so $j = 1, b = c$. Note that $a + c - ia \geq b$ by Lemma 5.1 and so $i = 1$. Consequently, $w_0 = w_1$ giving a contradiction. Hence, $\mathcal{G}_1^-(v) = \mathcal{G}_1(v)$. Similarly, we can show that if $\mathcal{G}_1^-(v) \geq 2$ then $\mathcal{G}_1(v) = \mathcal{G}_1^-(v)$. This completes the proof. □

Note that the game G_1 (and so G_2) is not strongly miserable. A counterexample is the position (1, 3, 3) which is a $\mathcal{P}$-position in G_1 and its misère version.

Acknowledgment

The author thanks the referee for his/her helpful comments and references. He thank his supervisor, Dr Grant Cairns, for careful reading of this paper.

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