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Persistence of the incompressible Euler equations in a Besov space $\mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d)$

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Abstract

The unique existence of a solution of the incompressible Euler equations in a critical Besov space $\mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d)$ for $d \geq 2$ is investigated. The global existence of a solution of two-dimensional Euler equations is also discussed.

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1 Main theorems and terminology

The non-stationary Euler equations of an ideal incompressible fluid

$$\begin{aligned} &\frac{\partial}{\partial t}u + (u,\nabla)u = -\nabla p, \\ &\operatorname{div} u = 0 \end{aligned} \tag{1}$$

are considered. Here $u(x,t) = (u^1, u^2, \dots, u^d)$ is the Eulerian velocity of a fluid flow and $(u,\nabla)u^k = \sum_{i=1}^d u^i \partial_i u^k$, $k = 1,2,\dots,d$ with $\partial_i \equiv \frac{\partial}{\partial x_i}$.

The best local existence and uniqueness results known for the Euler equations (1) in Besov spaces are a series of theorems on the space $\mathbf{B}^{d/p+1}_{p,1}(\mathbb{R}^d)$ with $1 < p \leq \infty$ (see the introductions in [1, 2] for details and the references therein). The local existence for the limit case of $p = 1$ has not been reported yet possibly due to the lack of L^1-estimates. On the other hand, the ill-posedness of the Euler equations in [3] for a range of Besov spaces has been recently studied, which signifies that it is worthwhile to clarify either the well-posedness or the ill-posedness of the solutions in some particular Besov spaces. This is why the existence problem in the space $\mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d)$ is not trivial even though it is smaller than the space $\mathbf{B}^1_{\infty,1}(\mathbb{R}^d)$.

This paper takes care of the local unique existence of the solution to the Euler equations (1) in a critical Besov space $\mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d)$ and of the global existence for a two-dimensional case. Our main results are the following.

Theorem 1.1 (Local existence and uniqueness) *For any divergence-free vector field* $u_0 \in \mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d)$, *there exists a positive time* T *for which the initial value problem* (1) *with* $u|_{t=0} = u_0$ *has a unique solution* u *in the space* $C([0,T];\mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d))$.

Furthermore,

Theorem 1.2 (2-D global existence) *For any divergence-free vector field* $u_0 \in \mathbf{B}^3_{1,1}(\mathbb{R}^2)$, *there exists a unique solution* $u \in C([0,\infty); \mathbf{B}^3_{1,1}(\mathbb{R}^2))$ *of the problem* (1) *with* $u|_{t=0} = u_0$.

The compactness argument was used for the main literature in the proof employed by the authors in [2], and one of the primary difficulties of estimates was to get rid of the L^1-singularity. To take care of it, we present some new estimates together with substantial modifications of the identities and of the estimates proved in [2]. The essential tools for *a priori* estimates are Bony's para-product formula and Littlewood-Paley decomposition. An Osgood-type ordinary differential inequality is solved to complete the compactness argument.

For the proof of global existence in two-dimensional case, the limiting case of Beale-Kato-Majda inequality in 2-D is fetched, which is a modification of the inequality originally proved in [4] by Vishik. In [5], Chae proved the global existence of velocity in the Triebel-Lizorkin spaces $F^3_{1,q}(\mathbb{R}^2)$, $1 < q < \infty$, for the 2-D Euler equations, and also discussed the vorticity existence in the spaces $F^2_{1,q}(\mathbb{R}^2)$, $q = 1$ or ∞. Since the spaces $B^s_{q,q}(\mathbb{R}^d)$ are equivalent to the spaces $F^s_{q,q}(\mathbb{R}^d)$, our 2-D global velocity existence theorem is similar to Chae's theorem displayed in [5].

Here are some notations. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz class of rapidly decreasing functions. Consider a nonnegative radial function $\chi \in \mathcal{S}(\mathbb{R}^d)$ satisfying $\operatorname{supp}\chi \subset \{\xi \in \mathbb{R}^d : |\xi| \le \frac{5}{6}\}$ and $\chi = 1$ for $|\xi| \le \frac{3}{5}$. Set $h_j(\xi) \equiv \chi(2^{-j-1}\xi) - \chi(2^{-j}\xi)$, and it can be easily seen that

$$\chi(\xi) + \sum_{j=0}^{\infty} h_j(\xi) = 1 \quad \text{for } \xi \in \mathbb{R}^d.$$

Let φ_j and Φ be functions defined by $\varphi_j \equiv \mathcal{F}^{-1}(h_j)$, $j \ge 0$ and $\Phi \equiv \mathcal{F}^{-1}(\chi)$, where $\mathcal{F}$ represents the Fourier transform on $\mathbb{R}^d$. Note that φ_j is a mollifier of φ_0, that is, $\varphi_j(x) \equiv 2^{jd}\varphi_0(2^j x)$ (or $\hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j}\xi)$). One can readily check that

$$\Phi(x) + \sum_{j=0}^{k-1} \varphi_j(x) = 2^{kd}\Phi(2^k x) \quad \text{for } k \ge 1.$$

For $f \in \mathcal{S}'(\mathbb{R}^d)$, denote $\Delta_j f \equiv h_j(D)f = \varphi_j * f$ if $j \ge 0$, $\Delta_{-1} f \equiv \Phi * f$ and $\Delta_j f = 0$ if $j \le -2$. The partial sums are also defined: $S_k f \equiv \sum_{j=-\infty}^{k} \Delta_j f$ for $k \in \mathbb{Z}$. Assume that $s \in \mathbb{R}$, and $1 \le p, q \le \infty$. The Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ are defined by

$$f \in B^s_{p,q}(\mathbb{R}^d) \quad \Leftrightarrow \quad \{\|2^{js}\Delta_j f\|_{L^p}\}_{j\in\mathbb{Z}} \in l^q.$$

The corresponding spaces of vector-valued functions are denoted by the bold faced symbols. For example, the product space $L^1(\mathbb{R}^d)^d$ is denoted by $\mathbf{L}^1(\mathbb{R}^d)$ and the corresponding triple Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ by $\mathbf{B}^s_{p,q}(\mathbb{R}^d) \equiv B^s_{p,q}(\mathbb{R}^d)^d$. Note that the classical Hölder spaces $\mathbf{C}^s(\mathbb{R}^d)$ are equivalent to the Besov spaces $\mathbf{B}^s_{\infty,\infty}(\mathbb{R}^d)$ (if $s \in \mathbb{R} - \mathbb{N}$); see, for example, p.26 in [6] or [7].

Notation Throughout this paper (especially in Section 4), the notation $X \lesssim Y$ means that $X \le CY$, where C is a fixed but unspecified constant. Unless explicitly stated otherwise,

C may depend on the dimension d and various other parameters (such as exponents), but not on the functions or variables $(u, v, f, g, x_i, \dots)$ involved.

2 Local existence and uniqueness of the solution

The proof of Theorem 1.1 is presented in this section. Initial velocity $u|_{t=0} = u_0 \in \mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d)$ is given. In order to prove that the velocity $u(t)$ (representing the solution of the Euler equations (1)) stays (locally) in the function space $\mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d)$, we start with defining a sequence $\{u_m\}_{m\in\mathbb{N}}$ of vector fields depending on time by means of the following restrictions on each initial vector field:

$$u_m|_{t=0} = S_m u_0, \quad m = 1, 2, 3, \dots,$$
$$\frac{\partial}{\partial t} u_m + (u_m, \nabla) u_m = -\nabla p_m. \tag{2}$$

Then we first note that $u_m(0) \in \mathbf{C}^s(\mathbb{R}^d)$ for any $s \in (1, \infty)$. Therefore, for fixed $s \in (d+2, \infty)$, classical results (see [6]) say that for each m, there exist a maximal time $T^*_m \in (0, \infty]$ and a solution u_m to the Euler equations (2) in $C([0, T^*_m); \mathbf{C}^s(\mathbb{R}^d))$. In case of dimension 2, it is well-known that $T^*_m = \infty$. That is, there is a global solution $u_m \in C([0, \infty); \mathbf{C}^s(\mathbb{R}^2))$.

