# $p$-J-Generator And $p_{1}-$ J-Generator In Bitopology 

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#### Abstract

In this article, we investigate several relations between $p$-J-generator, $p_{1}$-J-generator with $p$-Lindelöf and $p_{1}$-Lindelöf spaces by using $\tau_{i}$-codense, $(i, j)$ meager set, $(i, j)$-nowhere dense set and perfect mapping of bitopological space. Various relations between $p$-compactness, $p$-Lindelöfness, $p_{1}$-Lindelöfness, topological ideal, $(i, j)$-meager, $(i, j)$-Baire space in bitopological space are investigated. Some properties are studied by using perfect mapping in a product bitopological space. It is found that bitopological space has many applications in real life.


Key Words: opological ideal, $p$-Lindelöf, $p_{1}$-Lindelöf, pairwise weakly Lindelöf, pairwise almost Lindelöf.

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## 1. Introduction, motivation and scopes of bitopological space in other areas of mathematics and natural science.

Kelly [1] introduced bitopological space via quasi-pseudo metric and systematically investigated its various important properties. It has drawn direct and indirect attentions of many point set topologists, fuzzy topologists, engineers, researchers of medical science, computer scientists, economists etc. for its applications in their respective areas.

Definition of topological ideal is well known. Topological ideal $\mathcal{J}$ and $\sigma$-ideal can be found in Dontchev et al.[2]. Ideal of all nowhere dense sets and ideal of all

[^0]meager sets of an ideal topological space $(X, \tau, \mathcal{J})$ are denoted by $\mathcal{N}$ and $\mathcal{N}$; respectively. Throughout this paper, no separation axiom is considered unless otherwise stated.

Kuratowski [3] introduced the notion of local function of $A \subseteq X$ in $(X, \tau)$ with respect to $\mathcal{J}$ and $\tau$ (briefly $A^{*}$ ). $A^{*}(\mathcal{J})$ or $A^{*}=\{x \in X \mid U \cap A \notin \mathcal{J}, x \in U$ for all $U \in \tau\}$.

It is well known that $c l^{*}(A)=A^{*} \cup A$; defines a Kuratowski closure operator for a topology $\tau^{*}(\mathcal{J})$ finer than $\tau$.

Throughout this paper, the word "bitopological space" will be denoted by BS.
A cover $\mathcal{U}$ of a $\mathrm{BS}\left(X, \tau_{1}, \tau_{2}\right)$ is called $\tau_{1} \tau_{2}$-open (Swart [4], Definition 4.1) if $\mathcal{U} \subseteq \tau_{1} \cup \tau_{2}$. If in addition, $\mathcal{U}$ contains atleast one non-empty member of $\tau_{1}$ and atleast one nonempty member of $\tau_{2}$; then it is called pairwise open (see for instance Fletcher et al. [5]). Pairwise compactness was defined by Fletcher et al. [5]. $p$-compact, $p_{1}$ - compact, $p$-Lindelöf and $p_{1}$-Lindelöf were defined by Kilićckman and Salleh [6]. According to Reilly [7]; $\left(X, \tau_{1}, \tau_{2}\right)$ is pairwise Lindelöf ( resp. pairwise compact ) if each pairwise open cover has a countable ( resp. finite) subcover. Cooke and Reilly [8] investigated relation between semi-compactness and pairwise compactness in bitopological space.

Kilićkman and Salleh [9-11] also studied various properties of pairwise Lindelöfness. Cocompactness, cotopology, $(i, j)$-Baire space etc. were studied by Dvalishvili [12].

Frolik [13] introduced weakly Lindelöf space, Willard and Dissanayake [14] introduced almost Lindelöf space in a topological space and their bitopological versions were studied by Kilićkman and Salleh [9]. In the last two decades, various developments have been observed in bitopological space. Still a little progress has been observed in case of generalized closed sets of bitopological space and related areas. Fuzzy versions of some generalized closed sets and related structures from both topology and bitopology have been studied ( one may refer to [15-17] ). Fuzzy version of topological ideal was introduced by Sarkar [18].

Bitopological space and its properties have many useful applications in real world. In 2010, Salama [19] used lower and upper approximations of Pawlak's rough sets; by using a class of generalized closed set of bitopological space for data reduction of rheumatic fever data sets. Fuzzy topology integrated support vector machine (FTSVM)-classification method for remotely sensed images based on standard support vector machine (SVM) was introduced by Zhang et al. [20]. For some recent indirect applications of topology or bitopology as fuzzy versions, one may refer to [19-21]. Ideal topological space has many applications. Recently Tripathy and Acharjee [22] introduced a class of generalized closed set in bitopo-
logical space using topological ideal, two expansion operators and local functions. The application of this set can be found in market price equilibrium [23]. There are maximum nine out of eleven strategies; under which expected price of a daily useful commodity, which is decided by a consumer and price; which is decided by government; are equal. Other two strategies are special cases. These are useful from the view point that; no one will have to face poverty in year 2017 if she has price lists of these commodities for 2016 and 2015. She has freedom to choose her daily useful commodities; according to her preferences.

One may refer to [41]; for interrelated research works on topology, orderings and utility theory of mathematical economics. In this paper one may find; how concepts of countability, compactness, normality, Lindelöfness etc. of general topology and order (i.e. LOTS etc.) have been used for countable representation of utility function. One may refer to Bosi and Mehta [43]; for their interlink of bitopology and choice via utility function.

