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Research Article

Stability of Compressible Hollow Jet Pervaded by a Transverse Varying Magnetic Field

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The magnetohydrodynamic stability of an ordinary compressible hollow cylinder pervaded by a transverse varying magnetic field, under the influence of capillary, inertia, and Lorentz force, has been developed. The problem is modelized. The basic equations formulated, solved, and, upon applying appropriate boundary conditions, the singular solutions are excluded. The eigenvalue relation has been derived and discussed. The capillary force has destabilizing influence only for long wavelengths in the axisymmetric perturbation but it is stabilizing in the rest and also so in the nonaxisymmetric perturbations. The compressibility increases the stable domains and simultaneously decreases those of instability. The electromagnetic force has different effects due to the axial uniform field and varying transverse one. The axial field is stabilizing for all wavelengths in all kinds of perturbations. The transverse field is stabilizing or not according to restrictions. Here, the high compressibility increases rapidly the magnetodynamic stable domains and leads to shrinking those of instability.

1. Introduction

The stability of a fluid cylinder subjected to different external forces has been published and reported by Rayleigh [1] and Chandrasekhar [2]. The instability of the mirror case of a gas cylinder surrounded by an ideal fluid endowed with surface tension is envised by

Chandrasekhar [2] for axisymmetric perturbation. See also Drazin and Reid [3] and Cheng [4]. Kendall [5] made very neat experiments to check the breaking-up of this model due to its instability as the inertia force of the exterior liquid is predominant over that of the gas cylinder. Moreover, he did write about its applications in several domains of science and attracted the attention of researchers for elaborating the stability of this model. One has to infer here that the analytical results given by Cheng [4], in (2.4) and (2.5), are incorrect in the third term. In fact, the term $(1-s^2-k^2R_0^2)$ must be in the numerator as it is clear from (2.3). See also (5.1)–(6.1) in the present work and Drazin and Reid's result [3, page 16], and also Chandrasekhar's dispersion relation [2, page 538 and page 540], ((147) and (155) there). For other works extending some of the previous results, we may refer to Radwan [6–9]. In all these foregoing studies, the fluids are considered to be incompressible and nonviscous. See also Chen and Lin [10].

In reality, in the model of a gas jet injected into a liquid, we have to consider the fluid is compressible. In [11, 12], Radwan et al. tried to investigate the instability of that model pervaded by constant magnetic field.

The stability of different cylindrical models under the action of selfgravitating force in addition to other forces has been elaborated by Radwan and Hasan [13, 14]. Hasan [15] has discussed the stability of oscillating streaming fluid cylinder subjected to combined effect of the capillary, selfgravitating, and electrodynamic forces for all axisymmetric and nonaxisymmetric perturbation modes.

There are many applications of magnetohydrodynamic stability in several fields of science such as the following.

(i) Geophysics: the fluid of the core of the Earth and other theorized to be a huge MHD dynamo that generates the Earth's magnetic field due to the motion of the liquid iron.

(ii) Astrophysics: MHD applies quite well to astrophysics since 99% of baryonic matter content of the universe is made of plasma, including stars, the interplanetary medium, nebulae and jets, stability of spiral arm of galaxy, and so forth. Many astrophysical systems are not in local thermal equilibrium and, therefore, require an additional kinematic treatment to describe all phenomena within the system.

(iii) Engineering applications: there are many forms in engineering sciences include oil and gas extraction process if it is surrounded by electric field or magnetic field, gas and steam turbines, MHD power generation systems, magneto-flow meters, and so forth.

Here, we present a complete analysis of the stability of compressible hollow cylinder pervaded by a transverse varying magnetic field for all axisymmetric and nonaxisymmetric modes of perturbation where the fluid velocity is not solenoidal any more in the present analysis.

2. Formulation of the Problem

Consider a gas cylinder of radius R_0 surrounded by a non-viscous, compressible and perfectly conducting liquid. The liquid is pervaded by the uniform magnetic field $\underline{H}_0 = (0,\ 0,\ H_0)$ while the gas cylinder is penetrated by the transverse varying magnetic field $\underline{H}_0^g = (0,\ \beta H_0 r/R_0,\ 0)$, where β is a parameter satisfying certain restrictions and H_0 is the intensity of the magnetic field in the liquid region, see Figure 1. We shall use the cylindrical coordinates

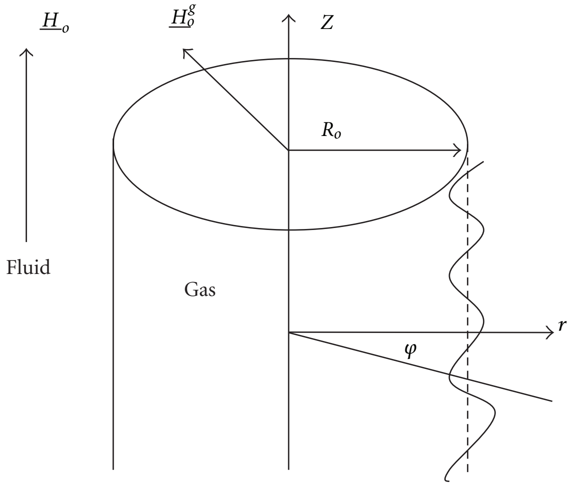

Figure 1: Sketch for MHD hollow jet.

$(r,\ \varphi, z)$ with the z-axis coinciding with the axis of the gas cylinder; the components of $\underline{H}_0$ and $\underline{H}_0^g$ are considered along the coordinates $(r,\ \varphi, z)$. The model is acting upon the electromagnetic, capillary, inertia and pressure gradient forces such that the liquid inertia force is paramount over that of the gas cylinder. The fundamental equations for such a study are the combination of the ordinary fluid dynamics, Maxwell electromagnetic equation and those of the perfect gas. Under the present circumstances these equations are given as follows.