2.1 Compactness of the sequence $\{u_m\}_{m\in\mathbb{N}}$

Take the Δ_j operator and add the term $(S_j u_m, \nabla)\Delta_j u_m$ on both sides of (2) to have

$$\frac{\partial}{\partial t} \Delta_j u_m + (S_j u_m, \nabla)\Delta_j u_m = (S_j u_m, \nabla)\Delta_j u_m - \Delta_j (u_m, \nabla) u_m - \Delta_j \nabla p_m.$$

Consider the *trajectory flow* $\{X^m_j(x, t)\}$ *along* $S_j u_m$ defined by the solutions of the ordinary differential equations

$$\begin{cases} \frac{\partial}{\partial t} X^m_j(x, t) = (S_j u_m)(X^m_j(x, t), t), \\ X^m_j(x, 0) = x \end{cases}$$

(observe $\operatorname{div} S_j u_m = 0$ implies that $x \mapsto X^m_j(x, t)$ is a volume preserving mapping) to get

$$\begin{aligned} \big\|\Delta_j u_m(t)\big\|_{\mathbf{L}^1} &\le \|\Delta_j S_m u_0\|_{\mathbf{L}^1} + \int_0^t \|\Delta_j \nabla p_m\|_{\mathbf{L}^1}\, d\tau \\ &\quad + \int_0^t \big\|(S_j u_m, \nabla)\Delta_j u_m - \Delta_j\big((u_m, \nabla) u_m\big)\big\|_{\mathbf{L}^1}\, d\tau. \end{aligned} \tag{3}$$

Multiply $2^{j(d+1)}$ on both sides and sum up together to achieve

$$\begin{aligned} \big\|u_m(t)\big\|_{\mathbf{B}^{d+1}_{1,1}} &\le \|S_m u_0\|_{\mathbf{B}^{d+1}_{1,1}} + \int_0^t \|\nabla p_m\|_{\mathbf{B}^{d+1}_{1,1}}\, d\tau \\ &\quad + \int_0^t \sum_{j\ge -1} 2^{j(d+1)} \big\|(S_{j-2} u_m, \nabla)\Delta_j u_m - \Delta_j\big((u_m, \nabla) u_m\big)\big\|_{\mathbf{L}^1}\, d\tau. \end{aligned}$$

Propositions 4.4 and 4.5 in Section 4 yield

$$\big\|u_m(t)\big\|_{\mathbf{B}^{d+1}_{1,1}} \le C_0 \|u_0\|_{\mathbf{B}^{d+1}_{1,1}} + C_0 \int_0^t \big\|\nabla u_m(\tau)\big\|_{\mathbf{B}^{d}_{1,1}} \big\|u_m(\tau)\big\|_{\mathbf{B}^{d+1}_{1,1}}\, d\tau \tag{4}$$

for some constant $C_0 > 0$. By virtue of Gronwall's inequality, this leads to

$$\sup_{0\le\tau\le t}\|u_m(\tau)\|_{\mathbf{B}^{d+1}_{1,1}} \le C_0\|u_0\|_{\mathbf{B}^{d+1}_{1,1}} \exp\left\{C_0\int_0^t \sup_{0\le\tau'\le\tau}\|u_m(\tau')\|_{\mathbf{B}^{d+1}_{1,1}}\,d\tau\right\}. \tag{5}$$

Let $\lambda(\cdot)$ satisfy the following ordinary differential equation:

$$\frac{d}{dt}\lambda = C_0\lambda^2, \qquad \lambda(0) = C_0\|u_0\|_{\mathbf{B}^{d+1}_{1,1}}, \tag{6}$$

and let

$$\lambda_1(t) \equiv C_0\|u_0\|_{\mathbf{B}^{d+1}_{1,1}} \exp\left\{C_0\int_0^t \sup_{0\le\tau'\le\tau}\|u_m(\tau')\|_{\mathbf{B}^{d+1}_{1,1}}\,d\tau\right\}.$$

Then from (5) it can be noticed that

$$\frac{d}{dt}\lambda_1 \le C_0\lambda_1^2, \qquad \lambda_1(0) = C_0\|u_0\|_{\mathbf{B}^{d+1}_{1,1}}. \tag{7}$$

The time $T_1 > 0$ is chosen to be less than the blow-up time for (6). Then, by solving the separable ordinary differential inequality (7), we see that $\lambda_1(t) \le \lambda(t)$ for $t \in [0, T_1]$. Indeed, (7) leads to $-\frac{d}{dt}(\frac{1}{\lambda_1}) \le C_0$. This yields that for $t \in [0, T_1]$,

$$\lambda_1(t) \le \frac{C_0\|u_0\|_{\mathbf{B}^{d+1}_{1,1}}}{1 - tC_0^2\|u_0\|_{\mathbf{B}^{d+1}_{1,1}}} = \lambda(t).$$

Hence we have that

$$\sup_{0\le\tau\le t}\|u_m(\tau)\|_{\mathbf{B}^{d+1}_{1,1}} \le \lambda(t), \quad t \in [0, T_1] \tag{8}$$

for all $m \in \mathbb{N}$, that is, the sequence $\{u_m\}_{m\in\mathbb{N}}$ is uniformly bounded in $L^\infty([0, T_1]; \mathbf{B}^{d+1}_{1,1})$. From the blow-up criterion on p.77 in [6], saying that

$$T_m^* < \infty \quad \Rightarrow \quad \int_0^{T_m^*}\|u_m(\tau)\|_{\mathbf{C}^1}\,d\tau = \infty,$$

and from the fact that $\|u_m(\tau)\|_{\mathbf{C}^1} \le \|u_m(\tau)\|_{\mathbf{B}^{d+1}_{1,1}}$, we see that $T_1 > 0$ is a lower bound of $\{T_m^* : m \in \mathbb{N}\}$.

We close this section by explaining the continuity of u_m on $[0, T_1]$ with values in the Besov space $\mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d)$.

Lemma 2.1 (Temporal regularity) *Suppose that v is a solution for the Euler equations* (1) *staying inside of $L^\infty([0, T]; \mathbf{B}^{d+1}_{1,1})$ with initial velocity $v_0 \in \mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d)$. Then v is continuous on $[0, T]$ with values in $\mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d)$, that is, $v \in C([0, T]; \mathbf{B}^{d+1}_{1,1})$.*

Proof First, applying Propositions 4.2 and 4.5 in Section 4 to the Euler equations (1), we can deduce that $\frac{\partial}{\partial t}v \in L^\infty([0, T]; \mathbf{B}^{d}_{1,1})$, and so $v \in W^{1,\infty}([0, T]; \mathbf{B}^{d}_{1,1}) \subset C([0, T]; \mathbf{B}^{d}_{1,1})$.

For any $\ell \in \mathbb{N}$, we put $w_\ell \equiv S_\ell v$. We will demonstrate that the sequence $\{w_\ell\}_{\ell\in\mathbb{N}}$ converges to v in $L^\infty([0,T];\mathbf{B}^{d+1}_{1,1})$. As in the beginning of Section 2.1, we obtain that

$$\frac{\partial}{\partial t}\Delta_j v + (S_j v, \nabla)\Delta_j v = (S_j v, \nabla)\Delta_j v - \Delta_j (v,\nabla)v - \Delta_j \nabla p$$

for $j \in \mathbf{N}$. The interchange of the two operators $\frac{\partial}{\partial t}$ and Δ_j on the left-hand side follows from the fact that $\frac{\partial}{\partial t}v \in L^\infty([0,T];\mathbf{L}^\infty(\mathbb{R}^d))$. Since $\Delta_j v$ is absolutely continuous on $[0,T]$ with values in $\mathbf{L}^1(\mathbb{R}^d)$, we get

$$\begin{aligned}\big\|\Delta_j v(t)\big\|_{\mathbf{L}^1} \le \|\Delta_j v_0\|_{\mathbf{L}^1} &+ \int_0^t \|\Delta_j \nabla p\|_{\mathbf{L}^1}\,d\tau \\ &+ \int_0^t \big\|(S_j v,\nabla)\Delta_j v - \Delta_j\big((v,\nabla)v\big)\big\|_{\mathbf{L}^1}\,d\tau.\end{aligned}$$

