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A magnetic regularity criterion for the 2D MHD equations with velocity dissipation

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Abstract

In this paper we consider an initial value problem of the 2D MHD equations with velocity dissipation and without magnetic diffusion. We establish a new magnetic regularity criterion in terms of the magnetic field. In contrast to the magnetic regularity criterion $\nabla b \in L^1(0,T;BMO(\mathbb{R}^2))$, our regularity criterion $\int_0^T (\|b\otimes b(s)\|_{B^0_{\infty,1}(\mathbb{R}^2)} + \|b\otimes b(s)\|_{L^2(\mathbb{R}^2)})\,ds < \infty$ is different; for example, our simplified regularity criterion $\int_0^T \|b(s)\|^2_{B^{\varepsilon}_{\infty,1}(\mathbb{R}^2)}\,ds < \infty$ requires higher time integrability and lower regularity of space.

MSC: 35A07; 35Q53

Keywords: 2D magnetohydrodynamic equations; velocity dissipation; regularity criterion

1 Introduction

In this paper we consider the global regularity on the 2D incompressible magnetohydrodynamic (MHD) equations with velocity dissipation and without magnetic diffusion,

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = b\cdot\nabla b + \Delta u, & x\in\mathbb{R}^2, t>0,\\ \partial_t b + u\cdot\nabla b = b\cdot\nabla u, & x\in\mathbb{R}^2, t>0,\\ \nabla\cdot u = \nabla\cdot b = 0, & x\in\mathbb{R}^2, t\geq 0,\\ u(0,x) = u_0(x), \quad b(0,x) = b_0(x), & x\in\mathbb{R}^2, \end{cases} \tag{1.1}$$

where $u = u(t,x)$ stands for the velocity of the fluid, $b = b(t,x)$ for the magnetic field, and $p = p(t,x)$ for the scalar pressure. Due to the lack of magnetic diffusion, the global well-posedness is extremely difficult and remains open.

The study of basic equations in fluid kinematics is one of the interesting fields. For example, we have the MHD equations [1, 2], the Benjamin-Bona-Mahony equations [3, 4], and the quasi-geostrophic equations [5–7]. Since the MHD equations have strong physical backgrounds [1, 2], the study of the MHD equations has attracted considerable interest and much progress has been made in the last few years. One of the fundamental problems regarding the MHD equations is that they develop singularities. This is due to the nonlinear coupling between the Navier-Stokes equations with a forcing induced by the magnetic field and the induction equation (see [8–10]).

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We first recall some of the recent progress in this direction. The local well-posedness for the system (1.1) has been proved in [11–13], respectively. Furthermore, Jiu and Niu [13] proved that the solution keeps smoothness up to time T if

$$b \in L^p\big(0,T;W^{2,q}\big(\mathbb{R}^2\big)\big) \quad \text{with } \frac{2}{p}+\frac{1}{q} \le 2, 1 \le p \le \frac{4}{3}, 2 < q \le \infty.$$

Jiu and Liu [14] discussed the global regularity for the 3D axisymmetric MHD equations with horizontal dissipation and vertical magnetic diffusion. Fan and Ozawa [15], and Zhou and Fan [16] obtained a regularity criterion on the velocity field $\nabla u \in L^1(0,T;L^\infty(\mathbb{R}^2))$ and the magnetic field $\nabla b \in L^1(0,T;BMO(\mathbb{R}^2))$, respectively. To the best of our knowledge, for the MHD equations (1.1), when the indicator of dissipation is larger than 1 and without zero magnetic diffusion the global well-posedness is also open. Due to lack of the magnetic diffusion, it is very difficult to get global estimates of the local solution in any Sobolev spaces. Very recently, Jiu, Niu, and Wu *et al.* [17] gave a new regularity criterion by the Besov space technique. Motivated by the ideas described in [17] and [18], the main goal of this paper is to establish another regularity criterion in terms of the condition on the magnetic field. Our main result is the following.

Theorem 1.1 *Assume that* $(u_0(x),b_0(x)) \in H^s(\mathbb{R}^2)$ $(s > 2)$ *with* $\nabla\cdot u_0(x) = \nabla\cdot b_0(x) = 0$. *Let* $(u(t,x),b(t,x))$ *be a local smooth solution of system* (1.1). *Then* $(u(t,x),b(t,x))$ *can be extended beyond time T if*

$$\int_0^T \big(\|b\otimes b(s)\|_{B^0_{\infty,1}(\mathbb{R}^2)} + \|b\otimes b(s)\|_{L^2(\mathbb{R}^2)}\big)\,ds < \infty. \tag{1.2}$$

Remark 1.1 $\forall\varepsilon > 0$, $B^\varepsilon_{\infty,1}$ is a Banach algebra and the embedding $B^\varepsilon_{\infty,1} \hookrightarrow B^0_{\infty,1} \hookrightarrow L^\infty$ hold, the condition (1.2) can be replaced by

$$\int_0^T \|b(s)\|^2_{B^\varepsilon_{\infty,1}(\mathbb{R}^2)}\,ds < \infty. \tag{1.3}$$

Based on the above observation, the condition (1.3) demands higher time integrability and lower regularity of space than the regularity condition imposed by Zhou and Fan [16].