Equations of motion:

$$\frac{\partial u_r}{\partial t} + (\underline{u} \cdot \nabla) u_r - \frac{\mu}{\rho}(\underline{H} \cdot \nabla) H_r = -\frac{\partial \Pi}{\partial r}, \tag{2.1}$$

$$\frac{\partial u_\varphi}{\partial t} + (\underline{u} \cdot \nabla) u_\varphi - \frac{\mu}{\rho}(\underline{H} \cdot \nabla) H_\varphi = -\frac{1}{r}\frac{\partial \Pi}{\partial \varphi}, \tag{2.2}$$

$$\frac{\partial u_z}{\partial t} + (\underline{u} \cdot \nabla) u_z - \frac{\mu}{\rho}(\underline{H} \cdot \nabla) H_z = -\frac{\partial \Pi}{\partial z}, \tag{2.3}$$

where $\rho\Pi$ is the total magnetohydrodynamic pressure which is the sum of kinetic and magnetic pressures, given by

$$\rho\Pi = P + \left(\frac{\mu}{2}\right)(\underline{H} \cdot \underline{H}). \tag{2.4}$$

Equation of conservation of flux:

$$\nabla \cdot \underline{H} = 0. \tag{2.5}$$

Equation of conservation of mass of a compressible fluid:

$$\frac{\partial \rho}{\partial t} + (\underline{u} \cdot \nabla)\rho + \rho(\nabla \cdot \underline{u}) = 0. \tag{2.6}$$

Equation of state gives the relation between the pressure P and density ρ:

$$P = \rho R^c T, \tag{2.7}$$

R^c is a general constant of gas.

Equation of conservation of energy:

$$\rho C_v \left(\frac{\partial}{\partial t} + \underline{u} \cdot \nabla \right) T + P(\nabla \cdot \underline{u}) = 0. \tag{2.8}$$

Evaluation equations of a magnetic field (derived from the electromagnetic Maxwell equation) in a perfectly conducting fluid:

$$\frac{\partial H_r}{\partial t} + (\underline{u} \cdot \nabla) H_r + (\nabla \cdot \underline{u}) H_r = (\underline{H} \cdot \nabla) u_r, \tag{2.9}$$

$$\begin{gathered} \frac{\partial H_\varphi}{\partial t} + (\underline{u} \cdot \nabla) H_\varphi + (\nabla \cdot \underline{u}) H_\varphi = (\underline{H} \cdot \nabla) u_\varphi, \\ \frac{\partial H_z}{\partial t} + (\underline{u} \cdot \nabla) H_z + (\nabla \cdot \underline{u}) H_z = (\underline{H} \cdot \nabla) u_z. \end{gathered} \tag{2.10}$$

In the gas cylinder region, surrounded by a liquid, there is no current flow:

$$\nabla \cdot \underline{H}^g = 0, \tag{2.11}$$

$$\nabla \Lambda \underline{H}^g = 0. \tag{2.12}$$

Along the gas liquid interface, the curvature pressure P_s due to the capillary force is given by

$$P_s = S\left(\frac{1}{r_1} + \frac{1}{r_2} \right), \tag{2.13}$$

with

$$r_1^{-1} + r_2^{-1} = \nabla \cdot \underline{N}, \tag{2.14}$$

where S is the surface tension coeffient, while r_1 and r_2 are the principle radii of curvature of the gas fluid and $\underline{N}$ the unit outward drawn normal vector to the perturbed interface,

$F(r,\ \varphi,\ z, t) = 0$, given by

$$\underline{N} = \frac{\nabla F(r, \varphi, z, t)}{|\nabla F(r, \varphi z, t)|}. \tag{2.15}$$

Here, (u_r, u_φ, u_z) and (H_r, H_φ, H_z) are, respectively, the components of the velocity vector $\underline{u}$ of the fluid and the magnetic field intensity $\underline{H}$, while $\rho,\ P,\ C_v$, and T are the fluid mass density, kinetic pressure, specific heat at a constant volume, and temperature of the fluid, and $\underline{H}^g$ is the intensity of the magnetic field in the gas cylinder.

The unperturbed state is studied by simplifying the fundamental equation in view of $\underline{u}_0 = 0$, $\partial/\partial\varphi = 0$ and $\partial/\partial z = 0$ and integrating the resulting differential equations. Finally, by means of the continuity of the normal component of the stress tensor across the gas-fluid interface at $r = R_0$, the kinetic pressure of the fluid in the unperturbed state is obtained, namely,

$$P_0 = P_0^g + \left(\frac{\mu}{2}\right)\left(\beta^2 - 1\right)H_0^2 - \frac{S}{R_0}. \tag{2.16}$$

Equation (2.16) gives the kinetic pressure P_0 of the fluid and it is a simple linear combination of the contribution of the different forces effects.

(a) $-S/R_0$ is the contribution due to the capillary force.

(b) $(\mu H_0^2/2)\beta^2$ is the contribution due to the magnetodynamic force acting in the gas region.

(c) $(\mu H_0^2/2)(-1)$ is the contribution due to the magnetodynamic force acting in the fluid region.

(d) P_0^g is the gas kinetic pressure and it must be suitably strong otherwise the model collapses and the gas spreads into fluid region surrounding the gas cylinder.

One has to refer here that the net magnetodynamic force acting on the model has no contribution in the balance of the total pressure (cf. (2.16)) in the unperturbed state in the following cases:

(1) as $H_0 = 0$,

(2) as $H_0 \neq 0$, $\beta = 1$.

To maintain and keep the model without collapsing in the unperturbed state, it must be $P_0 > 0$, this means that the contribution of the capillary force S/R_0 must be less than the gas pressure P_0^g, otherwise, the model will collapse and the gas may spread through the fluid region.