This implies that for $t \in [0,T]$,

$$\begin{aligned}&\big\|v(t) - w_\ell(t)\big\|_{\mathbf{B}^{d+1}_{1,1}} \\ &\quad\lesssim \sum_{j\ge\ell} 2^{j(d+1)}\big\|\Delta_j v(t)\big\|_{\mathbf{L}^1} \\ &\quad\lesssim \sum_{j\ge\ell} 2^{j(d+1)}\|\Delta_j v_0\|_{\mathbf{L}^1} + \int_0^t \sum_{j\ge\ell} 2^{j(d+1)}\|\Delta_j\nabla p\|_{\mathbf{L}^1}\,d\tau \\ &\qquad + \int_0^t \sum_{j\ge\ell} 2^{j(d+1)}\big\|(S_{j-2}v,\nabla)\Delta_j v - \Delta_j\big((v,\nabla)v\big)\big\|_{\mathbf{L}^1}\,d\tau.\end{aligned}$$

The first term of the right-hand side converges to zero as ℓ tends to infinity because $v_0 \in \mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d)$. By virtue of Propositions 4.4 and 4.5 in Section 4 and the fact that $v(t) \in \mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d)$, the second and third terms of the right-hand side also converge to zero as ℓ tends to infinity. Hence the sequence $\{w_\ell\}_{\ell\in\mathbb{N}}$ converges to v in $L^\infty([0,T];\mathbf{B}^{d+1}_{1,1})$.

From the estimate

$$\begin{aligned}\big\|w_\ell(s) - w_\ell(t)\big\|_{\mathbf{B}^{d+1}_{1,1}} &= \big\|S_\ell\big(v(s)-v(t)\big)\big\|_{\mathbf{B}^{d+1}_{1,1}} \\ &\lesssim \sum_{j=-1}^{\ell+1} 2^{j(d+1)}\big\|\Delta_j\big(v(s)-v(t)\big)\big\|_{\mathbf{L}^1} \\ &\lesssim 2^{\ell+1}\big\|v(s)-v(t)\big\|_{\mathbf{B}^d_{1,1}}\end{aligned}$$

together with the fact that $v \in C([0,T];\mathbf{B}^d_{1,1})$, we can deduce that each w_ℓ is continuous on $[0,T]$ with values in $\mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d)$. In all, the limit v is continuous on $[0,T]$ with values in $\mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d)$. □

From this lemma, we observe that $u_m \in C([0,T_1];\mathbf{B}^{d+1}_{1,1})$. We also notice that the sequence $\{\frac{\partial}{\partial t}u_m\}_{m\in\mathbb{N}}$ is *uniformly* bounded in $L^\infty([0,T_1];\mathbf{B}^d_{1,1})$, thanks to Propositions 4.2 and 4.5.

2.2 Convergence of the sequence $\{u_m\}_{m\in\mathbb{N}}$

We now select a strictly positive time T_2 depending on $\|u_0\|_{\mathbf{B}^{d+1}_{1,1}}$ so that the sequence $\{u_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty([0,T_2];\mathbf{B}^d_{1,1})$. To do this, subtract the two relations on (2) to get

$$\begin{aligned}
&\frac{\partial}{\partial t}(u_{m+\ell}-u_m)+(u_{m+\ell},\nabla)(u_{m+\ell}-u_m)\\
&\qquad +(u_{m+\ell}-u_m,\nabla)u_m\\
&\quad =\pi(u_{m+\ell}-u_m,u_{m+\ell})+\pi(u_m,u_{m+\ell}-u_m),\\
&(u_{m+\ell}-u_m)|_{t=0}=\Delta_{m+\ell}u_0,
\end{aligned}\tag{9}$$

where we set

$$\pi(u,v)\equiv\sum_{i,j=1}^{d}\nabla\Delta^{-1}\partial_i u^j\partial_j v^i$$

(refer to Section 2.5 in [6]). Take the Δ_j operator, and add $(S_{j-2}u_{m+\ell},\nabla)\Delta_j(u_{m+\ell}-u_m)$ on both sides of (9) to have

$$\begin{aligned}
&\big\|(u_{m+\ell}-u_m)(t)\big\|_{\mathbf{B}^d_{1,1}}\\
&\quad\le\|\Delta_{m+\ell}u_0\|_{\mathbf{B}^d_{1,1}}\\
&\qquad+\int_0^t\sum_{j\ge-1}\big\|(S_{j-2}u_{m+\ell},\nabla)\Delta_j(u_{m+\ell}-u_m)-\Delta_j(u_m,\nabla)(u_{m+\ell}-u_m)\big\|_{\mathbf{L}^1}\,d\tau\\
&\qquad+\int_0^t\big\|(u_{m+\ell}-u_m,\nabla)u_m\big\|_{\mathbf{B}^d_{1,1}}\,d\tau\\
&\qquad+\int_0^t\big\|\pi(u_{m+\ell}-u_m,u_{m+\ell})+\pi(u_m,u_{m+\ell}-u_m)\big\|_{\mathbf{B}^d_{1,1}}\,d\tau.
\end{aligned}$$

Propositions 4.2-4.5 and estimate (8) can be used as before to get

$$\begin{aligned}
\|u_{m+\ell}-u_m\|_{L^\infty([0,T];\mathbf{B}^d_{1,1})}\le{}&\|\Delta_{m+\ell}u_0\|_{\mathbf{B}^d_{1,1}}\\
&+C_1T\lambda(T)\|u_{m+\ell}-u_m\|_{L^\infty([0,T];\mathbf{B}^d_{1,1})},
\end{aligned}$$

for $T\le T_1$ and some constant $C_1>0$ independent of $\|u_0\|_{\mathbf{B}^{d+1}_{1,1}}$, where $\lambda(\cdot)$ is defined in (6). Choose $T_2>0$ small enough to ensure $T_2<\min\{T_1,\frac{1}{2C_1\lambda(T_1)}\}$, and we have

$$\|u_{m+\ell}-u_m\|_{L^\infty([0,T_2];\mathbf{B}^d_{1,1})}\le 2\|\Delta_{m+\ell}u_0\|_{\mathbf{B}^d_{1,1}}\le 2^{-m+\ell}\|\Delta_{m+\ell}u_0\|_{\mathbf{B}^{d+1}_{1,1}}.$$

This implies that $\{u_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T_2];\mathbf{B}^d_{1,1}(\mathbb{R}^d))$. Hence there exists a strong limit u of the sequence $\{u_m\}_{m\in\mathbb{N}}$ in the space $C([0,T_2];\mathbf{B}^d_{1,1}(\mathbb{R}^d))$.