The plan of this paper is as follows. In the next section, we state some notations and preliminary results in the standard theory of Besov spaces. In the third section, we first establish all tools needed to get a magnetic regularity, then we divide the proof into three steps to get the magnetic regularity criterion.

2 Notations and preliminaries

We first introduce the following notations. C denotes a positive constant which may vary from line to line. $X \lesssim Y$ means that there exists a positive harmless constant C such that $X \le CY$. We use the sub-indices (like C_s or $\lesssim_s$) to indicate the parameter dependence of the constant C. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz class of rapidly decreasing functions and $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions. For all $u \in \mathcal{S}'$, the Fourier transform $\mathcal{F}u$, also denoted by $\hat{u}$, is defined by

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}u(x)\,dx, \quad \forall\xi \in \mathbb{R}^d.$$

The inverse Fourier transform allows us to recover u from $\hat{u}$:

$$u(x) = \mathcal{F}^{-1}\hat{u}(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\,\hat{u}(\xi)\,d\xi.$$

$[A,B]$ stands for the commutator operator $AB - BA$, where A and B are any pair of operators on some Banach space.

We now recall some standard theories of Besov space (more details see [19]).

Let $\mathcal{C}$ denote the annulus $\{\xi \in \mathbb{R}^d : 3/4 \le |\xi| \le 8/3\}$. There exist two radial functions $\chi \in C_c^\infty(\mathcal{B}(0,4/3))$ and $\varphi \in C_c^\infty(\mathcal{C})$ both taking values in $[0,1]$ such that

$$\chi(\xi) + \sum_{j\ge 0}\varphi\bigl(2^{-j}\xi\bigr) = 1, \quad \forall \xi \in \mathbb{R}^d.$$

For every $u \in \mathcal{S}'(\mathbb{R}^d)$, the inhomogeneous dyadic blocks $\triangle_j$ are defined as follows: $\triangle_{-1}u = \chi(D)u$ and $\triangle_j u = \varphi(2^{-j}D)u$, $\forall j \ge 0$. The inhomogeneous low-frequency cut-off operator S_j is defined by

$$S_j u = \sum_{q=-1}^{j-1}\triangle_q u.$$

In the inhomogeneous case, the following Littlewood-Paley decomposition makes sense:

$$u = \sum_{j\ge -1}\triangle_j u, \quad u \in \mathcal{S}'\bigl(\mathbb{R}^d\bigr).$$

Let $s \in \mathbb{R}$ and $p,q \in [1,\infty]$, the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^d)$ is defined by

$$B^s_{p,q} = \left\{u \in \mathcal{S}'\bigl(\mathbb{R}^d\bigr), \|u\|_{B^s_{p,q}(\mathbb{R}^d)} := \left(\sum_{j\ge -1} 2^{qjs}\|\triangle_j u\|^q_{L^p}\right)^{1/q} < \infty\right\}.$$

In this paper, two kinds of the coupled space-time Besov spaces $L^r_T B^s_{p,q}$ and $\widetilde{L}^r_T B^s_{p,q}$ $(r \ge 1)$ are defined, respectively, as follows:

$$L^r_T B^s_{p,q} = \bigl\{u \in \mathcal{S}'\bigl(\mathbb{R}^d\bigr), \|u\|_{L^r_T B^s_{p,q}(\mathbb{R}^d)} := \bigl\|\bigl(2^{js}\|\triangle_j u\|_{L^p}\bigr)_{l^q}\bigr\|_{L^r_T} < \infty\bigr\},$$
$$\widetilde{L}^r_T B^s_{p,q} = \bigl\{u \in \mathcal{S}'\bigl(\mathbb{R}^d\bigr), \|u\|_{\widetilde{L}^r_T B^s_{p,q}(\mathbb{R}^d)} := \bigl(2^{js}\|\triangle_j u\|_{L^r_T L^p}\bigr)_{l^q} < \infty\bigr\}.$$

The following links between these spaces are direct results due to the Minkowski inequality:

$$L^r_T B^s_{p,q} \hookrightarrow \widetilde{L}^r_T B^s_{p,q}, \quad \text{if } q \ge r; \quad \text{and} \quad \widetilde{L}^r_T B^s_{p,q} \hookrightarrow L^r_T B^s_{p,q}, \quad \text{if } r \ge q. \tag{2.1}$$

In particular,

$$L^q_T B^s_{p,q} \approx \widetilde{L}^q_T B^s_{p,q}. \tag{2.2}$$

Bernstein's inequalities are fundamental in the analysis involving Besov spaces and these inequalities trade integrability for derivatives.

Lemma 2.1 [19] *Let $\mathcal{C}$ be an annulus and $\mathcal{B}$ a ball. Then there is a constant such that for all $k \in \mathbb{N} \cup \{0\}$, $1 \leq p \leq q \leq \infty$, and $f \in L^p$, we have*

$$\operatorname{supp}\hat{f} \subset \lambda\mathcal{B} \quad \Rightarrow \quad \|D^k f\|_q = \sup_{|\alpha|=k} \|\partial^\alpha f\|_q \leq C\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|f\|_p, \tag{2.3}$$

$$\operatorname{supp}\hat{f} \subset \lambda\mathcal{C} \quad \Rightarrow \quad C^{-1}\lambda^k\|f\|_p \leq \|D^k f\|_p \leq C\lambda^k\|f\|_p. \tag{2.4}$$

The Biot-Savart law will be used often to get the control between the gradient of velocity and the vorticity.