In the general case, in order that $P_0 \geq 0$, the gas kinetic pressure P_0^g in the initial state must satisfy the restriction:

$$P_0^g \geq \frac{S}{R_0} + \frac{\mu H_0^2}{2}\left(1 - \beta^2\right). \tag{2.17}$$

Otherwise, the model collapses and will be a homogeneous fluid medium.

3. Perturbation Analysis

For small departures from the unperturbed state, due to the perturbation along the gas-fluid interface, every variable quantity $Q(r,\varphi,z,t)$ may be expressed as its unperturbed part plus a fluctuation part (see [11, 12], viz.)

$$Q(r,\varphi,z,t) = Q_0(r) + \delta(t)Q_1(r,\varphi,z). \tag{3.1}$$

Here, Q stands for each of $\underline{u}$, $\underline{H}$, ρ, P, T, and the perturbed cross-section radius distance of the gas-fluid interface. The amplitude δ of the perturbation is given by

$$\delta = \delta_0 \exp(\sigma t), \tag{3.2}$$

where $\delta_0(=\delta$ at $t = 0)$ is the initial amplitude and σ is the temporal amplification at any instant of time t.

Consider a sinusoidal wave along the gas-fluid interface, for a single Fourier term, the perturbed cylindrical radial distance is described by

$$r = R_0 + R_1, \tag{3.3}$$

with

$$R_1 = \delta_0 \exp\left[\sigma t + i(kz + m\varphi)\right] \tag{3.4}$$

being the elevation of the surface wave measured from the unperturbed position, where k and m are the axial and transverse wave numbers. From the view point of the foregoing expansions (3.1)–(3.4), the relevant perturbation equations of the present case could be deduced from the fundamental equations (2.1)–(2.15) as follows.

For the fluid surrounding the gas jet,

$$\sigma \underline{u}_1 - \left(\frac{\mu}{\rho_0}\right)(\underline{H}_0 \cdot \nabla)\underline{H}_1 = -\nabla\Pi_1, \tag{3.5}$$

$$\rho\Pi_1 = P_1 + \left(\frac{\mu}{2}\right)(2\underline{H}_0 \cdot \underline{H}_1), \tag{3.6}$$

$$\nabla \cdot \underline{H}_1 = 0, \tag{3.7}$$

$$\sigma\rho_1 + \rho_0(\nabla \cdot \underline{u}_1) = 0, \tag{3.8}$$

$$\rho_0 a^2(\nabla \cdot \underline{u}_1) = -\sigma P_1, \tag{3.9}$$

$$\frac{P_1}{P_0} = \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0}, \tag{3.10}$$

$$\sigma \underline{H}_1 = \nabla\Lambda(\underline{u}_1 \Lambda \underline{H}_0), \tag{3.11}$$

where a is the sound speed in the gas defined by $a = (\gamma P_0/\rho_0)^{1/2}$, and $\gamma = C_p/C_v$ is the ratio of specific heats of the fluids.

For the gas region surrounded by the fluid,

$$\nabla \cdot \underline{H}_1^g = 0, \tag{3.12}$$

$$\nabla \Lambda \underline{H}_1^g = 0 \quad \text{(there is no current)}. \tag{3.13}$$

Along the gas-liquid perturbed interface,

$$P_{1s} = \frac{S}{R_0^2}\left(R_1 + R_0^2 \frac{\partial^2 R_1}{\partial z^2} + \frac{\partial^2 R_1}{\partial \varphi^2}\right). \tag{3.14}$$

By the aid of the series expansion (3.1) and the time-space dependence (3.4), based on the linear perturbation technique, every physical quantity $Q(r, \varphi, z, t)$ could be expressed as

$$Q_1(r, \varphi, z, t) = Q_1^*(r) \exp[\sigma t + i(kz + m\varphi)]. \tag{3.15}$$

This means that any perturbed quantity could be expressed as an amplitude function of r times the space-time dependence $\exp[\sigma t + i(kz + m\varphi)]$.

Upon utilizing the expansion, (3.15), (3.5), and (3.11) yield

$$\left(\sigma + \frac{\Omega_A^2}{\sigma}\right)\underline{u}_1 = -\nabla \Pi_1 + \frac{i\Omega_A^2 P_1}{ka^2 \rho_0} \underline{e}_z, \tag{3.16}$$

$$\sigma \underline{H}_1 = ikH_0 \underline{u}_1 + \frac{\sigma H_0}{\rho_0 a^2} P_1 \underline{e}_z, \tag{3.17}$$

where $\underline{e}_z$ is a unit vector in the z-direction and Ω_A is the Alfven wave frequency defined in terms of H_0 as

$$\Omega_A = \left(\frac{\mu H_0^2 k^2}{\rho_0}\right)^{1/2}. \tag{3.18}$$

Combining the z-components of (3.16) and (3.17) which are

$$\begin{gathered} \sigma\left[1 + \left(\frac{\Omega_A}{\sigma}\right)^2\right] u_{1z} = -ik\Pi_1 + \frac{i\Omega_A^2}{ka^2 \rho_0} P_1, \\ \sigma H_{1z} = H_0\left(iku_{1z} + \frac{\sigma P_1}{a^2 \rho_0}\right), \end{gathered} \tag{3.19}$$

we obtain

$$H_{1z} = \frac{(ik)^2 H_0}{(\sigma^2 + \Omega_A^2)}\left[-\Pi_1 + \frac{\mu H_0^2}{a^2 \rho_0} P_1\right] + \frac{H_0}{a^2 \rho_0} P_1. \tag{3.20}$$