We point out that the sequence of pressures $\{p_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty([0,T_2]; H^1(\mathbb{R}^d))$. In fact, since each p_m can be represented by

$$p_m = \sum_{i,j=1}^{d} (-\Delta)^{-1} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u_m^i u_m^j = \sum_{i,j=1}^{d} R_i R_j u_m^i u_m^j,$$

where R_i's are the d-dimensional Riesz transforms, we have, for $0 \le t \le T_2$,

$$\begin{aligned} \|p_{m+\ell}(t) - p_m(t)\|_{L^2} &\lesssim \sum_{i,j=1}^{d} \|u_{m+\ell}^i(t) u_{m+\ell}^j(t) - u_m^i(t) u_m^j(t)\|_{L^2} \\ &\lesssim \left(\|u_{m+\ell}(t)\|_{\mathbf{L}^\infty} + \|u_m(t)\|_{\mathbf{L}^\infty}\right) \|u_{m+\ell}(t) - u_m(t)\|_{\mathbf{B}_{1,1}^d} \end{aligned}$$

and

$$\begin{aligned} &\|\nabla p_{m+\ell}(t) - \nabla p_m(t)\|_{\mathbf{L}^2} \\ &\quad \lesssim \left(\|u_{m+\ell}(t)\|_{\mathbf{B}_{1,1}^{d+1}} + \|u_m(t)\|_{\mathbf{B}_{1,1}^{d+1}}\right) \|u_{m+\ell}(t) - u_m(t)\|_{\mathbf{B}_{1,1}^d}. \end{aligned}$$

Hence there exists $p \in L^\infty([0,T_2]; H^1(\mathbb{R}^d))$ such that

$$p_m \to p \quad \text{strongly in } L^\infty\big([0,T_2]; H^1\big(\mathbb{R}^d\big)\big). \tag{10}$$

2.3 Local existence of a solution

We first claim that the limit u stays in $L^\infty([0,T_2]; \mathbf{B}_{1,1}^{d+1}(\mathbb{R}^d))$. Since the sequence $\{u_m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty([0,T_2]; \mathbf{B}_{1,1}^{d+1}(\mathbb{R}^d))$ and the sequence $\{\frac{\partial}{\partial t} u_m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty([0,T_2]; \mathbf{B}_{1,1}^{d}(\mathbb{R}^d))$ (see the last paragraph of Section 2.1), we can find two vector fields $v \in L^\infty([0,T_2]; \mathbf{B}_{1,1}^{d+1}(\mathbb{R}^d))$ and $w \in L^\infty([0,T_2]; \mathbf{B}_{1,1}^{d}(\mathbb{R}^d))$ satisfying (after possibly choosing subsequences)

$$u_m \rightharpoonup v \quad \text{weakly* in } L^\infty\big([0,T_2]; \mathbf{B}_{1,1}^{d+1}\big(\mathbb{R}^d\big)\big)$$

and

$$\frac{\partial}{\partial t} u_m \rightharpoonup w \quad \text{weakly* in } L^\infty\big([0,T_2]; \mathbf{B}_{1,1}^{d}\big(\mathbb{R}^d\big)\big).$$

On the other hand, since the sequence $\{u_m\}_{m\in\mathbb{N}}$ converges strongly to u in $C([0,T_2]; \mathbf{B}_{1,1}^{d}(\mathbb{R}^d))$, we have that

$$u_m \rightharpoonup u \quad \text{weakly* in } L^\infty\big([0,T_2]; \mathbf{B}_{1,1}^{d}\big(\mathbb{R}^d\big)\big).$$

Due to the fact that the two limits should coincide, we have $u = v \in L^\infty([0,T_2]; \mathbf{B}_{1,1}^{d+1}(\mathbb{R}^d))$, and $\frac{\partial}{\partial t} u = w \in L^\infty([0,T_2]; \mathbf{B}_{1,1}^{d}(\mathbb{R}^d))$. This implies that u is absolutely continuous on $[0,T_2]$ with values in $\mathbf{B}_{1,1}^{d}(\mathbb{R}^d)$.

We now verify that u satisfies the Euler equations (1). We notice that the convergence of the sequence $\{u_m\}_{m\in\mathbb{N}}$ to the limit u in $C([0,T_2]; \mathbf{B}_{1,1}^{d}(\mathbb{R}^d))$ implies that

$$u_m \to u \quad \text{strongly in } L^\infty\big([0,T_2]; \mathbf{L}^2\big(\mathbb{R}^d\big)\big). \tag{11}$$

(See the inclusion (34).) We also note that u is absolutely continuous on $[0, T_2]$ with values in $\mathbf{L}^2(\mathbb{R}^d)$. For any test vector field $\rho \in \mathcal{S}(\mathbb{R}^d)$ and test function $\theta \in C^\infty([0, T_2])$ with $\theta(T_2) = 0$, apply the $L^2(\mathbb{R}^d)$-inner product $\langle \cdot, \cdot \rangle$ to the Euler equations corresponding to (2) to get the functional formulation

$$\left\langle \frac{\partial}{\partial t} u_m(t), \rho \right\rangle \theta(t) + \big\langle \big(u_m(t), \nabla\big) u_m(t), \rho \big\rangle \theta(t) = -\langle \nabla p_m, \rho \rangle \theta(t).$$

Integrate both sides with respect to time to achieve

$$\int_0^{T_2} \left\langle \frac{\partial}{\partial t} u_m(\tau), \rho \right\rangle \theta(\tau)\, d\tau + \int_0^{T_2} \big\langle (u_m, \nabla) u_m, \rho \big\rangle \theta(\tau)\, d\tau = -\int_0^{T_2} \langle \nabla p_m, \rho \rangle \theta(\tau)\, d\tau.$$

As m tends to infinity, $u_m(0) = S_m u_0$ converges to u_0 in the sense of distribution (p.20 in [6]). Hence integration by parts and an application of (11) establish the limit of the first term:

$$\begin{aligned} \int_0^{T_2} \left\langle \frac{\partial}{\partial t} u_m(\tau), \rho \right\rangle \theta(\tau)\, d\tau &= -\big\langle u_m(0), \rho \big\rangle \theta(0) - \int_0^{T_2} \big\langle u_m(\tau), \rho \big\rangle \frac{\partial}{\partial t} \theta(\tau)\, d\tau \\ &\to -\langle u_0, \rho \rangle \theta(0) - \int_0^{T_2} \big\langle u(\tau), \rho \big\rangle \frac{\partial}{\partial t} \theta(\tau)\, d\tau \\ &= \int_0^{T_2} \left\langle \frac{\partial}{\partial t} u(\tau), \rho \right\rangle \theta(\tau)\, d\tau \end{aligned}$$

as $m \to \infty$. The last equality comes from the uniqueness of the limit since the sequence $\{\frac{\partial}{\partial t} u_m\}_{m \in \mathbb{N}}$ weak*-converges to $\frac{\partial}{\partial t} u$ in $L^\infty([0, T_2]; \mathbf{L}^2(\mathbb{R}^d))$. Hence the absolute continuity of the function $\langle u, \rho \rangle \theta$ on $[0, T_2]$ yields $u(0) = u_0$. Green's formula and (11) take the second term to

$$\begin{aligned} \int_0^{T_2} \big\langle \big(u_m(\tau), \nabla\big) u_m(\tau), \rho \big\rangle \theta(\tau)\, d\tau &= -\int_0^{T_2} \big\langle u_m(\tau), \big(u_m(\tau), \nabla\big) \rho \big\rangle \theta(\tau)\, d\tau \\ &\to -\int_0^{T_2} \big\langle u(\tau), \big(u(\tau), \nabla\big) \rho \big\rangle \theta(\tau)\, d\tau \\ &= \int_0^{T_2} \big\langle \big(u(\tau), \nabla\big) u(\tau), \rho \big\rangle \theta(\tau)\, d\tau \end{aligned}$$

as $m \to \infty$. On the other hand, (10) can be used to make the right-hand side get to

$$\begin{aligned} -\int_0^{T_2} \langle \nabla p_m, \rho \rangle \theta(\tau)\, d\tau &= \int_0^{T_2} \langle p_m, \operatorname{div} \rho \rangle \theta(\tau)\, d\tau \\ &\to \int_0^{T_2} \langle p, \operatorname{div} \rho \rangle \theta(\tau)\, d\tau = -\int_0^{T_2} \langle \nabla p, \rho \rangle \theta(\tau)\, d\tau \end{aligned}$$

as $m \to \infty$. Therefore we obtain the limit formulation

$$\int_0^{T_2} \left\langle \frac{\partial}{\partial t} u(\tau) + \big(u(\tau), \nabla\big) u(\tau) + \nabla p(\tau), \rho \right\rangle \theta(\tau)\, d\tau = 0$$

for all $\rho \in \mathcal{S}(\mathbb{R}^d)$ and $\theta \in C^\infty([0,T_2])$ with $\theta(T_2)=0$. Also, the constraint $\operatorname{div} u_m = 0$ turns into

$$0 = \langle \operatorname{div} u_m, \varrho\rangle = -\langle u_m, \nabla\varrho\rangle \to \langle u, \nabla\varrho\rangle = 0 \quad \text{for any } \varrho \in \mathcal{S}\big(\mathbb{R}^d\big)$$

as $m \to \infty$. Therefore we get that $\operatorname{div} u = 0$. In all, it has been shown that the limit u satisfies the Euler equations and the initial condition

$$\begin{cases} \frac{\partial}{\partial t}u + (u,\nabla)u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases}$$

So far, we have shown that u is a solution for the Euler equations (1) located in $L^\infty([0,T_2];\mathbf{B}^{d+1}_{1,1})$. Hence, by virtue of Lemma 2.1, we conclude that u belongs to $C([0,T_2];\mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d))$. We may continue to use this argument until the value $\|u(T^*)\|_{\mathbf{B}^{d+1}_{1,1}}$ blows up, that is, $\lim_{t\uparrow T^*}\|u(t)\|_{\mathbf{B}^{d+1}_{1,1}} = \infty$. This completes the proof of the local existence.