Lemma 2.2 [19] *For any divergence-free vector field u, there exists a universally positive constant C such that, for any $1 < p < \infty$, we have*

$$\|\nabla u\|_{L^p} \leq C\frac{p^2}{p-1}\|w\|_{L^p}, \tag{2.5}$$

here $w = \operatorname{curl} u = \nabla \times u$ is the vorticity.

Next, we state a commutator estimate involving the Riesz operator $\mathcal{R} = (-\Delta)^{-1}\operatorname{curldiv}$.

Lemma 2.3 [20] *The standard Riesz operator $\mathcal{R} = (-\Delta)^{-1}\operatorname{curldiv}$ is continuous and linear, it maps $L^p(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$. In particular, for all $f \in L^p(\mathbb{R}^d)$ the following estimate holds true*:

$$\|\mathcal{R}f\|_{L^p(\mathbb{R}^d)} \leq C_{d,p}\|f\|_{L^p(\mathbb{R}^d)}. \tag{2.6}$$

Lemma 2.4 [20] *If u is a smooth divergence-free vector field of $\mathbb{R}^2$ with vorticity, and f is a smooth function, then for all $p \in (1,\infty)$,*

$$\|[\mathcal{R}, u\cdot\nabla]f\|_{L^p} \leq C\|\nabla u\|_{L^p}\left(\|f\|_{L^2} + \|f\|_{B^0_{\infty,1}}\right). \tag{2.7}$$

Proof For the sake of convenience, we sketch the proof. Without loss of generality, we assume that the functions $u \in C_c^\infty(\mathbb{R}^2)$ and $f \in C_c^\infty(\mathbb{R}^2)$. It is easy to verify directly that

$$[\mathcal{R}, u\cdot\nabla]f = \operatorname{div}_x\big(\mathcal{R}(uf)\big) - \operatorname{div}_x(u\mathcal{R}f) = \sum_{j=1}^{2}\partial_j u f_j - \sum_{i,j=1}^{2}\partial_i u\mathcal{R}_{i,j}f_j. \tag{2.8}$$

Due to Hölder's inequality, Bernstein's inequality, and the embedding $B^0_{\infty,1} \hookrightarrow L^\infty$, we get

$$\begin{aligned}
\|[\mathcal{R}, u\cdot\nabla]f\|_{L^p} &\leq C\|\nabla u\|_{L^p}\left(\|f\|_{L^\infty} + \|\mathcal{R}f\|_{L^\infty}\right) \\
&\leq C\|\nabla u\|_{L^p}\left(\|f\|_{B^0_{\infty,1}} + \|\Delta_{-1}\mathcal{R}f\|_{L^\infty} + \sum_{j=0}^{\infty}\|\Delta_j\mathcal{R}f\|_{L^\infty}\right) \\
&\leq C\|\nabla u\|_{L^p}\left(\|f\|_{B^0_{\infty,1}} + \|\mathcal{R}f\|_{L^2} + \sum_{j=0}^{\infty}\|\Delta_j f\|_{L^\infty}\right) \\
&\leq C\|\nabla u\|_{L^p}\left(\|f\|_{L^2} + \|f\|_{B^0_{\infty,1}}\right).
\end{aligned} \tag{2.9}$$

Moreover, it is easy to see that both inequalities (2.8) and (2.9) can be extended to all functions by a simple density argument. □

3 Proof of Theorem 1.1

In this section we prove our main result Theorem 1.1. The strategy of the proof is as follows. We first prove the global a priori bounds for $\|w\|_{H^1}$ and $\|j\|_{H^1}$. Then we divide the proof into three steps: (1) the L^p $(2 < p < \infty)$ estimate of the vorticity ω, (2) the gradient estimate of the velocity u, (3) the energy estimate of the vorticity ω and j.

Now we act with the operator curl on the velocity equation in (1.1) and obtain the following vorticity equation:

$$\partial_t w + u \cdot \nabla w - \Delta w = \operatorname{curldiv}(b \otimes b). \tag{3.1}$$

Multiplying the ith component of the magnetic equation (1.1) by b_j, we obtain

$$(\partial_t b_i)b_j + u \cdot \nabla b_i b_j = (b \cdot \nabla u_i)b_j, \tag{3.2}$$

similarly, multiplying the jth component of the magnetic equation (1.1) by b_i, we have

$$(\partial_t b_j)b_i + u \cdot \nabla b_j b_i = (b \cdot \nabla u_j)b_i. \tag{3.3}$$

Adding (3.2) and (3.3), we know that the (i,j)th component of $b \otimes b$ satisfies

$$\partial_t(b_i b_j) + u \cdot \nabla(b_i b_j) = \bigl(\nabla u(b \otimes b)\bigr)_{i,j} + \bigl((b \otimes b)\nabla^\top u\bigr)_{i,j}, \quad i,j = 1,2, \tag{3.4}$$

equivalently,

$$\partial_t(b \otimes b) + u \cdot \nabla(b \otimes b) = \nabla u(b \otimes b) + (b \otimes b)\nabla^\top u, \tag{3.5}$$

where $\nabla^\top u$ denotes the transposed matrix to ∇u.