By substituting (3.6) into (3.20), we finally obtain

$$H_{1z} = \frac{(\sigma^2 + a^2k^2)H_0}{\sigma^2 a^2 \rho_0} P_1. \tag{3.21}$$

Again, inserting (3.21) into (3.6) yields

$$\Pi_1 = \left(\frac{\xi}{\rho_0}\right) P_1. \tag{3.22}$$

With

$$\xi = 1 + \frac{\mu H_0^2 (\sigma^2 + a^2k^2)}{\sigma^2 a^2 \rho_0}. \tag{3.23}$$

Consequently, the components u_{1r}, $u_{1\varphi}$, and u_{1z} of $\underline{u}_1$ are given, from the vector equation (3.16), by

$$\begin{aligned}
\left(\sigma^2 + \Omega_A^2\right) u_{1r} &= -\sigma \frac{\partial \Pi_1}{\partial r}, \\
\left(\sigma^2 + \Omega_A^2\right) u_{1\varphi} &= -\frac{im\sigma}{r} \Pi_1, \\
\left(\sigma^2 + \Omega_A^2\right) u_{1z} &= -\frac{ik\sigma\left(-a^2\xi + \mu H_0^2\right)}{a^2 \xi} \Pi_1.
\end{aligned} \tag{3.24}$$

By inserting (3.24) into the nonsolenoid equation (3.9) which is

$$\frac{1}{r}\frac{\partial (r u_{1r})}{\partial r} + \frac{1}{r}\frac{\partial u_{1\varphi}}{\partial \varphi} + \frac{\partial u_{1z}}{\partial z} = \frac{-\sigma P_1}{a^2 \rho_0}, \tag{3.25}$$

and by taking into account the space (φ, z)-dependence (3.15), we get

$$\frac{1}{r}\frac{d}{dr}\left(r \frac{d\Pi_1}{dr}\right) - \left(\frac{m^2}{r^2} + \eta^2\right)\Pi_1 = 0, \tag{3.26}$$

with

$$\eta^2 = k^2 + \frac{\sigma^2}{a^2 \xi}. \tag{3.27}$$

Equation (3.26) is an ordinary second-order differential equation; its solution is given in terms of the ordinary Bessel functions of order m with imaginary argument. For the problem under consideration, apart from the singular solution, the finite solution of (3.26) is given by

$$\Pi_1(r) = A K_m(\eta r). \tag{3.28}$$

Therefore, $\Pi_1(r,\varphi,z,t)$ is being

$$\Pi_1(r,\varphi,z,t) = AK_m(\eta r)\exp[\sigma t + i(kz + m\varphi)], \tag{3.29}$$

where A is a constant of integration to be determined, while $K_m(\eta r)$ is modified Bessel function of second kind of order m.

It is worthwhile to mention here that by means of (3.29), the components of $\underline{u}_1(= (u_{1r}, u_{1\varphi}, u_{1z}))$ and $\underline{H}_1(= (H_{1r}, H_{1\varphi}, H_{1z}))$ and also P_1 could be identified from (3.16), (3.17), and (3.22) explicitly.

Equation (3.13) means that the magnetic field $\underline{H}_1^g$ in the perturbation state can be derived by means of a scalar function that, by using (3.12), satisfies Laplace's equation. The latter, by means of the expansion (3.15), transforms to an ordinary second-order differential equation whose solution is given in terms of cylindrical functions. For the problem under consideration, the nonsingular solution gives $\underline{H}_1^g$ in the form:

$$\underline{H}_1^g = B\nabla[I_m(kr)\exp[\sigma t + i(kz + m\varphi)]], \tag{3.30}$$

where B is an arbitrary constant of integration to be identified, while $I_m(kr)$ is the modified Bessel function of the first kind of order m.

Based on the space dependence (3.15), the curvature pressure due to the capillary force along the gas-fluid interface is given, from (3.14), in the form:

$$P_{1s} = \frac{S}{R_0^2}\left(1 - m^2 - k^2R_0^2\right)R_1. \tag{3.31}$$

4. Boundary Conditions

The solution of the relevant perturbation equations (3.5)–(3.14) represented by (3.15)–(3.31) and the solution of the unperturbed system of equations represented by (2.16) must satisfy appropriate boundary conditions. Under the present circumstances for the problem at hand, these boundary conditions are given as follows.

(i) The normal component of the magnetic field must be continuous across the gas-fluid interface (3.3) at $r = R_0$. This condition may be formulated as follows:

$$\underline{N}_o \cdot \underline{H}_1 + \underline{N}_1 \cdot \underline{H}_o = \underline{N}_o \cdot \underline{H}_1^g + \underline{N}_1 \cdot \underline{H}_o^g, \tag{4.1}$$

where $\underline{N}(= \underline{N}_0 + \varepsilon\underline{N}_1)$ is a unit outward vector normal to the gas-fluid interface given by (2.15), with

$$F(r,\varphi,z,t) = r - R_0 - R_1 = 0 \tag{4.2}$$

being the equation of the perturbed interface. Then,

$$\underline{N}_o = (1,0,0), \qquad \underline{N}_1 = \left(0, \frac{-im}{R_o}, -ik\right)R_1. \tag{4.3}$$

By substituting from (4.3) and the required expressions concerning $\underline{H}_0$, $\underline{H}_1$, $\underline{H}_0^g$, and $\underline{H}_1^g$ into (4.1), we obtain

$$B = \frac{im\beta H_0}{xI'_m(x)}, \tag{4.4}$$

where $x(= kR_0)$ is the dimensionless longitudinal wave number.