Hence, there is a need to study different types of pairwise compactness, pairwise Lindelöfness from the point of view of topological ideal, $(i, j)$-meager set and ( $i, j$ )-Baire space.

In this paper, we try to give some possible answers of the following questions.
(i) Is there any relation between different forms of pairwise Lindelöfness, $(i, j)$ meager set and pairwise Baire space in a bitopological space?
(ii) Is there any relation between different forms of pairwise Lindelöfness and topological ideal in a bitopological space?
(iii) What are the results related to pairwise Lindelöfness in product bitopology using Datta's perfect mapping?

In this paper, we consider two types of pairwise Lindelöfnss. They are $p$-Lindelöf due to Kilićkman and Salleh [2] and $p_{1}$-Lindelöf due to Birsan [24] ( as defined by Kilićkman and Salleh [2] ). Dvalishvili [25] defined ( $i, j$ )-nowhere dense set. Dontchev et al. [26] studied ideal irresoluteness in topology. Datta [27] defined perfect map from the view point of bitopology. Researchers have studied Khalimsky digital line by considering generalized closed sets in topological space ( [28-30]). Many topologists are now focusing on ideal and its various consequences. Systematic study on pairwise Lindelöfness can be found in Salleh and Kilićkman [31]. Throughout this paper; we will consider $i, j \in\{1,2\}, i \neq j$

From the above; it is clear that bitopological space has drawn attentions as an applied branch for research. Many researchers have used bitopological properties as their tools to solve problems of mechanical engineering, medicine, economics etc. Hence, the above questions may play significant roles in near future. Often it is
easy to assume results of bitopology as extensions of results of general topology; which is it not true in general. This can be understood from the fact that bitopology has many definitions of Lindelöfness using only pairwise open sets etc.

Variations of $i$ and $j$ between 1 and 2 often signify different properties in a bitopological space; which general topology never follows. In [44], Acharjee et al. answered some open questions and one suitable counterexample.

Lemma 1.1. ([6], Lemma 1) Every pairwise closed subset of a $p$-Lindelöf bitopological space is $p$-Lindelöf.

Lemma 1.2. ([6], Lemma 4) Every pairwise closed subset of a $p_{1}$-Lindelöf bitopological space is $p_{1}$-Lindelöf.

Lemma 1.3. ([44], Theorem 3.1.) Let $\left(X, \tau_{1}, \tau_{2}\right)$ be a contra second countable bitopological space, then it is $p_{1}$-Lindelöf.

Lemma 1.4. ([44], Corollary 3.1.) Every pairwise closed subset of a contra second countable bitopological space is $p_{1}$-Lindelöf.

## 2. Some preliminary definitions

Definition 2.1. ([9], Definition 2.7) A BS $\left(X, \tau_{1}, \tau_{2}\right)$ is said to be $(i, j)$-nearly Lindelöf (resp. ( $i, j$ )-almost Lindelöf, ( $i, j$ )-weakly Lindelöf), if every $\tau_{i}$-open cover $\left\{U_{\alpha} \mid \alpha \in \Delta\right\}$ of $X$, there exists a countable subcollection $\left\{U_{\alpha_{n}} \mid n \in N\right\}$; such that $X=\underset{n \in N}{\bigcup} \tau_{i} i n t \tau_{j} c l\left(U_{\alpha_{n}}\right)\left(\right.$ resp. $\left.X=\bigcup_{n \in N} \tau_{j} c l\left(U_{\alpha_{n}}\right), X=\tau_{j} c l\left(\bigcup_{n \in N} U_{\alpha_{n}}\right)\right)$.
$\left(X, \tau_{1}, \tau_{2}\right)$ is said to be pairwise nearly Lindelöf, if it is both $(i, j)$-nearly Lindelöf and $(j, i)$-nearly Lindelöf. Similarly, one can define pairwise almost Lindelöf, pairwise weakly Lindelöf.

Definition 2.2. ([25], Definition 1.1) A subset $A$ of a $\mathrm{BS}\left(X, \tau_{1}, \tau_{2}\right)$ is termed as $(i, j)$-nowhere dense, if $\tau_{i} i n t \tau_{j} c l(A)=\emptyset$. The family of all $(i, j)$-nowhere dense subsets of $X$ is denoted by $(i, j)-\mathcal{N D}(X)$.

Let $\mathcal{J}$ be a topological ideal, then $\mathcal{J} \neq \emptyset$ and $\mathcal{J}$ is said to be codense [2] for a topological space $(X, \tau)$; if and only if $\mathcal{J} \cap \tau=\{\emptyset\}$. Similarly, one can define $\tau_{i^{-}}$ codense; $i \in\{1,2\}$ for a $\operatorname{BS}\left(X, \tau_{1}, \tau_{2}\right)$. An ideal $\mathcal{J}$ is said to be pairwise codense, if it is both $\tau_{1}$-codense and $\tau_{2}$-codense. We denote ideal of $(i, j)$-nowhere dense subsets of BS $\left(X, \tau_{1}, \tau_{2}\right)$ by $\mathcal{J}_{i} \mathcal{N}_{j}(X)$

Definition 2.3. ([12], Definition 1.6) A subset $A$ of a $\mathrm{BS}\left(X, \tau_{1}, \tau_{2}\right)$ is termed as $(i, j)$-first category (or $(i, j)$-meager), if $A=\bigcup_{n=1}^{\infty} A_{n}$; where $A_{n} \in(i, j)-\mathcal{N D}(X)$; for
every $n \in N$ and $A$ is of $(i, j)$-second category (or $(i, j)$-non meager), if it is not of $(i, j)$-first category. The family of all sets of $(i, j)$-first category (or $(i, j)$-second categories) in $X$ is denoted by $(i, j)-\operatorname{Catg}_{I}(X)\left((i, j)-\operatorname{Catg}_{I I}(X)\right)$.