Applying $\mathcal{R} = (-\Delta)^{-1}\operatorname{curldiv}$ to (3.5) yields

$$\begin{aligned} &\partial_t\bigl(\mathcal{R}(b \otimes b)\bigr) + u \cdot \nabla\mathcal{R}(b \otimes b) \\ &\quad = -[\mathcal{R}, u \cdot \nabla](b \otimes b) + \mathcal{R}\bigl(\nabla u(b \otimes b) + (b \otimes b)\nabla^\top u\bigr). \end{aligned} \tag{3.6}$$

Set $G = w - \mathcal{R}(b \otimes b)$. Combining (3.1) and (3.6), we get

$$\partial_t G + u \cdot \nabla G - \Delta G = [\mathcal{R}, u \cdot \nabla](b \otimes b) - \mathcal{R}(\nabla u(b \otimes b) + (b \otimes b)\bigl(\nabla^\top u\bigr) := f. \tag{3.7}$$

By the transport-diffusion equation (3.7), we can obtain the following desired bounded estimate.

Lemma 3.1 *Assume that* $(u_0(x), b_0(x))$ *fulfills the conditions in Theorem* 1.1. *Let* $(u(t,x),$ $b(t,x))$ *be the corresponding solution of the initial value problem* (1.1). *Then, for* $p \in (2,\infty)$

and for any $T > 0$, we have

$$\|w\|_{L^p(\mathbb{R}^2)} \leq C, \tag{3.8}$$

where C is a positive constant depending only on T and the initial data.

Proof Multiplying equation (3.7) by $|G|^{p-2}G$ and integrating over $\mathbb{R}^2$, using the integration by parts and $\operatorname{div} u = 0$, we have

$$\frac{1}{p}\frac{d}{dt}\|G\|_{L^p}^p + \int_{\mathbb{R}^2}(-\Delta)G|G|^{p-2}G\,dx = \int_{\mathbb{R}^2} f|G|^{p-2}G\,dx. \tag{3.9}$$

Due to the pointwise inequality $\int_{\mathbb{R}^2}(-\Delta)G|G|^{p-2}G\,dx \geq 0$ and the Hölder inequality, we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|G\|_{L^p}^p &\leq \int_{\mathbb{R}^2} f|G|^{p-1}G\,dx \leq \|f\|_{L^p}\|G\|_{L^p}^{p-1}\\ &\leq \big\|[\mathcal{R}, u\cdot\nabla](b\otimes b) - \mathcal{R}\big(\nabla u(b\otimes b) + (b\otimes b)\nabla^\top u\big)\big\|_{L^p}\|G\|_{L^p}^{p-1}.\end{aligned} \tag{3.10}$$

Since the singular integral type operator $\mathcal{R}$ is bounded on $L^p(\mathbb{R}^2)$ $(1 < p < \infty)$, we have

$$\begin{aligned}\frac{d}{dt}\|G\|_{L^p} &\leq \big\|[\mathcal{R}, u\cdot\nabla](b\otimes b) - \mathcal{R}\big(\nabla u(b\otimes b) + (b\otimes b)\nabla^\top u\big)\big\|_{L^p}\\ &\leq \big\|[\mathcal{R}, u\cdot\nabla](b\otimes b)\big\|_{L^p} + \big\|\mathcal{R}\big(\nabla u(b\otimes b) + (b\otimes b)\nabla^\top u\big)\big\|_{L^p}\\ &\leq \big\|[\mathcal{R}, u\cdot\nabla](b\otimes b)\big\|_{L^p} + \big\|\nabla u(b\otimes b) + (b\otimes b)\nabla^\top u\big\|_{L^p}\\ &\leq \big\|[\mathcal{R}, u\cdot\nabla](b\otimes b)\big\|_{L^p} + \|\nabla u\|_{L^p}\|b\otimes b\|_{L^\infty}.\end{aligned} \tag{3.11}$$

Due to Lemma 2.4, we have

$$\big\|[\mathcal{R}, u\cdot\nabla](b\otimes b)\big\|_{L^p} \leq C\|\nabla u\|_{L^p}\big(\|b\otimes b\|_{B^0_{\infty,1}} + \|b\otimes b\|_{L^2}\big). \tag{3.12}$$

Putting (3.12) into (3.11) and using the classical embedding $B^0_{\infty,1} \hookrightarrow B^0_{\infty,2} \hookrightarrow L^\infty$, we get

$$\frac{d}{dt}\|G\|_{L^p} \leq C\|\nabla u\|_{L^p}\big(\|b\otimes b\|_{B^0_{\infty,1}} + \|b\otimes b\|_{L^2}\big),$$

by the Biot-Savart law (Lemma 2.2), we have

$$\frac{d}{dt}\|G\|_{L^p} \leq C\|\omega\|_{L^p}\big(\|b\otimes b\|_{B^0_{\infty,1}} + \|b\otimes b\|_{L^2}\big),$$

as $\omega = G + \mathcal{R}(b\otimes b)$, we have

$$\begin{aligned}\frac{d}{dt}\|G\|_{L^p} &\leq C\big(\|G\|_{L^p} + \big\|\mathcal{R}(b\otimes b)\big\|_{L^p}\big)\big(\|b\otimes b\|_{B^0_{\infty,1}} + \|b\otimes b\|_{L^2}\big)\\ &\leq C\big(\|G\|_{L^p} + \|b\otimes b\|_{L^p}\big)\big(\|b\otimes b\|_{B^0_{\infty,1}} + \|b\otimes b\|_{L^2}\big),\end{aligned} \tag{3.13}$$

where the L^p boundedness of Riesz operator has been used in the last inequality.