(ii) The normal component u_r of the velocity $\underline{u}$ must be compatible with the velocity of the deformed gas-fluid interface (3.3) at $r = R_0$. This condition reads

$$\underline{N}_0 \cdot \underline{u}_1 + \underline{N}_1 \cdot \underline{u}_0 = \frac{\partial r}{\partial t}. \tag{4.5}$$

By substituting $\underline{u}_0$, $\underline{u}_1$, $\underline{N}_0$, $\underline{N}_1$,and r into (4.5), this yields

$$A = \frac{-(\sigma^2 + \Omega_A^2)}{\eta K'_m(\eta R_0)}. \tag{4.6}$$

As we seen, complete nonsingular solutions for the variables of the problem have been obtained. Here, for the aim of stability theory, one has to make one more step to identify the stability criterion upon applying some compatibility condition.

(iii) This compatibility condition states that the normal component of the total stress tensor concerning the kinetic and magnetic pressures must be discontinuous by the curvature pressure due to the capillary force, across the gas-fluid interface (3.3) at $r = R_0$.

Mathematically, this reads

$$\begin{aligned} &P_1 + R_1\frac{\partial P_0}{\partial r} + \left(\frac{\mu}{2}\right)\left[(2\underline{H}_0 \cdot \underline{H}_1) + R_1\frac{\partial}{\partial r}(\underline{H}_0 \cdot \underline{H}_0)\right] \\ &\quad = P_{1s} + \left(\frac{\mu}{2}\right)\left[\left(2\underline{H}_0^g \cdot \underline{H}_1^g\right) + R_1\frac{\partial}{\partial r}\left(\underline{H}_0^g \cdot \underline{H}_0^g\right)\right]. \end{aligned} \tag{4.7}$$

Upon substituting into the condition (4.7) about the different variables, the following eigenvalue relation is obtained:

$$\sigma^2 = \frac{-S}{\rho R_0^3}\left(1 - m^2 - x^2\right)\frac{yK'_m(y)}{K_m(y)} + \frac{\mu H_0^2}{\rho_o R_o^2}\left\{-x^2 + \left[-\beta^2 + (m\beta)^2\frac{I_m(x)}{xI'_m(x)}\right]\frac{yK'_m(y)}{K_m(y)}\right\}, \tag{4.8}$$

where $y(= \eta R_0)$ is the dimensionless longitudinal wave number due to compressibility.

5. Limiting Cases

Equation (4.8) is the required capillary instability eigenvalue relation for a compressible gas cylinder surrounded by a liquid and pervaded by a transverse varying magnetic field.

It relates the temporal amplification σ or alternatively the oscillation frequency ω (i.e., if $\sigma(= i\omega)$ is imaginary) with the dimensionless (ordinary) longitudinal x and y (compressible) wave numbers, azimuthal wave number m, the fundamental quantity $(S/\rho_0 R_0^3)^{-1/2}$, as well as $(\mu H_0^2/\rho_0 R_0^2)^{-1/2}$ as a unit of time, the modified Bessel functions I_m and K_m of the first and second kind of order m and their derivatives of different arguments, the parameter β of the magnetic field pervading the interior of the gas cylinder, and with the parameters S, ρ_0, R_0, μ, and H_0 of the problem.

Several reported works may be obtained as limiting cases from the general result (4.8).

For incompressible ($a \to \infty$), nonconducting ($\mu = 0$, $H_0 = 0$) fluid and axisymmetric perturbation $m = 0$, the relation (4.8) yields

$$\sigma^2 = -\frac{S}{\rho R_0^3}\left(\frac{xK_0'(x)}{K_0(x)}\right)\left(1 - x^2\right), \tag{5.1}$$

where ($a \to \infty$), we find ($y \to x$). Upon using the recurrence relation,

$$K_0'(x) = -K_1(x), \tag{5.2}$$

the relation (5.1) becomes

$$\sigma^2 = \frac{S}{\rho R_0^3}\left(\frac{xK_1(x)}{K_0(x)}\right)\left(1 - x^2\right). \tag{5.3}$$

The relation (5.3) is indicated by Chandrasekhar [2, page 540], for the first time as a dispersion relation for the mirror case of a full fluid cylinder surrounded by vacuum. For the discussion of this relation, we may refer to Radwan and Elazab [6] as we neglect the contribution of the viscosity there.

If we suppose that ($a \to \infty$) so ($y \to x$), $H_0 = 0$ for $m \neq 0$, the relation (4.8) gives

$$\sigma^2 = -\frac{S}{\rho R_0^3}\left(\frac{xK_m'(x)}{K_m(x)}\right)\left(1 - m^2 - x^2\right). \tag{5.4}$$

This relation coincides with the relation given by Drazin and Reid [3].

The capillary eigenvalue relation of a compressible hollow cylinder is given from (4.8) as $H_0 = 0$, in the form:

$$\sigma^2 = \frac{-S}{\rho R_0^3}\left(1 - m^2 - x^2\right)\frac{yK_m'(y)}{K_m(y)}. \tag{5.5}$$

The magnetodynamic eigenvalue relation of a compressible hollow cylinder is given from the general relation (4.8) as = 0, in the form

$$\sigma^2 = \frac{\mu H_0^2}{\rho_o R_o^2}\left\{-x^2 + \left[-\beta^2 + (m\beta)^2\frac{I_m(x)}{xI_m'(x)}\right]\frac{yK_m'(y)}{K_m(y)}\right\}. \tag{5.6}$$

6. Stability Discussions

The general eigenvalue relation (4.8) of the model under consideration is a quadratic relation in σ^2. Therefore, we have to distinguish between the following different cases.

(i) The model will be ordinary stable, as σ^2 is negative.

(ii) The model will be ordinary unstable, as σ^2 is positive.

(iii) The model will be marginally stable, as σ^2 is zero.

In order to judge such cases and investigate the stability of the present model, we have to write down about the character and behavior of the modified Bessel functions I_m and K_m and their derivatives.