2.4 Uniqueness of a solution

In order to prove the uniqueness, we consider two solutions u and v of the system (1) in $C([0,T_1);\mathbf{B}^{d+1}_{1,1}(\mathbb{R}^d))$ with the same initial velocity. Subtraction of the equations satisfied by them says that the vector field $u-v$ obeys

$$\frac{\partial}{\partial t}(u-v) + (u,\nabla)(u-v) = (v-u,\nabla)v + \pi(u,u) - \pi(v,v),$$
$$(u-v)|_{t=0} = 0,$$

where $\pi(u,v)$ was defined previously. Propositions 4.2-4.5 and the argument used in Section 2.2 yield

$$\|u-v\|_{\mathbf{B}^{d}_{1,1}} \le C_2\int_0^T \big(\|u\|_{\mathbf{B}^{d+1}_{1,1}} + \|v\|_{\mathbf{B}^{d+1}_{1,1}}\big)\|u-v\|_{\mathbf{B}^{d}_{1,1}}\,d\tau$$

for some constant $C_2>0$. Then, for sufficiently small $T>0$ (see (8) in Section 2.1), we have

$$\|u-v\|_{C([0,T_1];\mathbf{B}^{d}_{1,1})} \le 2C_2\lambda(T_1)T\|u-v\|_{C([0,T_1];\mathbf{B}^{d}_{1,1})},$$

which in turn implies the uniqueness of a solution for (1) in $C([0,T^*);\mathbf{B}^{d+1}_{1,1})$.

3 2-D global existence - Proof of Theorem 1.2

The 2-D vorticity equation corresponding to the Euler equations (1) is given by

$$\frac{\partial}{\partial t}\omega + (u,\nabla)\omega = 0, \tag{12}$$

where $\omega \equiv \operatorname{curl} u$ with the initial vorticity $\omega_0 \equiv \operatorname{curl} u_0$. We consider the trajectory flow $\{X(x,t)\}$ along u defined by the solution of

$$\begin{cases} \frac{\partial}{\partial t}X(x,t) = u(X(x,t),t), \\ X(x,0) = x. \end{cases} \tag{13}$$

Then it is well-known that the solution $\omega(x,t)$ of the 2-D vorticity equation can be represented by

$$\omega(x,t) = \omega_0\big(X^{-1}(x,t)\big), \quad x \in \mathbb{R}^2. \tag{14}$$

We now point out that the same argument used in Section 2.1 can be employed to give the estimate

$$\big\|u(t)\big\|_{\mathbf{B}^3_{1,1}} \le C\|u_0\|_{\mathbf{B}^3_{1,1}} \exp\left\{C\int_0^t \big\|\nabla u(\tau)\big\|_{\mathbf{L}^\infty} + \big\|u(\tau)\big\|_{\mathbf{L}^2}\,d\tau\right\}$$

(see the estimates (4), (5) and the estimate (26) in Proposition 4.5). From the fact that $\|\nabla u\|_{\mathbf{L}^\infty} \lesssim \|\omega\|_{\dot{B}^0_{\infty,1}} \lesssim \|\omega\|_{B^{2/p}_{p,1}}$ $(2 < p < \infty)$ and the conservation of kinetic energy $\|u(t)\|_{\mathbf{L}^2} = \|u_0\|_{\mathbf{L}^2}$ (see p.25 in [8]), we get

$$\big\|u(t)\big\|_{\mathbf{B}^3_{1,1}} \le C_0\|u_0\|_{\mathbf{B}^3_{1,1}} \exp\left\{C_0\int_0^t \big(\big\|\omega(\tau)\big\|_{B^{2/p}_{p,1}} + \|u_0\|_{\mathbf{L}^2}\big)\,d\tau\right\}. \tag{15}$$

The estimate (15) suggests that we focus on proving that $\|\omega(t)\|_{B^{2/p}_{p,1}}$ does not blow up for all time. For this, we recall the limiting case of Beale-Kato-Majda inequality in $B^{2/p}_{p,1}$.

Proposition 3.1 (Vishik's inequality) *For* $0 \le s < 1$ *and* $1 \le p \le \infty$, *we have*

$$\big\|\omega(t)\big\|_{B^s_{p,1}} \lesssim \big(1 + \log\big(\big\|\nabla_x X(\cdot,t)\big\|_{L^\infty}\big\|\nabla_x X^{-1}(\cdot,t)\big\|_{L^\infty}\big)\big)\|\omega_0\|_{B^s_{p,1}}.$$

The original version of the proposition was proved by Vishik in the space $B^0_{\infty,1}(\mathbb{R}^d)$ in [4], and Chae later generalized it to the Besov spaces $B^0_{p,q}(\mathbb{R}^d)$ and the Triebel-Lizorkin spaces $F^0_{p,q}(\mathbb{R}^d)$ in [5]. Our version (in $B^s_{p,1}(\mathbb{R}^d)$) can be considered as a slight generalization of those, and the proof is almost the same as the original proof except for inserting the differential index 2^{js}.

Vishik's inequality explains the exponential growth of $B^d_{1,1}$-norm of vorticity $\omega(t)$ as follows. The identity induced from (13)

$$\frac{\partial}{\partial t}\nabla_x X(x,t) = (\nabla u)\big(X(x,t),t\big)\cdot\nabla_x X(x,t)$$

implies

$$\big\|\nabla_x X(\cdot,t)\big\|_{L^\infty} \le 1 + \int_0^t \big\|\nabla u\big(X(\cdot,\tau),\tau\big)\big\|_{L^\infty}\big\|\nabla_x X(\cdot,\tau)\big\|_{L^\infty}\,d\tau.$$

Gronwall's inequality and subsequently Vishik's inequality yield that

$$\begin{aligned}
\big\|\nabla_x X(\cdot,t)\big\|_{L^\infty} &\le \exp\left\{\int_0^t \big\|\nabla u(\cdot,\tau)\big\|_{L^\infty}\,d\tau\right\} \\
&\le \exp\left\{C\int_0^t \big\|\omega(\tau)\big\|_{B^{2/p}_{p,1}}\,d\tau\right\} \\
&\le \exp\left\{C\|\omega_0\|_{B^2_{1,1}}\int_0^t \big(1 + \log\big(\big\|\nabla_x X(\tau)\big\|_{L^\infty}\big\|\nabla_x X^{-1}(\tau)\big\|_{L^\infty}\big)\big)\,d\tau\right\},
\end{aligned}$$

where $p > 2$, and in particular, the third inequality follows from the fact that $\|\omega_0\|_{B^{2/p}_{p,1}} \lesssim \|\omega_0\|_{B^2_{1,1}}$. Similar techniques can be used to have

$$\begin{aligned}&\|\nabla_x X^{-1}(\cdot,t)\|_{L^\infty}\\&\quad\le \exp\biggl\{C\|\omega_0\|_{B^2_{1,1}}\int_0^t \bigl(1+\log\bigl(\|\nabla_x X(\tau)\|_{L^\infty}\|\nabla_x X^{-1}(\tau)\|_{L^\infty}\bigr)\bigr)\,d\tau\biggr\}.\end{aligned}$$