If $X \in(i, j)-\operatorname{Catg}_{I}(X)$ ( resp. $\left.X \in(i, j)-\operatorname{Catg}_{I I}(X)\right)$, then it is abbreviated as $X$ is of $(i, j)$-Catg (resp. $(i, j)-$ Catg $\left._{I I}\right)$.

We denote $\sigma$-ideal [2] of $(i, j)$-meager subsets of a $\operatorname{BS}\left(X, \tau_{1}, \tau_{2}\right)$ by $\sigma_{i} \mathcal{M}_{j}(X)$.
Now, we define the following definitions.
Definition 2.4. A $\mathrm{BS}\left(X, \tau_{1}, \tau_{2}\right)$ is said to be $(i, j)$-non-nearly Lindelöf (resp. $(i, j)$-non-almost Lindelöf, $(i, j)$-non-weakly Lindelöf), if for every $\tau_{i}$-open cover $\left\{U_{\alpha} \mid \alpha \in \Delta\right\}$ of $X$; there exists a $\tau_{j}$-open countable sub-collection $\left\{U_{\alpha_{n}} \mid n \in N\right\}$ such that $X=\cup_{n \in N} \tau_{j} i n t \tau_{i} c l\left(U_{\alpha_{n}}\right)$ ( resp. $X=\bigcup_{n \in N} \tau_{i} c l\left(U_{\alpha_{n}}\right), X=\tau_{i} \operatorname{cl}\left({ }_{n \in N}^{\cup} U_{\alpha_{n}}\right)$ ).
( $X, \tau_{1}, \tau_{2}$ ) is said to be pairwise non-nearly Lindelöf, if it is both $(i, j)$-nonnearly Lindelöf and ( $j, i$ )-non-nearly Lindelöf. Similarly, we have pairwise nonalmost Lindelöf, pairwise non-weakly Lindelöf.

Kilićkman and Salleh defined p-Lindelöf ( [6] Definition 6 ) and Birsan defined $p_{1}$-Lindelöf ( One may refer to Definition 1 of [6] ).

Definition 2.5. ([44], Definition 3.1) Let $\left(X, \tau_{1}, \tau_{2}\right)$ be a bitopological space, then:
(i) $\left(X, \tau_{1}, \tau_{2}\right)$ is said to be an $(i, j)$-second countable bitopological space, if ( $X, \tau_{i}$ ) is second countable with respect to $\tau_{j}$.
(ii) $\left(X, \tau_{1}, \tau_{2}\right)$ is said to be a contra second countable bitopological space, if it is both $(1,2)$-second countable bitopological space and $(2,1)$-second countable bitopological space.

We state the following results those will be used in this paper.
Lemma 2.1. ([6], Theorem 6) If $\left(X, \tau_{1}, \tau_{2}\right)$ is second countable space, then $\left(X, \tau_{1}, \tau_{2}\right)$ is $p$-Lindelöf.

Definition 2.6. [7] A bitopological space $\left(X, \tau_{1}, \tau_{2}\right)$ is pairwise compact (resp. pairwise Lindelöf ), if each pairwise open cover of ( $X, \tau_{1}, \tau_{2}$ ) has a finite (resp. countable) subcover.

Definition 2.7. [46] ( $\left.X, \tau_{1}, \tau_{2}\right)$ is said to be pairwise countably compact, if every countable pairwise open cover of ( $X, \tau_{1}, \tau_{2}$ ) has a finite subcover.

Proposition 2.1. [7] In a pairwise Lindelöf space; pairwise countable compactness is equivalent to pairwise compactness.

Proposition 2.2. [7] Any second countable bitopological space is pairwise Lindelöf.

Proposition 2.3. [7] If ( $X, \tau_{1}, \tau_{2}$ ) is pairwise Lindelöf and $A$ is a proper subset of $X$ which is $\tau_{1}$-closed, then $A$ is pairwise Lindelöf and $\tau_{2}$-Lindelöf.

Proposition 2.4. [7] If ( $X, \tau_{1}, \tau_{2}$ ) is pairwise Lindelöf and pairwise regular; then it is pairwise normal.

## 3. Main results

In this section, we define two new classes in a bitopological space, which generate $p$-Lindelöf space and $p_{1}$-Lindelöf space respectively.