Multiplying equation (3.5) by $|b\otimes b|^{p-2}(b\otimes b)$ and integrating over $\mathbb{R}^2$, using integration by parts and $\operatorname{div} u = 0$, we have

$$\frac{1}{p}\frac{d}{dt}\|b\otimes b\|_{L^p}^p = \int_{\mathbb{R}^2}\big(\nabla u(b\otimes b) + (b\otimes b)(\nabla u)^\top\big)|b\otimes b|^{p-2}(b\otimes b)\,dx.$$

Hölder's inequality and the Biot-Savart law (Lemma 2.2) yield

$$\frac{d}{dt}\|b\otimes b\|_{L^p} \le C\big(\|G\|_{L^p} + \|b\otimes b\|_{L^p}\big)\big(\|b\otimes b\|_{B^0_{\infty,1}} + \|b\otimes b\|_{L^2}\big). \tag{3.14}$$

Combining the estimates (3.13) and (3.14), we get

$$\frac{d}{dt}\big(\|G\|_{L^p} + \|b\otimes b\|_{L^p}\big) \le C\big(\|G\|_{L^p} + \|b\otimes b\|_{L^p}\big)\big(\|b\otimes b\|_{B^0_{\infty,1}} + \|b\otimes b\|_{L^2}\big). \tag{3.15}$$

Assuming

$$\int_0^t \big(\|b\otimes b\|_{B^0_{\infty,1}} + \|b\otimes b\|_{L^2}\big)\,ds < \infty, \tag{3.16}$$

by Gronwall's inequality we have

$$\|G\|_{L^p} + \|b\otimes b\|_{L^p} \le C\exp\left(\int_0^t \big(\|b\otimes b\|_{B^0_{\infty,1}} + \|b\otimes b\|_{L^2}\big)\,ds\right) \le C. \tag{3.17}$$

This implies that, for any $2 < p < \infty$,

$$\|w\|_{L^p} \le \|G\|_{L^p} + \|b\otimes b\|_{L^p} \le C. \tag{3.18}$$

This completes the proof of Lemma 3.1. □

Next, we give a key bounded estimate which is crucial in the proof of Theorem 1.1.

Lemma 3.2 *Assume that $(u_0(x), b_0(x))$ fulfills the conditions in Theorem* 1.1. *Let $(u(t,x), b(t,x))$ be the corresponding solution of the initial value problem* (1.1). *Then, for $p \in (2,\infty)$ and for any $T > 0$,*

$$\int_0^T \big\|\nabla u(s)\big\|_{L^\infty}\,ds \le C, \tag{3.19}$$

where C is a positive constant depending only on T and the initial data.

Proof In view of (3.7), for $j \ge -1$, acting with the Littlewood-Paley operator $\triangle_j$ on (3.7), one has

$$\begin{aligned}&\partial_t\triangle_j G + \triangle_j(u\cdot\nabla G) - \Delta\triangle_j G \\ &\quad = \triangle_j[\mathcal{R}, u\cdot\nabla](b\otimes b) - \triangle_j\mathcal{R}\big(\nabla u(b\otimes b) + (b\otimes b)\nabla^\top u\big).\end{aligned} \tag{3.20}$$

For convenience, we take

$$f_j = \triangle_j[\mathcal{R}, u\cdot\nabla](b\otimes b) - \mathcal{R}\big(\nabla u(b\otimes b) + (b\otimes b)\nabla^\top u\big) - [\triangle_j, u\cdot\nabla]G.$$

Thus, equation (3.20) is written as

$$\partial_t\triangle_j G + u\cdot\nabla\triangle_j G - \Delta\triangle_j G = f_j. \tag{3.21}$$

Multiplying equation (3.21) by $|\triangle_j G|^{q-2}\triangle_j G$ with $q > 2$ and integrating over $\mathbb{R}^2$, with the help of the Hölder inequality and $\operatorname{div} u = 0$, we derive that

$$\frac{1}{q}\frac{d}{dt}\|\triangle_j G\|_{L^q}^q + \int_{\mathbb{R}^2}(-\Delta)\triangle_j G|\triangle_j G|^{q-2}\triangle_j G\,dx \le \|f_j\|_{L^q}\|\triangle_j G\|_{L^q}^{q-1}.$$

For $j \ge 0$, the Fourier transform of $\triangle_j G$ is supported away from the origin and the dissipative part possesses a lower bound,

$$\int_{\mathbb{R}^2}(-\Delta)\triangle_j G|\triangle_j G|^{q-2}\triangle_j G\,dx \ge c2^{2q}\|\triangle_j G\|_{L^q}^q,$$

where c is an absolute positive constant independent of q.

