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Separation properties for infinite iterated function systems

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Abstract
In this paper, we study the infinite iterated function systems (IFSs) of contractive similitudes with overlaps. We extend the notions of the weak separation condition and the generalized finite type condition for finite IFSs to the infinite case. We show that for an infinite IFS of contractive similitudes the generalized finite type condition implies the weak separation condition.

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1 Introduction

The separation properties are useful for studying the IFSs. At first, let's recall the separation property of finite IFSs. Suppose X is a nonempty compact subset of $\mathbf{R}^d$. Let S_i ($i = 1, 2, \ldots, m$) be a contractive self-map on $X \subset \mathbf{R}^d$. We call $\{S_i\}_{i=1}^m$ a finite similar IFS of contractive similitudes on $\mathbf{R}^d$ if there exists $C \in (0, 1)$ such that

$$\bigl|S_i(x) - S_i(y)\bigr| = C|x - y| \quad \text{for every } 1 \le i \le m.$$

There exists a nonempty subset K of X such that

$$K = \bigcup_{i=1}^m S_i(K).$$

K is called the invariant set or attractor of the system (see, *e.g.* [1–3]).

We say that $\{S_i\}_{i=1}^m$ satisfies the open set condition (OSC) if there exists a nonempty bounded open set $U \subseteq \mathbf{R}^d$ such that

$$S_i(U) \subseteq U \quad \text{and} \quad S_i(U) \cap S_j(U) = \emptyset \quad \text{for } i \ne j.$$

Such a U is called a basic open set for $\{S_i\}_{i=1}^m$. If, moreover, $U \cap K \ne \emptyset$, the $\{S_i\}_{i=1}^m$ is said to satisfy the strong open set condition (SOSC). It is a classical result (see [1, 4, 5]) that if a similar IFS satisfies the OSC, then it satisfies the SOSC. Fan *et al.* (see [6–8]) extended the result to finite conformal IFSs.

IFSs that do not satisfy the OSC are said to have overlaps. In this case, it is in general much harder to get acquainted with the structure of the corresponding invariant set K. The weak separation condition (WSC) and the generalized finite type condition are weaker than the OSC but still strong enough to obtain good results (see [9–14] *etc.*).

However, the circumstances for infinite IFSs are distinct [15]. Szardk and Wedrychowica [16] showed that for infinite IFSs, the OSC does not imply the SOSC. Moran [17] also showed that self-similar set generated by a countable system of similitudes may not be s-set even if the open set condition is satisfies. So it is necessary to look for some separation conditions for infinite IFSs that are weaker than the OSC. Moran defined a weak separation property for infinite IFSs [17]. Suppose $\{S_i\}_{i=1}^{\infty}$ is an infinite conformal IFS on an open set U_0. Moran [17] defined the infinite IFS $\{S_i\}_{i=1}^{\infty}$ satisfies the finite open set condition if for any integer n, there is a nonempty open set $U_n \subset U_0$ such that $S_i(U_n) \subset U_n$ for any $1 \leq i \leq n$ and $S_i(U_n) \cap S_j(U_n) = \emptyset$ for any $i, j \leq n$, $i \neq j$. The finite strong open set condition holds if furthermore $U_n \cap J \neq \emptyset$. It is easy to see that if the IFS $\{S_i\}_{i=1}^{\infty}$ satisfies the OSC then it satisfies the finite strong open set condition. Moran uses this separation property to study the Hausdorff dimension of invariant set with respect to the infinite IFS [17].

Our goal in this paper is to study the infinite iterated function systems (IFSs) of contractive similitudes with overlaps. We define the WSC and the generalized finite type condition for the infinite IFSs. Next, we study the relationship of the two separation conditions. We show that the generalized type condition implies the WSC. Our main result is Theorem 1.1.

Theorem 1.1 *Let $\{S_i\}_{i=1}^{\infty}$ be an infinite iterated function system of contractive similitude on $\mathbf{R}^d$. If $\{S_i\}_{i=1}^{\infty}$ is of generalized finite type condition, then it satisfies the weak separation condition.*

This paper is organized as follows. In Section 2, we define the weak separation condition for infinite IFSs and give some examples. In Section 3, we introduce the generalized finite type condition for infinite IFSs and provide examples of IFSs satisfying this condition. Finally, in Section 4, we prove that the generalized finite type condition implies the weak separation condition (*i.e.* Theorem 1.1).