Definition 3.1. A $\operatorname{BS}\left(X, \tau_{1}, \tau_{2}, \mathcal{J}\right)$ is said to be $\tau_{i}$-J-generator (resp. $\tau_{i}^{c}$ - $\mathcal{J}$ generator); if for every $\tau_{i}$-open cover $\left\{U_{\alpha} \mid \alpha \in \Delta\right\}$ of $X$, there exists a (resp. $\tau_{j}$-open) countable sub-collection $\left\{U_{\alpha_{n}} \mid n \in N\right\}$; such that $X \backslash \underset{n \in N}{\cup} U_{\alpha_{n}} \in \mathcal{J}$.
$\left(X, \tau_{1}, \tau_{2}, \mathcal{J}\right)$ is said to be $p$ - J-generator (resp. $p_{1}$-J-generator), if it is both $\tau_{i^{-}}$ $\mathcal{J}$-generator (resp. $\tau_{i}^{c}$-J-generator) and $\tau_{j}$-J-generator (resp. $\tau_{j}^{c}$-J-generator).

Remark 3.1. From definition of ideal, it is clear that $\mathcal{J} \neq \emptyset$. If $\mathcal{J}=\{\emptyset\}$, then Definition 3.1 reduces to $p$-Lindelöf (resp. $p_{1}$-Lindelöf) i.e. $p$ - $\{\emptyset\}$-generator $\Leftrightarrow p$ Lindelöf and $p_{1}-\{\emptyset\}$-generator $\Leftrightarrow p_{1}$-Lindelöf.

From $([2],[26])$, we know that a subset $S$ of $(X, \tau, \mathcal{J})$ is a topological subspace with ideal $\mathcal{J}_{S}=\{I \cap S: I \in \mathcal{J}\}$.

A subset $A$ of $X$ of $\left(X, \tau_{1}, \tau_{2}\right)$ is said to be pairwise clopen, if it is both $\tau_{1}$ clopen and $\tau_{2}$-clopen.

Theorem 3.1. (i) Let $\left(X, \tau_{1}, \tau_{2}, \mathcal{J}\right)$ be a $p$ - J-generator. If $A$ is a pairwise closed subset of $X$, then $\left(A,\left.\tau_{1}\right|_{A},\left.\tau_{2}\right|_{A}, \mathcal{J}_{A}\right)$ is also $p$-J $\mathcal{J}_{A}$-generator.
(ii) Let $\left(X, \tau_{1}, \tau_{2}, \mathcal{J}\right)$ be a $p_{1}$-J.generator. If $A$ is a pairwise clopen subset of $X$, then $\left(A,\left.\tau_{1}\right|_{A},\left.\tau_{2}\right|_{A}, \mathcal{J}_{A}\right)$ is also $p_{1}-\mathcal{J}_{A}$-generator.

Proof.(i) Let $\mathcal{U}_{A}=\left\{U_{\alpha} \cap A: U_{\alpha} \in \tau_{i}, \alpha \in \Delta\right\}$ be a $\left.\tau_{i}\right|_{A^{-o p e n}}$ cover of $A$. Thus, $\mathcal{U}=\left\{U_{\alpha}: \alpha \in \Delta\right\} \cup\{(X \backslash A)\}$ is $\tau_{i}$ open cover of $X$. Thus, $X$ has a countable sub-collection $\mathcal{V}=\left\{U_{\alpha_{n}}: U_{\alpha_{n}} \in \tau_{i}, n \in N\right\} \cup\{(X \backslash A)\}$ such that $X \backslash\left\{\cup_{n \in N} U_{\alpha_{n}} \cup(X \backslash A)\right\}=R($ say $) \in \mathcal{J}$.

Then, $A \subseteq \bigcup_{n \in N}\left\{U_{\alpha_{n}}: n \in N\right\} \cup R$. Thus, $A=\cup_{n \in N}\left(U_{\alpha_{n}} \cap A\right) \cup(R \cap A)$. So, we have $A \backslash \bigcup_{n \in N}^{\cup}\left\{\left(U_{\alpha_{n}} \cap A\right)\right\} \subseteq(R \cap A) \in \mathcal{J}_{A}$. Hence, $\mathcal{V}_{A}=\left\{U_{\alpha_{n}} \cap A: n \in N\right\}$ is satisfying the required condition for $p-\mathcal{J}_{A}$-generator. Hence the proof.
(ii) It can be established following the technique used in establishment of (i).

Remark 3.2. If $\mathcal{J}=\{\emptyset\}$, then $\mathcal{J}_{A}=\{\emptyset\}$. Then by Theorem 3.1, $A$ is $p$ - $\{\emptyset\}$ generator and it implies Lemma 1.1 of and vice-versa. Similarly, if $A$ is $p_{1}-\{\emptyset\}-$ generator, then it implies Lemma 1.2.

In view of Lemma 2.1 and Remark 3.1, we have the following result.
Corollary 3.1. Every second countable space is $p-\{\emptyset\}$-generator.
Theorem 3.2. (i) Let $\left(X, \tau_{1}, \tau_{2}\right)$ be a BS, then $X$ is pairwise weakly Lindelöf if and only if $X$ is both $\tau_{i}-\mathcal{J}_{j} \mathcal{N}_{i}$-generator and $\tau_{j}-\mathcal{J}_{i} \mathcal{N}_{j}$-generator.
(ii) Let $\left(X, \tau_{1}, \tau_{2}\right)$ be a BS, then $X$ is pairwise non-weakly Lindelöf if and only if $X$ is both $\tau_{i}-\mathcal{J}_{i} \mathcal{N}_{j}$-generator and $\tau_{j}-\mathcal{J}_{j} \mathcal{N}_{i}$-generator.