Consider the recurrence relations (cf. [16]):

$$\begin{gathered} 2I'_m(x) = I_{m-1}(x) + I_{m+1}(x), \\ 2K'_m(x) = -K_{m-1}(x) - K_{m+1}(x). \end{gathered} \tag{6.1}$$

For each nonzero real value of x. Also $I_m(x)$ is positive definite and monotonic increasing, while $K_m(x)$ is monotonic decreasing but never negative:

$$I_m(x) > 0, \qquad K_m(x) > 0. \tag{6.2}$$

Therefore, we may see, for a nonzero real value of x, that

$$I'_m(x) > 0, \qquad K'_m(x) < 0. \tag{6.3}$$

In view of (3.27), we see that $y \to x$ as $a \to \infty$, where the fluid in this case is an incompressible one. Consequently, the stability of the hollow cylinder under the action of the combined effects of the capillary and electromagnetic forces could be discussed.

As the model of a hollow cylinder is acted by the capillary force while the effect of the electromagnetic force is neglected, the eigenvalue relation (4.8) reduces to (5.5). By means of the relation, the capillary stable and unstable regions of the hollow cylinder could be identified. By an appeal to the relations (6.1) and the inequalities (6.2) and (6.3), the sign of σ^2 depends on the sign of the quantity $(1 - m^2 - x^2)$. Henceforth, we have the different cases:

$$\begin{gathered} \left.\begin{cases} \sigma^{*2} = 0, & \text{as } x = 1, \\ \sigma^{*2} < 0 & \text{as } 1 < x < \infty \\ \sigma^{*2} > 0 & \text{as } 0 < x < 1, \end{cases}\right. \quad \text{for } m = 0, \\ \sigma^{*2} < 0, \quad \text{as } 0 \leq x < \infty, \quad \text{for } m \neq 0, \end{gathered} \tag{6.4}$$

where

$$\sigma^* = \sigma \left(\frac{S}{\rho_0 R_0^3} \right)^{-1/2} \tag{6.5}$$

is the dimensionless growth rate at the instant of time t.

This means that the hollow cylinder is stable in the case $1 \leq x < \infty$ for $m = 0$ and $0 \leq x < \infty$ for $m \geq 1$, while it is unstable only as $0 \leq x \leq 1$ for $m = 0$. This results, of course, as $a^{-1} \rightarrow 0$. However, for ordinary values of a: in such case, we have the modified Bessel functions with argument y which in turn is a function of the compressibility factor $a(= (\gamma P_0/\rho_0)^{1/2})$. Now, for $x \neq 0$, $y \neq 0$ and $y > x$, we have

$$
\begin{aligned}
I_m(y) &> I_m(x), \qquad K_m(x) > K_m(y), \\
I'_m(y) &> I'_m(x), \qquad K'_m(x) > K'_m(y).
\end{aligned}
\tag{6.6}
$$

In view of these inequalities, the discussion of the relation (5.5) reveals that the compressibility has a stabilizing effect in all ($m = 0$ and $m \neq 0$) modes of perturbation for all wavelengths. The stabilizing effect becomes stronger for high compressibility and, therefore, it acts to overcome the capillary destabilizing influence.

As the hollow cylinder model is acted upon the electromagnetic force due to the pervading magnetic fields in the initial states and pressure gradient forces as $S = 0$, the eigenvalue relation of such case is given, in its general form, by the relation (5.6). The effect of the axial magnetic field in the liquid region is represented by the term $-x^2$ following $\mu H_0^2/\rho_0 R_0^2$. It contributes as a negative part in σ^2, that is, σ is imaginary in this case. This means that it has a stabilizing effect on the model. The transverse varying magnetic field pervaded into the gas region is presented by the terms $-\beta^2 y K'_m(y)/K_m(y)$ and $(m^2\beta^2)(y I_m(x) K'_m(y)/x I'_m(x) K_m(y))$ following the natural quantity $\mu H_0^2/\rho_0 R_0^2$. In the axisymmetric mode, $m = 0$, the transverse magnetic field is purely destabilizing. In the non-axisymmetric modes, $m \geq 1$, the transverse magnetic field is purely destabilizing in the term $-\beta^2 y K'_m(y)/K_m(y)$ while it is stabilizing due to the other terms. Therefore, we conclude that the transverse magnetic field is stabilizing or destabilizing according to restrictions.

Consequently, the electromagnetic forces in the gas and liquid regions are strongly stabilizing the model in the axisymmetric mode $m = 0$, while they have a destabilizing influence in $m \geq 1$ modes in the gas region only. Also, in such magnetodynamic case the compressibility has a stabilizing tendency not only in $m = 0$ mode but also in the non-axisymmetric modes $m \geq 1$. Its influence is decreasing the magnetodynamic unstable domain in $m \geq 1$ and simultaneously increasing those of stability in $m = 0$ mode.

One has to mention here it is argued that when the effects of materials compressibility are considered, the growth rate value may be reduced in comparison with the incompressible case. This is due to the fact that compression absorbs some of the energy which would, otherwise, go into fluid motion and causes more instability. For this reason, it is stated that the compressibility has a stabilizing tendency.

It is worth to mention here that the stabilizing effect of the electromagnetic force $\mu(\nabla \wedge \underline{H}) \wedge \underline{H}$ may be interpreted as follows.

This force is interpreted as arising from the action on the fluid of Maxwell's stresses: a magnetic tension $\mu(\underline{H} \cdot \underline{H})/2$ per unit area along the magnetic lines of force and equal magnetic pressure acting in all the directions in the conducting fluid. Taking into account that the latter is not perpendicular to the magnetic lines of force and acting in all directions because the diffusion term is neglected in the evolution equations of the magnetic field (2.9). Due to these stresses, the lines of force are able to endow the fluid with a sort of rigidity.

7. Numerical Analysis

The main purpose of the numerical analysis is that one could determine exactly where are the MHD stable and unstable domains of the model under consideration influenced by the combined effects of the electromagnetic and capillary forces. In addition, we may identify the critical points which separate the stable and unstable domains for different values of the problem parameters.