Therefore, we have

$$\frac{d}{dt}\|\triangle_j G\|_{L^q} + c2^{2q}\|\triangle_j G\|_{L^q} \le \|f_j\|_{L^q},$$

thus Gronwall's inequality implies that

$$\|\triangle_j G\|_{L^q} \lesssim e^{-ct2^{2q}}\|\triangle_j G_0\|_{L^q} + \int_0^t e^{-c(t-s)2^{2q}}\big\|f_j(s)\big\|_{L^q}\,ds. \tag{3.22}$$

Taking the $L^1[0,t]$ norm and using Young's inequality, we obtain

$$\begin{aligned}\|\triangle_j G\|_{L_t^1L^q} &\lesssim \big\|e^{-ct2^{2q}}\big\|_{L_t^1}\big(\|\triangle_j G_0\|_{L^q} + \|f_j\|_{L_t^1L^q}\big)\\ &\lesssim 2^{-2q}\left(\|\triangle_j G_0\|_{L^q} + \int_0^t\big\|f_j(s)\big\|_{L^q}\,ds\right).\end{aligned} \tag{3.23}$$

For $j = -1$, we have

$$\int_0^t\big\|\triangle_{-1}G(s)\big\|_{L^q}\,ds \le C\int_0^t\big\|G(s)\big\|_{L^q}\,ds \le C(t). \tag{3.24}$$

Gathering the above high-low-frequency estimations, multiplying the corresponding inequality by $2^{j\frac{2}{q}}$, and then summing over j from -1 to ∞, one has

$$\int_0^t\big\|G(s)\big\|_{B_{q,1}^{\frac{2}{q}}}\,ds = \|G\|_{L_t^1B_{q,1}^{\frac{2}{q}}},$$

due to the fact $L_t^1B_{q,1}^{\frac{2}{q}} \approx \widetilde{L}_t^1B_{q,1}^{\frac{2}{q}}$, we have

$$\int_0^t\big\|G(s)\big\|_{B_{q,1}^{\frac{2}{q}}}\,ds = \|G\|_{L_t^1B_{q,1}^{\frac{2}{q}}} \approx \|G\|_{\widetilde{L}_t^1B_{q,1}^{\frac{2}{q}}},$$

thus

$$\begin{aligned}
&\int_0^t \|G(s)\|_{B^{\frac{2}{q}}_{q,1}}\,ds \\
&\quad\lesssim \|G_0\|_{B^{\frac{2}{q}-2}_{q,1}} + \int_0^t \sum_{j=-1}^{\infty} 2^{j(\frac{2}{q}-2)} \|f_j(s)\|_{L^q}\,ds + C(t) \\
&\quad\lesssim \|G_0\|_{B^{\frac{2}{q}-2}_{q,1}} + \int_0^t \sum_{j=-1}^{\infty} 2^{j(\frac{2}{q}-2)} \|[\triangle_j, u\cdot\nabla]G(s)\|_{L^q}\,ds + C(t) \\
&\qquad + \int_0^t \|[\mathcal{R}, u\cdot\nabla](b\otimes b) - \mathcal{R}(\nabla u(b\otimes b) + (b\otimes b)\nabla^\top u)\|_{L^q}\,ds.
\end{aligned} \tag{3.25}$$

Next, we estimate the last term in the right hand side of the above inequality. Due to the commutator estimate in Lemma 2.4 and the boundedness of $\mathcal{R}$ in L^p $(1 < p < \infty)$, we have

$$\begin{aligned}
&\int_0^t \|[\mathcal{R}, u\cdot\nabla](b\otimes b) - \mathcal{R}(\nabla u(b\otimes b) + (b\otimes b)\nabla^\top u)(s)\|_{L^q}\,ds \\
&\quad\lesssim \int_0^t \|\nabla u\|_{L^q}\big(\|b\otimes b\|_{B^0_{\infty,1}} + \|b\otimes b\|_{L^2} + \|b\otimes b\|_{L^\infty}\big)\,ds \\
&\quad\lesssim 1.
\end{aligned} \tag{3.26}$$

For the second term, using the Bernstein inequality, we have

$$\begin{aligned}
I &= \int_0^t \sum_{j=-1}^{\infty} 2^{\frac{2}{q}-2} \|[\triangle_j, u\cdot\nabla]G(s)\|_{L^q}\,ds \\
&\lesssim \int_0^t \sum_{j=-1}^{\infty} 2^{j(\frac{2}{q}-2)} \big(\|\triangle_j(u\cdot\nabla G)\|_{L^q} + \|u\cdot\triangle_j\nabla G\|_{L^q}\big)\,ds \\
&\lesssim \int_0^t \sum_{j=-1}^{\infty} 2^{j(\frac{2}{q}-2)} \big(\|\triangle_j\nabla\cdot(uG)\|_{L^q} + \|u\cdot\nabla\triangle_j G\|_{L^q}\big)\,ds \\
&\lesssim \int_0^t \sum_{j=-1}^{\infty} 2^{j(\frac{2}{q}-2)} \big(2^j\|uG\|_{L^q} + 2^j\|u\|_{L^{2q}}\|G\|_{L^{2q}}\big)\,ds \\
&\lesssim \int_0^t \sum_{j=-1}^{\infty} 2^{j(\frac{2}{q}-1)} \|u\|_{L^{2q}}\|G\|_{L^{2q}}\,ds \\
&\le C(t),
\end{aligned} \tag{3.27}$$

where we have used the inequalities: $\|u\|_{L^{2q}} \le C(t)$ and $\|G\|_{L^{2q}} \le C(t)$ for $q > 2$. Putting the above estimates together, one has

$$\int_0^t \|G(s)\|_{B^{\frac{2}{q}}_{q,1}}\,ds \le C\|G_0\|_{B^{\frac{2}{q}-2}_{q,1}} + C(t). \tag{3.28}$$