## Proof. (i) Necessity.

We have only to show, if $X$ is $(i, j)$-weakly Lindelöf; then it is $\tau_{i}-\mathcal{J}_{j} \mathcal{N}_{i}$-generator.
Let us assume, $X$ be $(i, j)$-weakly Lindelöf and let $\mathcal{U}=\left\{U_{\alpha} \mid \alpha \in \Delta\right\}$ be a $\tau_{i^{-}}$ open cover of $X$. Then by Definition 2.1, there exists a countable sub-collection $\left\{U_{\alpha_{n}} \mid n \in N\right\}$ such that $X=\tau_{j} c l\left(\cup_{n \in N} U_{\alpha_{n}}\right)$. Then, $X \backslash \underset{n \in N}{\cup} U_{\alpha_{n}} \in \mathcal{J}_{j} \mathcal{N}_{i}(X)$. Similarly, it can be established for ( $j, i$ )-weakly Lindelöf case.

## Sufficiency.

We will only prove that if $X$ is $\tau_{i}-\mathcal{J}_{j} \mathcal{N}_{i}$-generator, then $X$ is $(i, j)$-weakly Lindelöf.

Let $\mathcal{U}=\left\{U_{\alpha} \mid \alpha \in \Delta\right\}$ be a $\tau_{i}$-open cover of $X$, then by Definition 3.1; there exists a countable sub-collection $\left\{U_{\alpha_{n}} \mid n \in N\right\}$ such that $X \backslash \underset{n \in N}{\bigcup} U_{\alpha_{n}} \in \mathcal{J}_{j} \mathcal{N}_{i}(X)$. Then, $X=\tau_{j} c l\left(\cup_{n \in N} U_{\alpha_{n}}\right)$. Thus, $X$ is $(i, j)$-weakly Lindelöf. Similarly, we can prove for $\tau_{j}-\mathcal{J}_{i} \mathcal{N}_{j}$-generator case.
(ii) It can be established by following the technique of proof of (i).

Theorem 3.3. (i) A BS $\left(X, \tau_{1}, \tau_{2}\right)$ is pairwise weakly Lindelöf; if and only if it is both $\tau_{i}$ - $R$-generator and $\tau_{j}$ - $S$-generator for some $\tau_{j}$-codense ideal $R$ and $\tau_{i^{-}}$ codense ideal $S$.
(ii) $\mathrm{A} \mathrm{BS}\left(X, \tau_{1}, \tau_{2}\right)$ is pairwise non-weakly Lindelöf; if and only if it is both $\tau_{i^{-}}$-$R$-generator and $\tau_{j}-S$-generator for some $\tau_{i}$-codense ideal $R$ and $\tau_{j}$-codense ideal $S$.

## Proof. (i) Necessity.

If ( $X, \tau_{1}, \tau_{2}$ ) is pairwise weakly Lindelöf, then by Theorem 3.2(i), $X$ is both $\tau_{i}-\mathcal{J}_{j} \mathcal{N}_{i}$-generator and $\tau_{j}-\mathcal{J}_{i} \mathcal{N}_{j}$-generator. It is easy to verify $\mathcal{J}_{j} \mathcal{N}_{i}(X) \cap \tau_{j}=\{\emptyset\}$. So, $\mathcal{J}_{j} \mathcal{N}_{i}(X)$ is $\tau_{j}$-codense. Similarly, we can establish the other case.

## Sufficiency.

Let $R$ be any $\tau_{j}$-codense ideal and $X$ is $\tau_{i}$ - $R$-generator. Let $\mathcal{U}=\left\{U_{\alpha} \mid \alpha \in \Delta\right\}$ be any $\tau_{i}$-open cover of $X$. Then, there is a countable subcover $\left\{U_{\alpha_{n}} \mid n \in N\right\}$ such that $X \backslash \underset{n \in N}{\cup} U_{\alpha_{n}} \in R$. Hence, $X=\tau_{j} c l\left(\cup_{n \in N} U_{\alpha_{n}}\right)$. Thus, $X$ is $(i, j)$-weakly Lindelöf. Similarly, we can prove for the other case. Thus, $X$ is pairwise weakly Lindelöf. Hence the proof.

Dvalishvili $([12],[25])$ defined $(i, j)$-Baire space and pairwise Baire space.
In next theorem, we establish the relation between pairwise weakly Lindelöf space and pairwise $\sigma$-ideal generator under certain condition.

Theorem 3.4. Let $\left(X, \tau_{1}, \tau_{2}\right)$ is a pairwise Baire space. Then,
(i) $\left(X, \tau_{1}, \tau_{2}\right)$ is pairwise weakly Lindelöf; if and only if $\left(X, \tau_{1}, \tau_{2}\right)$ is both $\tau_{i^{-}}$ $\sigma_{j} \mathcal{M}_{i}$-generator and $\tau_{j}-\sigma_{i} \mathcal{M}_{j}$-generator.
(ii) $\left(X, \tau_{1}, \tau_{2}\right)$ is pairwise non-weakly Lindelöf; if and only if $\left(X, \tau_{1}, \tau_{2}\right)$ is both $\tau_{i}-\sigma_{i} \mathcal{M}_{j}$-generator and $\tau_{j}-\sigma_{j} \mathcal{M}_{i}$-generator.