To perform that, we have to write down the eigenvalue relation (4.8) in dimensionless form in the most important mode $m = 0$ of perturbation. Now, since the natural quantity $(S/\rho_0 R_0^3)^{-1/2}$ has a unit of $(\text{time})^{-1}$, we have $\sigma^* = \sigma(S/\rho_0 R_0^3)^{-1/2}$ which is the dimensionless growth rate. For $m = 0$, the relation (4.8) leads to

$$\sigma^{*2} = \left(1 - x^2\right)\frac{yK_1(y)}{K_0(y)} + \left(\frac{H_0}{H_s}\right)^2\left\{-x^2 + \beta^2\frac{yK_1(y)}{K_0(y)}\right\}, \tag{7.1}$$

where σ^* is the nondimensional growth rate.

If σ^{*2} is negative, we write $\sigma^* = i\omega^*$ (with $i = \sqrt{-1}$ being the imaginary factor) where $\omega/2\pi$ is the oscillation frequency. The notation $H_s^2 = S/\mu R_0$ is with H_s as a unit of magnetic field, and we have used the notation $I_0' = I_1$ and $K_0' = -K_1$. The dispersion relation (7.1) has been computed for all short and long wavelengths in the new form in relation (7.2) which is

$$\sigma^{*2} = \left(1 - x^2\right)\frac{y\ K_1(y)}{K_0(y)} + M\left\{-x^2 + \beta^2\frac{yK_1(y)}{K_0(y)}\right\} \tag{7.2}$$

such calculations have been elaborated for different values of β and M for regular values of y. The values of σ^* corresponding the unstable domains and those of corresponding the stable domains are collected, tabulated, and presented graphically.

It is found that there are many features of interest in this numerical analysis as we see in the following.

(i) For $\beta = 0.7$ and $x = 1$, see Figure 2. Corresponding to $M = 0.5, 1, 1.5, 2, 2.5$, and 3, it is found that the unstable domains are $0 < y < 1.6209$, $0 < y < 1.6209$, $0 < y < 1.6209$, $0 < y < 1.6209$, $0 < y < 1.6209$, and $0 < y < 1.6209$ while the neighbouring stable domains are $1.6209 \le y < \infty$, $1.6209 \le y < \infty$, $1.6209 \le y < \infty$, $1.6209 \le y < \infty$, $1.6209 \le y < \infty$, and $1.6209 \le x < \infty$ where the equalities correspond to the marginal stability states.

(ii) For $\beta = 0.8$ and $x = 1$, see Figure 3. Corresponding to $M = 0.5, 1, 1.5, 2, 2.5$, and 3, it is found that the unstable domains are $0 < y < 1.1383$, $0 < y < 1.1383$, $0 < y < 1.1383$, $0 < y < 1.1383$, $0 < y < 1.1383$, and $0 < y < 1.1383$ while the neighbouring stable domains are $1.1383 \le y < \infty$, $1.1383 \le y < \infty$, $1.1383 \le y < \infty$, $1.1383 \le y < \infty$, $1.1383 \le y < \infty$, and $1.1383 \le x < \infty$ where the equalities correspond to the marginal stability states.

(iii) For $\beta = 0.9$ and $x = 1$, see Figure 4. Corresponding to $M = 0.5, 1, 1.5, 2, 2.5$, and 3, it is found that the unstable domains are $0 < y < 0.829$, $0 < y < 0.829$, $0 < y < 0.829$, $0 < y < 0.829$, $0 < y < 0.829$, and $0 < y < 0.829$ while the neighbouring stable domains are $0.829 \le y < \infty$, $0.829 \le y < \infty$, $0.829 \le y < \infty$, $0.829 \le y < \infty$,

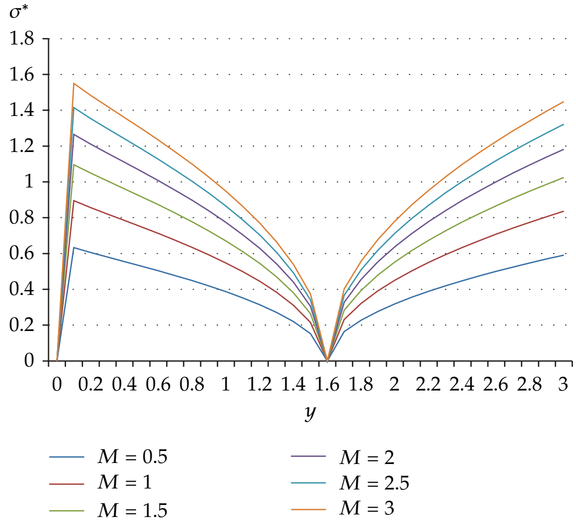

Figure 2: Stable and unstable domains for $\beta = 0.7$ and $x = 1$.

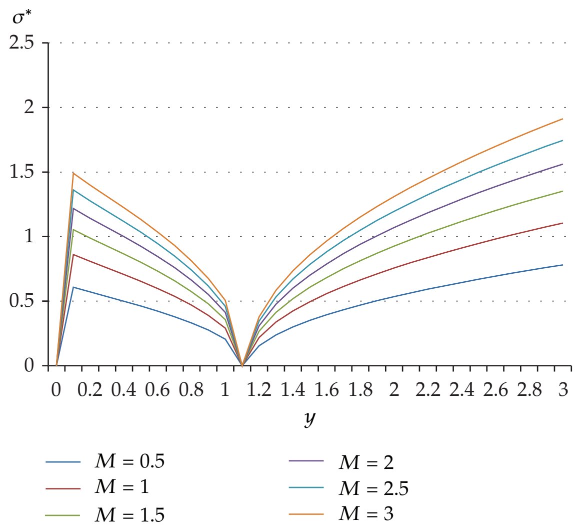

Figure 3: Stable and unstable domains for $\beta = 0.8$ and $x = 1$.