2 The weak separation condition

Let $\Sigma = \{1, 2, \ldots, n, \ldots\}$, $\Sigma^n = \{I = (i_1, i_2, \ldots, i_n) : i_n \in \Sigma\}$ and $\Sigma^* = \bigcup_{n\geq 0} \Sigma^n$. Let $\{S_i\}_{i=1}^{\infty}$ be an IFS of contractions defined on a compact subset $X \subset \mathbf{R}^d$ with $X^0 \neq \emptyset$. Let ρ_i be the contractive ratio of S_i, and $\rho_I = \rho_{i_1}\rho_{i_2}\cdots\rho_{i_n}$, for $I = (i_1, i_2, \ldots, i_n)$. We define $\rho_{\max} = \max_{i\geq 1} \rho_i$.

Definition 2.1 We say that an IFS $\{S_i\}_{i=1}^{\infty}$ satisfies the weak separation condition (WSC) if there exist $x_0 \in X$ and $\gamma \in N$ such that, for any $I \in \Sigma^*$, the ball of radius b contains at most γ points of $\{S(S_I(x_0)) : S \in \mathcal{A}_b\}$ for any $0 < b \leq 1$. Here we let

$$\mathcal{A}_b = \{S_I : I \in \tau_b\} \quad \text{and} \quad \tau_b = \big\{I = (i_1, i_2, \ldots, i_n) : \rho_I \leq b \leq \rho_{i_1 i_2 \cdots i_{n-1}}\big\}.$$

Remark 1 For any starting point $y \in X$, it is easy to see that $\{S_i\}_{i=1}^{\infty}$ will satisfy the WSC if there exists $n > 0$ such that, for any $J_1, J_2 \in \tau_b$, either

$$S_{J_1}(y) = S_{J_2}(y) \quad \text{or} \quad \big|S_{J_1}(y) - S_{J_2}(y)\big| \geq nb.$$

For any $a > 0$ and any bounded subsets $D \subseteq X$ and $U \subseteq \mathbf{R}^d$, we let

$$\mathcal{A}_{a,U,D} = \{S \in \mathcal{A}_{a|U|} : S(D) \cap U \neq \emptyset\}, \qquad \gamma_{a,D} = \sup_U \#\mathcal{A}_{a,U,D}.$$

Here $|U|$ denotes the diameter of U. We have two lemmas with respect to the definition of the weak separation condition which are needed to prove our main result.

Lemma 2.2 *Let $\{S_i\}_{i=1}^{\infty}$ be an infinite IFS of contractive similitudes on $\mathbf{R}^d$, for any $a > 0$ and any nonempty subset $D \subseteq X$, $\gamma_{a,D} < \infty$. Then $\{S_i\}_{i=1}^{\infty}$ satisfies the WSC.*

Proof Let $x_0 \in X$. Let $D \subseteq X$ be a nonempty subset and let $\mathcal{A}_b$ be defined as above. Then for any $I \in \Sigma^*$ and any ball B_b of radius b,

$$\begin{aligned}\#\{S(S_I(x_0)) \in B_b : S \in \mathcal{A}_b\} &\leq \#\{S \in \mathcal{A}_b : S(S_I(x_0)) \in B_b\} \\ &\leq \{S \in \mathcal{A}_{\frac{1}{2}|B_b|} : S(D) \cap B_b \neq \emptyset\} \\ &\leq \gamma_{\frac{1}{2},D} \\ &< \infty,\end{aligned}$$

which yields the statement. □

Lemma 2.3 *Let $\{S_i\}_{i=1}^{\infty}$ be an infinite IFS of contractive similitudes on a compact subset $X \subseteq \mathbf{R}^d$. If there exist a constant $\gamma \in \mathbf{N}$ and a subset $D \subseteq X$ with $D^\circ \neq \emptyset$, such that, for any $0 < b < 1$ and $x \in X$,*

$$\#\{S \in \mathcal{A}_b : x \in S(D)\} \leq \gamma.$$

Then $\{S_i\}_{i=1}^{\infty}$ satisfies the WSC.