Consequently, for any fixed $t > 0$, we get

$$\int_0^t \|G(s)\|_{B^{\frac{2}{q}}_{q,1}}\,ds \le C < \infty. \tag{3.29}$$

With the aid of the standard embedding $B^{\frac{2}{q}}_{q,1}(\mathbb{R}^2) \hookrightarrow B^0_{\infty,1}(\mathbb{R}^2)$, we have

$$\int_0^t \|G(s)\|_{B^0_{\infty,1}}\,ds \le \int_0^t \|G(s)\|_{B^{\frac{2}{q}}_{q,1}}\,ds \le C < \infty. \tag{3.30}$$

Furthermore, we have the following estimate:

$$\begin{aligned}\int_0^t \|w\|_{B^0_{\infty,1}}\,ds &\le \int_0^t \big(\|G\|_{B^0_{\infty,1}} + \big\|\mathcal{R}(b\otimes b)\big\|_{B^0_{\infty,1}}\big)\,ds \\ &\le \int_0^t \big(\|G\|_{B^0_{\infty,1}} + C\|b\otimes b\|_{B^0_{\infty,1}} + C\|b\otimes b\|_{L^2}\big)\,ds \\ &\le C. \end{aligned}\tag{3.31}$$

Consequently, we derive the following key bound:

$$\int_0^t \|\nabla u(s)\|_{L^\infty}\,ds \le C\int_0^t \big(\|u(s)\|_{L^2} + \|w\|_{B^0_{\infty,1}}\big)\,ds \le C < \infty. \tag{3.32}$$

This completes the proof of Lemma 3.2. □

With the aid of the boundedness of $\int_0^t \|\nabla u(s)\|_{L^\infty}\,ds$, we obtain the global bounds of $\|w\|_{H^1}$ and $\|j\|_{H^1}$.

Lemma 3.3 *If* $(u(t,x),b(t,x))$ *is a solution of system* (1.1), *then for any* $T>0$,

$$\|w(t)\|^2_{L^2} + \|j(t)\|^2_{L^2} + \int_0^t \|\nabla w\|^2_{L^2}\,ds \le C, \quad \forall t\in[0,T], \tag{3.33}$$

$$\|\nabla w(t)\|^2_{L^2} + \|\nabla j(t)\|^2_{L^2} + \int_0^t \|\Delta w\|^2_{L^2}\,ds \le C, \quad \forall t\in[0,T], \tag{3.34}$$

where C is a positive constant depending only on T and the initial data.

Proof Taking the inner products of the first equation in (1.1) with u and the second equation in (1.1) with b, respectively, adding the resulting equations, and integrating by parts, we obtain

$$\|u(t)\|^2_{L^2} + \|b(t)\|^2_{L^2} + \int_0^t \|\nabla u\|^2_{L^2}\,ds = \|u_0\|^2_{L^2} + \|b_0\|^2_{L^2}. \tag{3.35}$$

Now, w and j satisfy the equations

$$\partial_t w + u\cdot\nabla w - \Delta w = b\cdot\nabla j, \tag{3.36}$$

$$\partial_t j + u\cdot\nabla j = b\cdot\nabla w + T(\nabla u,\nabla b), \tag{3.37}$$

respectively, where

$$T(\nabla u,\nabla b) = 2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - 2\partial_1 u_1(\partial_1 b_2 + \partial_2 b_1). \tag{3.38}$$

Taking the inner product of (3.36) with w and (3.37) with j, respectively, adding the resulting equations and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\bigl[\bigl\|w(t)\bigr\|_{L^2}^2 + \bigl\|j(t)\bigr\|_{L^2}^2\bigr] + \|\nabla w\|_{L^2}^2 = \int_{\mathbb{R}^2} T(\nabla u, \nabla b) j\,dx \le \|\nabla u\|_{L^\infty}\|j\|_{L^2}^2. \tag{3.39}$$

With the help of the estimate (3.32) and Gronwall's inequality, we obtain

$$\bigl\|w(t)\bigr\|_{L^2}^2 + \bigl\|j(t)\bigr\|_{L^2}^2 + \int_0^t \|\nabla w\|_{L^2}^2\,ds \le C. \tag{3.40}$$

Taking the inner product of (3.36) with $-\Delta w$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\bigl\|\nabla w(t)\bigr\|_{L^2}^2 + \|\Delta w\|_{L^2}^2 &= \int_{\mathbb{R}^2} (-\nabla w \cdot \nabla u \cdot \nabla w)\,dx\,dy \\ &\quad + \int_{\mathbb{R}^2} \bigl(\nabla w \cdot \nabla b \cdot \nabla j + b \cdot \nabla(\nabla j) \cdot \nabla w\bigr)\,dx\,dy.\end{aligned} \tag{3.41}$$