Combine these estimates together to get

$$\begin{aligned}&\|\nabla_x X(\cdot,t)\|_{L^\infty}\|\nabla_x X^{-1}(\cdot,t)\|_{L^\infty}\\&\quad\le \exp\biggl\{C\|\omega_0\|_{B^2_{1,1}}\int_0^t \bigl(1+\log\bigl(\|\nabla_x X(\tau)\|_{L^\infty}\|\nabla_x X^{-1}(\tau)\|_{L^\infty}\bigr)\bigr)\,d\tau\biggr\}.\end{aligned}$$

Or

$$\begin{aligned}&\log\bigl(\|\nabla_x X(\cdot,t)\|_{L^\infty}\|\nabla_x X^{-1}(\cdot,t)\|_{L^\infty}\bigr)\\&\quad\lesssim \|\omega_0\|_{B^2_{1,1}}\int_0^t \bigl(1+\log\bigl(\|\nabla_x X(\cdot,\tau)\|_{L^\infty}\|\nabla_x X^{-1}(\cdot,\tau)\|_{L^\infty}\bigr)\bigr)\,d\tau.\end{aligned}$$

Now Gronwall's inequality can be adopted to get

$$\log\bigl(\|\nabla_x X(\cdot,t)\|_{L^\infty}\|\nabla_x X^{-1}(\cdot,t)\|_{L^\infty}\bigr)\lesssim \exp\bigl(C\|\omega_0\|_{B^2_{1,1}}t\bigr).$$

Placing this into Proposition 3.1, we have

$$\|\omega(t)\|_{B^{2/p}_{p,1}}\lesssim \bigl\{1+\exp\bigl(C\|\omega_0\|_{B^2_{1,1}}t\bigr)\bigr\}\|\omega_0\|_{B^{2/p}_{p,1}}\lesssim \|\omega_0\|_{B^2_{1,1}}\exp\bigl(C\|\omega_0\|_{B^2_{1,1}}t\bigr).$$

Then put this estimate into (15), and we can conclude that

$$\|u(t)\|_{\mathbf{B}^3_{1,1}}\le C_{u_0}\exp\bigl\{C_{u_0}\bigl(e^{C_{u_0}t}+t\bigr)\bigr\},$$

where the constant C_{u_0} depends only on u_0 (and so ω_0). This completes the 2-D global existence of a solution in $B^3_{1,1}(\mathbb{R}^2)$.

4 *A priori* estimates

This section presents some estimates which have been used for the proofs of the main theorems. We first recall *Bony's para-product formula* which decomposes the product fg of two functions f and g into three parts:

$$fg = T_f g + T_g f + R(f,g),$$

where $T_f g$ represents Bony's *para-product* of f and g defined by $T_f g \equiv \sum_j S_{j-2} f\,\Delta_j g$ and $R(f,g)$ denotes the remainder of the para-product $R(f,g)\equiv\sum_{|i-j|\le 1}\Delta_i f\,\Delta_j g$. The estimates of para-product parts in $B^s_{1,1}(\mathbb{R}^d)$ are provided as follows.

Lemma 4.1 (Para-product estimate) *Let $s \in \mathbb{R}$. For any $f, g \in B^s_{1,1}(\mathbb{R}^d)$, we have*

$$\|T_f g\|_{B^s_{1,1}} \lesssim \|f\|_{L^\infty} \|g\|_{B^s_{1,1}},$$

and for each $i = 1, 2, \ldots, d$, we also have

$$\|T_{\partial_i f} g\|_{B^s_{1,1}} \lesssim \|f\|_{L^\infty} \|\nabla g\|_{\mathbf{B}^s_{1,1}}.$$

Proof By considering the supports of $\mathcal{F}(S_{j'-2} f \Delta_{j'} g)$

$$\operatorname{supp} \mathcal{F}(S_{j'-2} f \Delta_{j'} g) \subset \left\{ \xi : \frac{11}{15} 2^{j'-2} \le |\xi| \le \frac{25}{3} 2^{j'-2} \right\},$$

it can be easily noted that

$$\Delta_j T_f g = \sum_{j'=j-1}^{j+3} \Delta_j \{S_{j'-2} f \Delta_{j'} g\}.$$

Some computations can yield that

$$\begin{aligned} \|T_f g\|_{B^s_{1,1}} &\lesssim \sum_{j=-1}^{\infty} \sum_{|j-j'| \le 3, j' \ge 1} \|S_{j'-2} f\|_{L^\infty} 2^{js} \|\Delta_{j'} g\|_{L^1} \\ &\lesssim \|f\|_{L^\infty} \sum_{j=-1}^{\infty} 2^{js} \|\Delta_j g\|_{L^1} \lesssim \|f\|_{L^\infty} \|g\|_{B^s_{1,1}}. \end{aligned}$$

The second assertion follows from Bernstein's lemma (see p.16 in [6])

$$\begin{aligned} \|T_{\partial_i f} g\|_{B^s_{1,1}} &\lesssim \sum_{j=-1}^{\infty} \sum_{|j-j'| \le 3, j' \ge 1} 2^j \|S_{j'-2} f\|_{L^\infty} 2^{js} \|\Delta_{j'} g\|_{L^1} \\ &\lesssim \|f\|_{L^\infty} \|\nabla g\|_{\mathbf{B}^s_{1,1}}. \end{aligned}$$

□

The para-product estimate implies the following pointwise product estimate in $B^s_{1,1}(\mathbb{R}^d)$.

Proposition 4.2 (Product formula) *Let $s > 0$. For any $f, g \in B^s_{1,1}(\mathbb{R}^d)$, we have*

$$\|fg\|_{B^s_{1,1}} \lesssim \|f\|_{L^\infty} \|g\|_{B^s_{1,1}} + \|f\|_{B^s_{1,1}} \|g\|_{L^\infty}$$

and

$$\|f \cdot \nabla g\|_{\mathbf{B}^s_{1,1}} \lesssim \|f\|_{L^\infty} \|\nabla g\|_{\mathbf{B}^s_{1,1}} + \|\nabla f\|_{\mathbf{B}^s_{1,1}} \|g\|_{L^\infty}.$$

Proof Lemma 4.1 leaves us to measure the remainder term

$$\Delta_j R(f, g) = \Delta_j \sum_{j=-1}^{\infty} \sum_{i=j-1}^{j+1} \Delta_i f \Delta_j g = \sum_{i=j-3}^{\infty} \sum_{k=i-1}^{i+1} \Delta_j (\Delta_i f \Delta_k g).$$

The second equality follows from computing the supports of $\Delta_j(\Delta_i f \Delta_k g)$. We now get

$$\begin{aligned}\|R(f,g)\|_{B^s_{1,1}} &\lesssim \|g\|_{L^\infty}\sum_{j=-1}^{\infty}\sum_{i=j-3}^{\infty} 2^{(j-i)s}\bigl\{2^{is}\|\Delta_j\Delta_i f\|_{L^1}\bigr\}\\ &\lesssim \|g\|_{L^\infty}\sum_{m=-3}^{\infty} 2^{-ms}\left\{\sum_{j=-1}^{\infty} 2^{(j+m)s}\|\Delta_{j+m} f\|_{L^1}\right\}\\ &\lesssim \|g\|_{L^\infty}\|f\|_{B^s_{1,1}}.\end{aligned}$$

This also implies the second inequality in the statement. □

Here is a commutator estimate in L^1.

Lemma 4.3 (Commutator estimate) *For any differentiable function f and any function g, we have the following commutator estimate*:

$$\|[f,\Delta_j]\partial_i g\|_{L^1} \lesssim \|\nabla f\|_{\mathbf{L}^\infty}\|g\|_{L^1}, \quad i = 1,2,\dots,d,$$

where the commutator $[f,\Delta_j]h$ is defined as $f\Delta_j h - \Delta_j(fh)$.