$0.829 \leq y < \infty$, and $0.829 \leq x < \infty$ where the equalities correspond to the marginal stability states.

(iv) For $\beta = 1$ and $x = 1$, see Figure 5. Corresponding to $M = 0.5, 1, 1.5, 2, 2.5$, and 3, it is found that the unstable domains are $0 < y < 0.6179$, $0 < y < 0.6179$, $0 < y < 0.6179$, $0 < y < 0.6179$, $0 < y < 0.6179$, and $0 < y < 0.6179$ while the neighbouring stable domains are $0.6179 \leq y < \infty$, $0.6179 \leq y < \infty$, $0.6179 \leq y < \infty$, $0.6179 \leq y < \infty$, $0.6179 \leq y < \infty$, and $0.6179 \leq x < \infty$ where the equalities correspond to the marginal stability states.

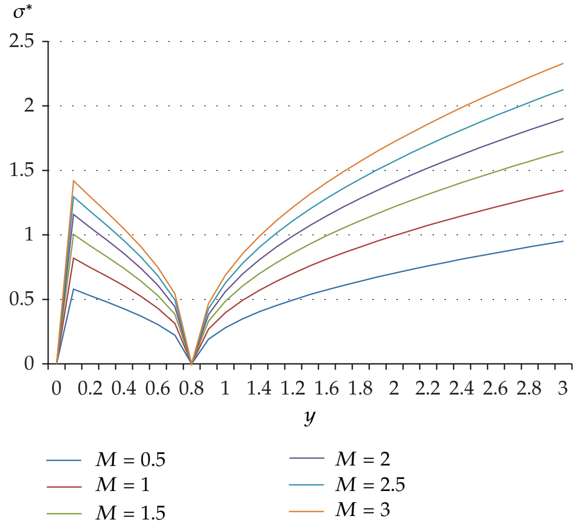

Figure 4: Stable and unstable domains for $\beta = 0.9$ and $x = 1$.

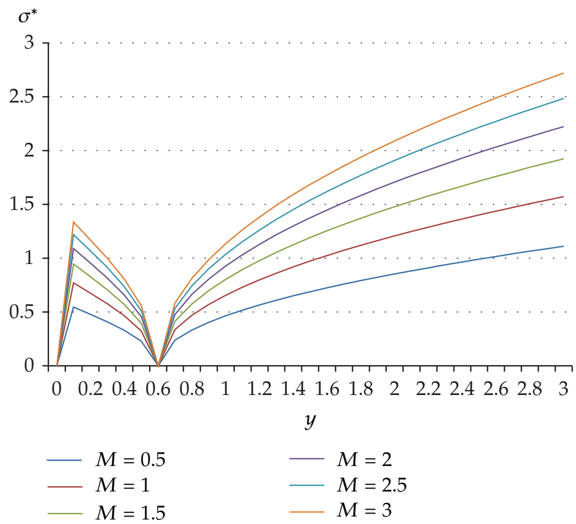

Figure 5: Stable and unstable domains for $\beta = 1$ and $x = 1$.

(v) For $\beta = 1.1$ and $x = 1$, see Figure 6. Corresponding to $M = 0.5, 1, 1.5, 2, 2.5$, and 3, it is found that the unstable domains are $0 < y < 0.4439$, $0 < y < 0.4439$, $0 < y < 0.4439$, $0 < y < 0.4439$, $0 < y < 0.4439$, and $0 < y < 0.4439$ while the neighbouring stable domains are $0.4439 \leq y < \infty$, $0.4439 \leq y < \infty$, $0.4439 \leq y < \infty$, $0.4439 \leq y < \infty$, $0.4439 \leq y < \infty$, and $0.4439 \leq x < \infty$ where the equalities correspond to the marginal stability states.

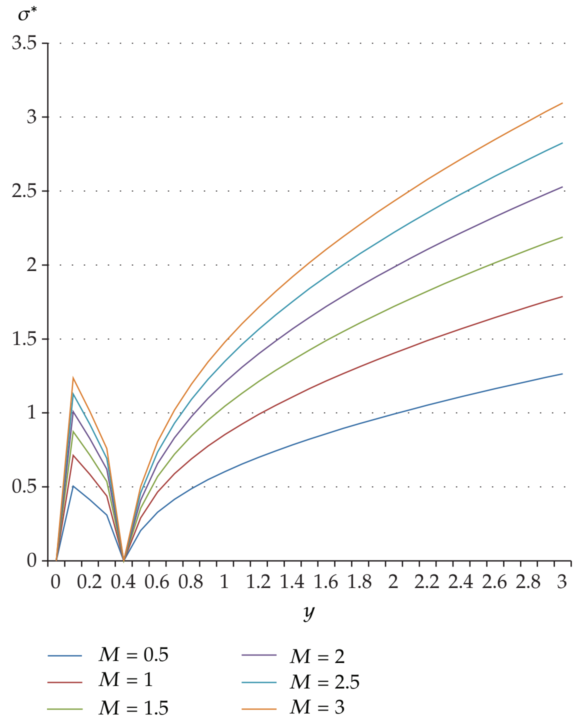

Figure 6: Stable and unstable domains for $\beta = 1.1$ and $x = 1$.

8. Conclusion

From the numerical discussion, we deduce that the compressibility has a strong stabilizing tendency for all wavelengths. The capillary force is destabilizing for a small domain of long wavelengths in the axisymmetric mode while it is stabilizing in all the rest. The electromagnetic force interior the gas is stabilizing. The electromagnetic force in the liquid region is stabilizing also. These results are in good agreement with the analytical discussions of relation (4.8).

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