Similarly, taking the inner products of (3.37) with $-\Delta j$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\bigl\|\nabla j(t)\bigr\|_{L^2}^2 &= \int_{\mathbb{R}^2} (-\nabla j \cdot \nabla u \cdot \nabla j + \nabla j \cdot \nabla b \cdot \nabla w)\,dx\,dy \\ &\quad + \int_{\mathbb{R}^2} \bigl(b \cdot \nabla(\nabla w) \cdot \nabla j + \nabla T(\nabla u, \nabla b)\nabla j\bigr)\,dx\,dy.\end{aligned} \tag{3.42}$$

Adding the above equations and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\bigl[\bigl\|\nabla w(t)\bigr\|_{L^2}^2 + \bigl\|\nabla j(t)\bigr\|_{L^2}^2\bigr] + \|\Delta w\|_{L^2}^2 \le \sum_{i=1}^{5} I_i, \tag{3.43}$$

where

$$\begin{aligned}I_1 &= \int_{\mathbb{R}^2} |\nabla u||\nabla w|^2\,dx\,dy; \\ I_2 &= \int_{\mathbb{R}^2} |\nabla u||\nabla j|^2\,dx\,dy; \\ I_3 &= \int_{\mathbb{R}^2} |\nabla w||\nabla b||\nabla j|\,dx\,dy; \\ I_4 &= \int_{\mathbb{R}^2} |\nabla j||\nabla b||\nabla w|\,dx\,dy; \\ I_5 &= \int_{\mathbb{R}^2} \bigl(\bigl|\nabla^2 u\bigr||\nabla b| + |\nabla u|\bigl|\nabla^2 b\bigr|\bigr)|\nabla j|\,dx\,dy.\end{aligned}$$

Obviously, $I_3 = I_4$. We only need to estimate the other four terms.

For the terms I_1 and I_2, by the Hölder inequality, we have

$$\begin{aligned}I_1 &\le \|\nabla u\|_{L^\infty}\|\nabla w\|_{L^2}^2, \\ I_2 &\le \|\nabla u\|_{L^\infty}\|\nabla j\|_{L^2}^2.\end{aligned}$$

For the term I_3, by the Hölder inequality, we have

$$I_3 \leq \|\nabla j\|_{L^2}\|\nabla w\|_{L^4}\|\nabla b\|_{L^4}.$$

By the Gagliardo-Nirenberg inequality $\|f\|_{L^4} \leq C\|f\|_{L^2}^{\frac{1}{2}}\|\nabla f\|_{L^2}^{\frac{1}{2}}$, one has

$$\begin{aligned} I_3 &\leq C\|\nabla j\|_{L^2}\|\nabla w\|_{L^2}^{\frac{1}{2}}\|\Delta w\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|\nabla j\|_{L^2}^{\frac{1}{2}} \\ &\leq \frac{1}{4}\|\Delta w\|_{L^2}^2 + C\|\nabla j\|_{L^2}^2\|\nabla w\|_{L^2}^{\frac{2}{3}}. \end{aligned}$$

For the term I_5, it is easy to obtain

$$I_5 \leq \frac{1}{4}\|\Delta w\|_{L^2}^2 + C\|\nabla j\|_{L^2}^2\big(\|\nabla w\|_{L^2}^{\frac{2}{3}} + \|\nabla u\|_{L^\infty}\big).$$

Adding the estimates of I_i $(i = 1,2,3,4,5)$, we get

$$\begin{aligned} &\frac{d}{dt}\big[\big\|\nabla w(t)\big\|_{L^2}^2 + \big\|\nabla j(t)\big\|_{L^2}^2\big] + \|\Delta w\|_{L^2}^2 \\ &\quad\leq C\big(\|\nabla w\|_{L^2}^{\frac{2}{3}} + \|\nabla u\|_{L^\infty}\big)\big\|\nabla w(t), \nabla j(t)\big\|_{L^2}^2. \end{aligned} \tag{3.44}$$

Due to Lemma 3.2, $\int_0^t \|\nabla w\|_{L^2}^2\,ds \leq C$, Gronwall's inequality immediately yields

$$\big\|\nabla w(t)\big\|_{L^2}^2 + \big\|\nabla j(t)\big\|_{L^2}^2 + \int_0^t \|\Delta w\|_{L^2}^2\,ds \leq C < \infty. \tag{3.45}$$

This completes the proof of Lemma 3.3. □

Proof of Theorem 1.1 By the estimate (3.45), we know that $\|\nabla w(t)\|_{L^2}^2 + \|\nabla j(t)\|_{L^2}^2 \leq C$, $\forall t \in [0,T]$. Due to the classical embedding $H^1(\mathbb{R}^2) \hookrightarrow BMO(\mathbb{R}^2)$, we obtain

$$\int_0^T \big(\big\|w(s)\big\|_{BMO} + \big\|j(s)\big\|_{BMO}\big)\,ds < \infty. \tag{3.46}$$

By an argument which generalizes the classical BKM criterion [21] to the MHD system, we complete the proof of Theorem 1.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YY carried out the theory of Besov spaces, XW carried out the Fourier localization technique and YT carried out the well-posedness. All authors read and approved the final manuscript.

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