Proof We denote by $B_b(x)$ the closed ball with radius b and center x. Let $\mathcal{L}$ denote the Lebesgue measure on $\mathbf{R}^d$. Let $S \in \mathcal{A}_b$ such that $S(D) \cap B_b(x) \neq \emptyset$, and $y \in D$ such that $S(y) \in B_b(x)$. Then for any $z \in D$,

$$\begin{aligned}|S(z) - x| &\leq |S(z) - S(y)| + |S(y) - x| \\ &\leq c|z - y| + b \\ &\leq b(1 + c_1|D|).\end{aligned}$$

So

$$S(D) \subseteq B_{b(1+c_1|D|)}(x).$$

Let $x_0 \in D$, $I \in \Sigma^*$, $0 < b < 1$, we have

$$\begin{aligned}\#\{S(S_I(x_0)) \in B : S \in \mathcal{A}_b\} &\leq \#\{S \in \mathcal{A}_b : S(S_I(x_0)) \in B\} \\ &\leq \#\{S \in \mathcal{A}_b : S(D) \cap B_b(x) \neq \emptyset\}.\end{aligned}$$

Suppose $\rho := \min\{\rho_i : 1 \le i \le n\}$. By assumption,

$$\begin{aligned}(b\rho)^d\mathcal{L}(D)\#\{S \in \mathcal{A}_b : S(D) \cap B_b(x) \neq \emptyset\} \\ \le \sum \{\mathcal{L}(S(D)) : S \in \mathcal{A}_b : S(D) \cap B_b(x) \neq \emptyset\} \\ \le \gamma \mathcal{L}(B_{b(1+c_1)}(x)).\end{aligned}$$

That means

$$\#\{S \in \mathcal{A}_b : S(D) \cap B_b(x) \neq \emptyset\} \le \gamma \frac{(1 + c_1|D|)^d}{\rho^d L(D)} := N.$$

This completes the proof of the lemma. □

Lemma 2.4 *Suppose $\{S_i\}_{i=1}^{\infty}$ is an infinite IFS on a compact subset $X \subseteq \mathbf{R}^d$, and it satisfies the OSC. Then it satisfies the WSC.*

Proof Suppose U is an open set guaranteed by the open set condition. For any $0 < b \le 1$, we write $\mathcal{A}_b = \{S_I : I \in \tau_b\}$ and

$$\tau_b = \{I = (i_1, i_2, \dots, i_n) : \rho_I \le b \le \rho_{i_1 i_2 \cdots i_{n-1}}\}.$$

For any $I_1, I_2 \in \mathcal{A}_b$ with $I_1 \neq I_2$, the open set condition implies that $S_{I_1}(U) \cap S_{I_2}(U) = \emptyset$. Since

$$\rho \cdot b|U| \le |S_{I_i}(U)| \le b|U| \quad (i = 1, 2).$$

So $|S_{I_1}(0) - S_{I_2}(0)| \ge \rho b$. Then the remark implies that $\{S_i\}_{i=1}^{\infty}$ satisfies the WSC. □

Example 2.5 $S_1(x) = \frac{1}{3}x$, $S_2(x) = \frac{1}{3}x + \frac{1}{6}$, $S_i(x) = \frac{1}{2}(\frac{1}{2^{i-2}} + \frac{1}{3^{i-2}}x) + \frac{1}{2}$ $(i \ge 3)$. This IFS satisfies the WSC. It is easy to see that this infinite IFS does not satisfy the OSC. We know that $S_1 \cap S_i = \emptyset$ $(i \ge 3)$, $S_2 \cap S_i = \emptyset$ $(i \ge 3)$, $S_j \cap S_{j'} = \emptyset$ $(j, j' \ge 3$, and $j \neq j')$. By Lemma 2.4 and the example in [18] $\{S_i\}_{i=1}^{\infty}$ satisfies the WSC.