Proof. (i) $\left(X, \tau_{1}, \tau_{2}\right)$ is $(i, j)$-Baire space and $(j, i)$-Baire space $\Rightarrow X$ is $(i, j)$ Catg $_{I I}$ and $(j, i)-$ Catg $_{I I}$.
$\left(X, \tau_{1}, \tau_{2}\right)$ is $(i, j)$-Baire space and $(j, i)$-Baire space $\Leftrightarrow \sigma_{i} \mathcal{N}_{j}(X)$ is $\tau_{i}$-codense and $\sigma_{j} \mathcal{M}_{i}(X)$ is $\tau_{j}$-codense. Then, the proof follows from Theorem 3.3(i). Hence the proof.

A BS $\left(X, \tau_{1}, \tau_{2}\right)$ is said to have property ${ }^{*}$; if $\tau_{i} c l\left(\tau_{j} c l(U)\right)=\tau_{j} c l(U)$, whenever $U \subseteq X$ and $i, j \in\{1,2\}, i \neq j$.

We state the following result without proof.
Theorem 3.5.(i) If ( $X, \tau_{1}, \tau_{2}$ ) is pairwise almost Lindelöf with property ${ }^{*}$; then it is both $\tau_{i}-\sigma_{j} \mathcal{M}_{i}$-generator and $\tau_{j}-\sigma_{i} \mathcal{N}_{j}$-generator.
(ii) If $\left(X, \tau_{1}, \tau_{2}\right)$ is pairwise non-almost Lindelöf with property ${ }^{*}$; then it is both $\tau_{i}-\sigma_{i} \mathcal{M}_{j}$-generator and $\tau_{j}-\sigma_{j} \mathcal{M}_{i}$-generator.

In view of Theorem 3.4 and Theorem 3.5, we state the following result.
Corollary 3.2. (i) If a $\mathrm{BS}\left(X, \tau_{1}, \tau_{2}\right)$ is pairwise almost Lindelöf with property * and pairwise Baire space, then it is pairwise weakly Lindelöf.
(ii) If a $\mathrm{BS}\left(X, \tau_{1}, \tau_{2}\right)$ is pairwise non-almost Lindelöf with property $*$ and pairwise Baire space, then it is pairwise non-weakly Lindelöf.

The following result is a consequence of Theorem 3.1 and Theorem 3.2.
Corollary 3.3. (i) If $A$ be a pairwise clopen subset of a pairwise weakly Lindelöf space $\left(X, \tau_{1}, \tau_{2}\right)$, then $\left(A,\left.\tau_{1}\right|_{A},\left.\tau_{2}\right|_{A}\right)$ is pairwise weakly Lindelöf.
(ii) If $A$ be a pairwise clopen subset of a pairwise non-weakly Lindelöf space ( $X, \tau_{1}, \tau_{2}$ ), then $\left(A,\left.\tau_{1}\right|_{A},\left.\tau_{2}\right|_{A}\right)$ is pairwise non-weakly Lindelöf.

During the preparation of this paper with refer to Kilic̀man and Salleh [6], some open questions were raised. Some answers of these questions are affirmative and one counter example is proved by Acharjee et al. in [44]; using interlocking and nest in a bitopological space. Notions of interlocking and nest can be found in [ 45]. The two main questions are stated below.
(i) What type of a countable space in a bitopological space is a $p_{1}$-Lindelöf space?
(ii) Does every $p_{1}$-Lindelöf space imply countable space of (i)?.

Theorem 3.6. Let $\left(X, \tau_{1}, \tau_{2}\right)$ be a contra second countable bitopological space, then it is $p_{1-}\{\emptyset\}$-generator.

Proof. The proof follows from Remark 3.1. and Lemma 1.1.
Theorem 3.7. Every pairwise closed subset of a contra second countable bitopological space is $p_{1}-\{\emptyset\}$-generator.

Proof. It can be proved by Lemma 1.4 and Remark 3.1.

## 4. Relations of $p$-J-generator and $p_{1}$-J-generator with perfect mapping

The following definition of perfect mapping is due to Datta [27].

Definition 4.1. ([27], Definition 2.1) A mapping $f:\left(X, \tau_{1}, \tau_{2}\right) \longrightarrow\left(Y, \psi_{1}, \psi_{2}\right)$ is said to be perfect if,
(i) $f$ is continuous i.e. $f$ is $\tau_{1}-\psi_{1}$-continuous and $\tau_{2}-\psi_{2}$-continuous.
(ii) $f$ is compact i.e. the inverse image of every point of $Y$ is $\tau_{1}$-compact, $\tau_{2^{-}}$ compact and pairwise compact.
(iii) $f$ is closed i.e. the image of every $\tau_{1}$-closed (resp. $\tau_{2}$-closed) subset of $X$ is $\psi_{1}$-closed (resp. $\psi_{2}$-closed) subset of $Y$.