Proof The desired estimate comes from the following observation:

$$\begin{aligned}[f,\Delta_j]\partial_i g(x) &= \int_{\mathbb{R}^d}\varphi_j(x-y)\bigl\{f(x)-f(y)\bigr\}\partial_i g(y)\,dy\\ &= 2^{j(d+1)}\int_{\mathbb{R}^d}\partial_i\varphi\bigl(2^j(x-y)\bigr)\bigl\{f(x)-f(y)\bigr\}g(y)\,dy + (\Delta_j\partial_i f)g(x)\\ &= 2^{j(d+1)}\int_{\mathbb{R}^d}\partial_i\varphi\bigl(2^j(x-y)\bigr)\left(\int_0^1\nabla f\bigl(x+\tau(y-x)\bigr)\cdot(x-y)\,d\tau\right)g(y)\,dy\\ &\quad + (\Delta_j\partial_i f)g(x)\\ &= \int_{\mathbb{R}^d}\partial_i\varphi(z)\left(z\cdot\int_0^1\nabla f\bigl(x-\tau 2^{-j}z\bigr)\,d\tau\right)g\bigl(x-2^{-j}z\bigr)\,dz + (\Delta_j\partial_i f)g(x).\end{aligned}$$ □

We now present the primary estimates which have been used for the proof of the main theorem.

Proposition 4.4 *Let $s\in\mathbb{R}$. For any vector field $u = (u_1,u_2,\dots,u_d)$ and a function g, we have*

$$\begin{aligned}&\sum_{j=-\infty}^{\infty} 2^{js}\|(S_j u,\nabla)\Delta_j g - \Delta_j(u,\nabla)g\|_{L^1}\\ &\quad\lesssim \|u\|_{\mathbf{B}^s_{1,1}}\|\nabla g\|_{\mathbf{L}^\infty} + \|g\|_{B^s_{1,1}}\|\nabla u\|_{\mathbf{L}^\infty}.\end{aligned}\tag{16}$$

We also have the estimate

$$\sum_{j=-\infty}^{\infty} 2^{js} \big\| (S_{j-2}u, \nabla)\Delta_j g - \Delta_j(u,\nabla)g \big\|_{L^1}$$
$$\lesssim \|\nabla u\|_{\mathbf{B}^s_{1,1}} \|g\|_{L^\infty} + \|g\|_{B^s_{1,1}} \|\nabla u\|_{\mathbf{L}^\infty}. \tag{17}$$

Proof We first observe that

$$(S_j u, \nabla)\Delta_j g - \Delta_j(u,\nabla)g = -\sum_{i=1}^{d} \Delta_j T_{\partial_i g} u_i + \sum_{i=1}^{d} T_{\partial_i \Delta_j g} S_j u_i \tag{18}$$
$$+ \sum_{i=1}^{d} [T_{u_i}\partial_i, \Delta_j]g \tag{19}$$
$$- \sum_{i=1}^{d} \big\{ \Delta_j R(u_i, \partial_i g) - R(S_j u_i, \Delta_j \partial_i g) \big\}, \tag{20}$$

where the bracket operator $[\cdot,\cdot]$ is defined as $[A,B] \equiv AB - BA$. In fact, use Bony's para-product formula to expand $\Delta_j(u,\nabla)g$ as follows:

$$\begin{aligned}
\Delta_j(u,\nabla)g &= \sum_{i=1}^{d} \Delta_j T_{\partial_i g} u_i + \Delta_j \sum_{i=1}^{d} \big\{ T_{u_i}\partial_i g + R(u_i, \partial_i g) \big\} \\
&= \sum_{i=1}^{d} \Delta_j T_{\partial_i g} u_i - \sum_{i=1}^{d} [T_{u_i}\partial_i, \Delta_j]g \\
&\quad + \sum_{i=1}^{d} T_{u_i - S_j u_i} \partial_i \Delta_j g - \sum_{i=1}^{d} T_{\partial_i \Delta_j g} S_j u_i \\
&\quad + \sum_{i=1}^{d} \big\{ \Delta_j R(u_i, \partial_i g) - R(S_j u_i, \Delta_j \partial_i g) \big\} + (S_j u, \nabla)\Delta_j g.
\end{aligned}$$

By reflecting the supports of functions in the expression, it can be seen that the term $\sum_{i=1}^{d} T_{u_i - S_j u_i} \partial_i \Delta_j g$ vanishes. Therefore, to complete the estimate, it suffices to assess the four terms in equations (18)-(20). First, Lemma 4.1 can be used to deliver the desired estimates (16) and (17) for the first two para-product terms of the right-hand side of (18). Now consider the supports, and we can see that

$$\begin{aligned}
\sum_{i=1}^{d} [T_{u^i}\partial_i, \Delta_j]g &= \sum_{i=1}^{d} \sum_{j'=-\infty}^{\infty} \big\{ S_{j'-2}u^i \Delta_{j'}(\partial_i \Delta_j g) - \Delta_j\big(S_{j'-2}u^i \partial_i \Delta_{j'} g\big) \big\} \\
&= \sum_{i=1}^{d} \sum_{j'=j-3}^{j+3} \big\{ S_{j'-2}u^i \Delta_j(\partial_i \Delta_{j'} g) - \Delta_j\big(S_{j'-2}u^i \partial_i \Delta_{j'} g\big) \big\}.
\end{aligned} \tag{21}$$

Therefore Lemma 4.3 leads to the estimates (16) and (17) for the third term (19). It remains to estimate the last term (20). We split (20) into two parts as follows:

$$\begin{aligned}
(20) &= -\sum_{i=1}^{d}\bigl\{\Delta_j R(u_i,\partial_i g) - R(S_j u_i,\Delta_j\partial_i g)\bigr\} \\
&= -\sum_{i=1}^{d}\bigl\{\Delta_j R\bigl(u^i - S_j u^i,\partial_i g\bigr)\bigr\} \\
&\quad -\sum_{i=1}^{d}\bigl\{\Delta_j R\bigl(S_j u^i,\partial_i g\bigr) - R\bigl(S_j u^i,\Delta_j\partial_i g\bigr)\bigr\} \\
&= \mathbf{I}_j + \mathbf{II}_j.
\end{aligned}$$

The first term $\mathbf{I}_j$ can be treated for two cases separately. That is, when $j = -1$, Bernstein's lemma yields

$$\|\mathbf{I}_{-1}\|_{L^1} = \Biggl\|\sum_{i=1}^{d}\Delta_j\sum_{j''=-1}^{0}\Delta_{-1}\Delta_0 u^i\Delta_{j''}\partial_i g\Biggr\|_{L^1} \lesssim \sum_{j''=-1}^{0}\|\Delta_0\nabla u\|_{\mathbf{L}^\infty}\|\Delta_{j''}g\|_{L^1}. \tag{22}$$

For the cases when $j \geq 0$, it suffices to assume $j \leq j'$ by considering the supports of functions in $\mathbf{I}_j$. Hence we have

$$\begin{aligned}
2^{js}\|\mathbf{I}_j\|_{L^1} &\lesssim \sum_{i=1}^{d}2^{js}\Biggl\|\Delta_j\sum_{|j'-j''|\leq 1, j'\geq 0}\Delta_{j'}\bigl(u^i - S_j u^i\bigr)\Delta_{j''}\partial_i g\Biggr\|_{L^1} \\
&\lesssim \sum_{i=1}^{d}\sum_{j'\geq j, j'\geq 0}2^{(j-j')s}\sum_{j''=j'-1}^{j'+1}2^{j''}\bigl\|\Delta_{j'}u^i\bigr\|_{L^\infty}2^{j's}\|\Delta_{j''}g\|_{L^1} \\
&\lesssim \sum_{j'\geq j}2^{(j-j')s}\sum_{j''=j'-1}^{j'+1}\|\Delta_{j'}\nabla u\|_{\mathbf{L}^\infty}2^{j''s}\|\Delta_{j''}g\|_{L^1} \\
&\lesssim \|\nabla u\|_{\mathbf{L}^\infty}\sum_{j'\geq j}2^{(j-j')s}\sum_{j''=j'-1}^{j'+1}2^{j''s}\|\Delta_{j''}g\|_{L^1}.
\end{aligned}$$