Let $f:\left(X, \tau_{1}, \tau_{2}, \mathcal{J}\right) \longrightarrow\left(Y, \psi_{1}, \psi_{2}, \mathcal{J}\right)$ be a function, then we denote $f(\mathcal{J})=$ $\{f(I) \mid I \in \mathcal{J}\}$ and $f^{-1}(\mathcal{J})=\left\{f^{-1}(J) \mid J \in \mathcal{J}\right\}$. Hence, $f(\mathcal{J})$ and $f^{-1}(\mathcal{J})$ are ideals of $Y$ and $X$ respectively.

Theorem 4.1 (i) Let $f:\left(X, \tau_{1}, \tau_{2}, \mathcal{J}\right) \longrightarrow\left(Y, \psi_{1}, \psi_{2}\right)$ be a continuous function and surjection. If $\left(X, \tau_{1}, \tau_{2}, \mathcal{J}\right)$ is $p$-J-generator, then $\left(Y, \psi_{1}, \psi_{2}\right)$ is also $p$ - $f(\mathcal{J})$-generator.
(ii) Let $f:\left(X, \tau_{1}, \tau_{2}, \mathcal{J}\right) \longrightarrow\left(Y, \psi_{1}, \psi_{2}\right)$ be a continuous function and surjection. If $\left(X, \tau_{1}, \tau_{2}, \mathcal{J}\right)$ is $p_{1}$-J-generator, then $\left(Y, \psi_{1}, \psi_{2}\right)$ is also $p_{1-} f(\mathcal{J})$-generator.

Proof.(i) It is enough to show, if $\left(X, \tau_{1}, \tau_{2}, \mathcal{J}\right)$ is $\tau_{i}$-J-generator, then $\left(Y, \psi_{1}, \psi_{2}\right)$ is also $\psi_{i^{-}} f(\mathcal{J})$-generator.

Let $\mathcal{U}=\left\{U_{\alpha} \mid \alpha \in \Delta\right\}$ be any $\psi_{i}$-open cover of $Y$. Then by Definition 4.1, $\mathcal{V}=\left\{f^{-1}\left(U_{\alpha}\right) \mid \alpha \in \Delta\right\}$ is $\tau_{i}$-open cover of $X$. So, we have a subcollection $\left\{f^{-1}\left(U_{\alpha_{n}}\right) \mid n \in N\right\}$ such that $X \backslash \bigcup_{n \in N} f^{-1}\left(U_{\alpha_{n}}\right) \in \mathcal{J}$. Suppose $f^{-1}\left(Y \backslash \bigcup_{n \in N} U_{\alpha_{n}}\right)=$ I. So, $\left(Y \backslash \underset{n \in N}{\cup} U_{\alpha_{n}}\right)=f(I) \in f(\mathcal{J})$ as $I \in \mathcal{J}$. Thus, we have the proof
(ii) It can be established following the technique used in establishing part(i).

We state the following result without proof.
Theorem 4.2. Let $f:\left(X, \tau_{1}, \tau_{2}\right) \longrightarrow\left(Y, \psi_{1}, \psi_{2}, \mathcal{J}\right)$ be a perfect, open and surjective function. Then,
(i) if $\left(Y, \psi_{1}, \psi_{2}, \mathcal{J}\right)$ is $p$ - $\mathcal{J}$-generator, then $\left(X, \tau_{1}, \tau_{2}\right)$ is $p-f^{-1}(\mathcal{J})$-generator.
(ii) if $\left(Y, \psi_{1}, \psi_{2}, \mathcal{J}\right)$ is $p_{1}$-J-generator, then $\left(X, \tau_{1}, \tau_{2}\right)$ is $p_{1-} f^{-1}(\mathcal{J})$-generator.

Lemma 4.1. If $f:\left(X, \tau_{1}, \tau_{2}\right) \longrightarrow\left(Y, \psi_{1}, \psi_{2}, \mathcal{J}\right)$ be an open function and surjective. If $\mathcal{J}$ is $\psi_{i}$-codense, then $f^{-1}(\mathcal{J})$ is $\tau_{i}$-codense.

Proof. Let $f^{-1}(\mathcal{J})$ is not $\tau_{i}$-codense. Let $f^{-1}(J) \in f^{-1}(\mathcal{J}) \cap \tau_{i} \neq\{\emptyset\}$. Then,
$f^{-1}(J) \in \tau_{i} \backslash\{\emptyset\}$. Due to surjective and open; $f\left(f^{-1}(J)\right)=J \in \psi_{i} \backslash\{\emptyset\}$. This contradicts the fact that $\mathcal{J}$ is $\psi_{i}$-codense. Hence the proof.

Corollary 4.1. Let $f:\left(X, \tau_{1}, \tau_{2}\right) \longrightarrow\left(Y, \psi_{1}, \psi_{2}\right)$ be a perfect,open and surjective function. Then,
(i) if $\left(Y, \psi_{1}, \psi_{2}\right)$ is $p$-Lindelöf, then $\left(X, \tau_{1}, \tau_{2}\right)$ is $p$-Lindelöf .
(ii) if $\left(Y, \psi_{1}, \psi_{2}\right)$ is $p_{1}$-Lindelöf, then $\left(X, \tau_{1}, \tau_{2}\right)$ is $p_{1}$-Lindelöf .