Therefore the estimate (22) together with the fact that

$$\begin{aligned}
&\sum_{j\geq -1}\sum_{j'\geq j}2^{(j-j')s}\sum_{j''=j'-1}^{j'+1}2^{j''s}\|\Delta_{j''}g\|_{L^1} \\
&\quad\lesssim \sum_{m\geq 0}2^{-m}\Biggl(3\sum_{j\geq -1}2^{(j+m)s}\|\Delta_{j+m}g\|_{L^1}\Biggr) \\
&\quad\lesssim \sum_{m\geq 0}2^{-m}\|g\|_{B^s_{1,1}} \lesssim \|g\|_{B^s_{1,1}}
\end{aligned}$$

leads to

$$\sum_{j=-1}^{\infty} 2^{js} \|\mathbf{I}_j\|_{L^1} \lesssim \|\nabla u\|_{\mathbf{L}^\infty} \|g\|_{B^s_{1,1}}. \tag{23}$$

A similar computation to that used above shows that

$$\sum_{j=-1}^{\infty} 2^{jd} \|\mathbf{I}_j\|_{L^1} \lesssim \|u\|_{\mathbf{B}^d_{1,1}} \|g\|_{B^{d+1}_{1,1}}. \tag{24}$$

The estimates (16) and (17) for $\mathbf{II}_j$ are obtained simply by using the commutator estimate (Lemma 4.3). By putting estimates together, the desired results can be achieved. □

The estimates for the pressure term ∇p are presented. From the Euler equations (1), we have that

$$-\Delta p = \sum_{i,j=1}^{d} \frac{\partial u^j}{\partial x_i} \frac{\partial u^i}{\partial x_j}. \tag{25}$$

Proposition 4.5 *Let $s > 0$. For any pair of divergence-free vector fields u and v, we have*

$$\|\pi(u,v)\|_{\mathbf{B}^{s+1}_{1,1}} \lesssim \|\nabla u\|_{\mathbf{L}^\infty} \|v\|_{\mathbf{B}^{s+1}_{1,1}} + \|u\|_{\mathbf{B}^{s+1}_{1,1}} \bigl(\|\nabla v\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^2}\bigr). \tag{26}$$

We also have

$$\|\pi(u,v)\|_{\mathbf{B}^{s}_{1,1}} \lesssim \|\nabla u\|_{\mathbf{L}^\infty} \|v\|_{\mathbf{B}^{s}_{1,1}} + \|u\|_{\mathbf{B}^{s+1}_{1,1}} \|v\|_{\mathbf{L}^\infty}, \tag{27}$$

and

$$\|\pi(u,v)\|_{\mathbf{B}^{s}_{1,1}} \lesssim \|\nabla v\|_{\mathbf{L}^\infty} \|u\|_{\mathbf{B}^{s}_{1,1}} + \|v\|_{\mathbf{B}^{s+1}_{1,1}} \|u\|_{\mathbf{L}^\infty}, \tag{28}$$

where

$$\pi(u,v) \equiv \sum_{i,j=1}^{d} \nabla \Delta^{-1} \partial_i u^j \partial_j v^i = \nabla \Delta^{-1} \operatorname{div}\bigl((u,\nabla)v\bigr).$$

Proof For $j \geq 0$, Bernstein's lemma can be used to obtain

$$\|\Delta_j \pi(u,v)\|_{\mathbf{L}^1} \lesssim 2^{-j} \sum_{i,k=1}^{d} \|\Delta_j \partial_i u^k \partial_k v^i\|_{L^1}$$

(refer to p.17 in [6]). Proposition 4.2 yields

$$\sum_{j=0}^{\infty} 2^{j(s+1)} \|\Delta_j \pi(u,v)\|_{\mathbf{L}^1} \lesssim \|\nabla u\|_{\mathbf{L}^\infty} \|\nabla v\|_{\mathbf{B}^{s}_{1,1}} + \|\nabla u\|_{\mathbf{B}^{s}_{1,1}} \|\nabla v\|_{\mathbf{L}^\infty} \tag{29}$$

and also

$$\sum_{j=0}^{\infty} 2^{js} \big\| \Delta_j \pi(u,v) \big\|_{\mathbf{L}^1} \lesssim \|\nabla u\|_{\mathbf{L}^\infty} \|v\|_{\mathbf{B}^s_{1,1}} + \|\nabla u\|_{\mathbf{B}^s_{1,1}} \|v\|_{\mathbf{L}^\infty}. \tag{30}$$

For $j = -1$, we note that

$$\begin{aligned}
\big\| \Delta_{-1}\pi(u,v) \big\|_{\mathbf{L}^1} &= \left\| \sum_{i,j=1}^{d} (\nabla \Phi) * \left(\Delta^{-1} \partial_i u^j \partial_j v^i\right) \right\|_{\mathbf{L}^1} \\
&= \left\| \sum_{i,j=1}^{d} \left(\Delta^{-1} \nabla \Phi\right) * \partial_j \partial_i \left(u^j v^i\right) \right\|_{\mathbf{L}^1} \\
&\le \frac{1}{d(d-2)\alpha(d)} \left\| \sum_{i,j=1}^{d} \partial_i \partial_j \left(\nabla \frac{1}{|x|^{d-2}} * \Phi \right) \right\|_{\mathbf{L}^1} \|u \otimes v\|_{\mathbf{L}^1},
\end{aligned} \tag{31}$$

where $\alpha(d)$ denotes the volume of the unit ball in $\mathbb{R}^d$. For the case of dimension $d = 2$, the factor $\frac{1}{d(d-2)\alpha(d)} \| \sum_{i,j=1}^{d} \partial_i \partial_j (\nabla \frac{1}{|x|^{d-2}} * \Phi) \|_{\mathbf{L}^1}$ in (31) ought to be replaced by $\frac{1}{2\pi} \| \sum_{i,j=1}^{2} \partial_i \partial_j (\nabla \log |x| * \Phi) \|_{\mathbf{L}^1}$. Observe that

$$\begin{aligned}
&\left\| \sum_{i,j=1}^{d} \partial_i \partial_j \left(\nabla \frac{1}{|x|^{d-2}} * \Phi \right) \right\|_{\mathbf{L}^1} \\
&\quad \le \sum_{i,j=1}^{d} \left(\left\| \left(\eta \nabla \frac{1}{|x|^{d-2}} \right) * (\partial_i \partial_j \Phi) \right\|_{\mathbf{L}^1} + \left\| \partial_i \partial_j \left((1-\eta) \nabla \frac{1}{|x|^{d-2}} \right) * \Phi \right\|_{\mathbf{L}^1} \right) \le C,
\end{aligned}$$

where $\eta \in C_0^\infty(\mathbb{R}^d)$ is a radial cut-off function satisfying $\eta(x) = 1$, $|x| \le 1$ and $\eta(x) = 0$, $|x| \ge 2$. Then we can conclude

$$\big\| \Delta_{-1}\pi(u,v) \big\|_{\mathbf{L}^1} \lesssim \|u\|_{\mathbf{L}^2} \|v\|_{\mathbf{L}^2} \tag{32}$$

or

$$\big\| \Delta_{-1}\pi(u,v) \big\|_{\mathbf{L}^1} \lesssim \|u\|_{\mathbf{L}^1} \|v\|_{\mathbf{L}^\infty}. \tag{33}$$

The fact that

$$\|u\|_{\mathbf{L}^2} \le \|u\|_{\mathbf{B}^0_{2,1}} \lesssim \|u\|_{\mathbf{B}^{1/2}_{1,1}} \lesssim \|u\|_{\mathbf{B}^{s+1}_{1,1}} \tag{34}$$

and the estimate (29) yield the first estimate (26). Also, the second estimate (27) follows from the estimate (30) together with (33). The last estimate (28) follows from the symmetry of π; $\pi(u,v) = \pi(v,u)$. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
Each of the authors made a comparable contribution to this paper. All authors read and approved the final manuscript.

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