Proof. (i) $\left(Y, \psi_{1}, \psi_{2}\right)$ is $p$-Lindelöf implies it is $p$ - $\{\emptyset\}$-generator. Then, the proof follows from Theorem 4.2(i) and Remark 3.1.
(ii) Proof follows similar to the case (i)

Applying Theorem 3.3 and Lemma 4.1, one can get the following result.
Corollary 4.2. Let $f:\left(X, \tau_{1}, \tau_{2}\right) \longrightarrow\left(Y, \psi_{1}, \psi_{2}\right)$ be a perfect, open and surjective function.
(i) If $\left(Y, \psi_{1}, \psi_{2}\right)$ is pairwise weakly Lindelöf, then $\left(X, \tau_{1}, \tau_{2}\right)$ is pairwise weakly Lindelöf .
(ii) If $\left(Y, \psi_{1}, \psi_{2}\right)$ is pairwise non-weakly Lindelöf, then $\left(X, \tau_{1}, \tau_{2}\right)$ is pairwise non-weakly Lindelöf .

## 5. On product bitopology

It is well known, every continuous mapping between $p$-compact spaces is $p$ compact in bitopological space. One may refer to Datta ([27], page no: 124)

Theorem 5.1. (i) If $\left(X, \tau_{1}, \tau_{2}, \mathcal{J}\right)$ is $p$ - J-generator and $\left(Y, \psi_{1}, \psi_{2}\right)$ is $p$-compact, then $\left(X \times Y, \tau_{1} \times \psi_{1}, \tau_{2} \times \psi_{2}\right)$ is $p-\pi^{-1}(\mathcal{J})$-generator; where $\pi: X \times Y \longrightarrow X$ is a projection map.
(ii) If $\left(X, \tau_{1}, \tau_{2}, \mathcal{J}\right)$ is $p_{1}$-J.generator and $\left(Y, \psi_{1}, \psi_{2}\right)$ is $p$-compact, then $(X \times$ $\left.Y, \tau_{1} \times \psi_{1}, \tau_{2} \times \psi_{2}\right)$ is $p_{1}-\pi^{-1}(\mathcal{J})$-generator; where $\pi: X \times Y \longrightarrow X$ is a projection map.

Proof. The projection map is perfect. Hence, the rest follows from Theorem 4.2.
The following result is a consequence of Theorem 3.3, Lemma 4.1 and Theorem 5.1.

Corollary 5.1. (i) If $\left(X, \tau_{1}, \tau_{2}\right)$ is pairwise weakly Lindelöf and $\left(Y, \psi_{1}, \psi_{2}\right)$ is $p$ compact, then $\left(X \times Y, \tau_{1} \times \psi_{1}, \tau_{2} \times \psi_{2}\right)$ is pairwise weakly Lindelöf.
(ii) If $\left(X, \tau_{1}, \tau_{2}\right)$ is pairwise non-weakly Lindelöf and $\left(Y, \psi_{1}, \psi_{2}\right)$ is $p$-compact, then $\left(X \times Y, \tau_{1} \times \psi_{1}, \tau_{2} \times \psi_{2}\right)$ is pairwise non-weakly Lindelöf .

Corollary 5.2. (i) If $\left(X, \tau_{1}, \tau_{2}\right)$ is $p$-Lindelöf and $\left(Y, \psi_{1}, \psi_{2}\right)$ is $p$-compact, then $\left(X \times Y, \tau_{1} \times \psi_{1}, \tau_{2} \times \psi_{2}\right)$ is $p$-Lindelöf .
(ii) If $\left(X, \tau_{1}, \tau_{2}\right)$ is $p_{1}$-Lindelöf and $\left(Y, \psi_{1}, \psi_{2}\right)$ is $p$-compact, then $\left(X \times Y, \tau_{1} \times\right.$ $\left.\psi_{1}, \tau_{2} \times \psi_{2}\right)$ is $p_{1}$-Lindelöf .

Proof. (i) By Remark 3.1, $\left(X, \tau_{1}, \tau_{2}\right)$ is $p$-Lindelöf $\Leftrightarrow\left(X, \tau_{1}, \tau_{1}\right)$ is $p$ - $\{\emptyset\}$-generator. By Theorem 5.1(i), $\left(X \times Y, \tau_{1} \times \psi_{1}, \tau_{2} \times \psi_{2}\right)$ is $p-\{\emptyset\}$-generator. Hence the proof.

## 6. Conclusion

In this paper, we have shown that $p$-Lindelöfness and $p_{1}$-Lindelöfness can be derived by defining new classes of sets in bitopological space. We also proved results related to perfect mapping of bitopological space and used them in the area of product bitopology. We used perfect mapping to prove various results. One can follow from literatures of bitopology that; various types of pairwise mappings play crucial roles to contradict results related to various pairwise concepts. Our idea may be extended on other types of Lindelöf spaces of a bitopological space. These methods give short and concrete ways to prove various results in product of Lindelöf spaces. We hope, this paper will attract attentions of topologists, economists and researchers of other branches. The connection between countability and $p_{1^{-}}$ Lindelöfness or $p_{1-}\{\emptyset\}$-generator may help economists to use bitopological space, Lindelöfness etc. in their respective research areas as one may refer to ([41],[43]), where authors studied utility functions and various results based on compactness, Lindelöfness, order and other properties of bitopology and general topology.

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