3 The generalized finite type condition

In this section we promote the generalized finite type condition to infinite IFSs. The generalized finite type condition for infinite IFSs is slightly modified from that for finite IFSs [9]. The definition consists of two parts. The first part concerns the sequence of nested index sets. The second part entails the concept of neighborhood types.

Let $\{S_i\}_{i=1}^{\infty}$ be an infinite IFS of contractive similitudes on a compact subset $X \subseteq \mathbf{R}^d$, $\Sigma = \{1, 2, \dots, n, \dots\}$, $\Sigma^n = \{I = (i_1, i_2, \dots, i_n) : i_n \in \Sigma\}$ and $\Sigma^* = \bigcup_{n\ge 0} \Sigma^n$. For any $I = (i_1, i_2, \dots, i_m) \in \Sigma^m$, $J = (j_1, j_2, \dots, j_n) \in \Sigma^n$, we let $IJ = (i_1, i_2, \dots, i_m, j_1, j_2, \dots, j_n)$. For $I, J \in \Sigma^*$, if I is an initial segment of J or $I = J$, we write $I \preceq J$. We denote by $I \npreceq J$ if $I \preceq J$ does not hold. Consider a sequence of index sets $\{\mathcal{M}_k\}_{k=0}^{\infty}$, where for all $k \ge 0$, $\mathcal{M}_k$ is a finite subset of Σ^*. Let

$$\underline{m}_k = \underline{m}_k(\mathcal{M}_k) := \min\{|I| : I \in \mathcal{M}_k\}$$

and

$$\overline{m}_k = \overline{m}_k(\mathcal{M}_k) := \max\{|I| : I \in \mathcal{M}_k\}.$$

Definition 3.1 We say that $\{\mathcal{M}_k\}_{k=0}^{\infty}$ is a sequence of nested index sets if it satisfies the following conditions:

(i) Both $\{\underline{m}_k\}$ and $\{\overline{m}_k\}$ are nondecreasing, and $\lim_{k\to\infty} \underline{m}_k = \lim_{k\to\infty} \overline{m}_k = \infty$.
(ii) For each $k \geq 0$, $\mathcal{M}_k$ is an antichain in Σ^*.
(iii) For each $J \in \Sigma^*$ with $|J| > \overline{m}_k$, there exists $I \in \mathcal{M}_k$ such that $I \preceq J$.
(iv) For each $J \in \Sigma^*$ with $|J| < \underline{m}_k$, there exists $I \in \mathcal{M}_k$ such that $J \preceq I$.
(v) There exists a positive integer n, independent of k, such that, for all $I \in \mathcal{M}_k$ with $I \preceq J$, we have $|J| - |I| \leq n$.

By letting $\mathcal{M}_k = \Sigma^k$ for all $k \geq 0$, we obtain an example of sequence of nested index sets.

To define neighborhood types, we fix a sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^{\infty}$. For each integer $k \geq 0$, let $\mathcal{V}_k$ be the set of kth level vertices (with respect to $\{\mathcal{M}_k\}$) defined as

$$\mathcal{V}_0 := \{(I, 0)\} \quad \text{and} \quad \mathcal{V}_k := \{(S_I, k) : I \in \mathcal{M}_k\} \quad \text{for all } k \geq 1.$$

We call $(I, 0)$ the root vertex and denote it by $\mathbf{v}_{\text{root}}$. Let $\mathcal{V} = \bigcup_{k\geq 0} \mathcal{V}_k$ be the set of all vertices. For $\mathbf{v} = (S_I, k) \in \mathcal{V}_k$, we use the convenient notation $S_{\mathbf{v}} := S_I$.

Fix any nonempty bounded open set $\Omega \subseteq X$ which is invariant under $\{S_i\}_{i=1}^{\infty}$. Two kth level $\mathbf{v}, \mathbf{v}' \in \mathcal{V}_k$ (allowing $\mathbf{v} = \mathbf{v}'$) are said to be neighbors (with respect to Ω and $\{\mathcal{M}_k\}$) if $S_{\mathbf{v}}(\Omega) \cap S_{\mathbf{v}'}(\Omega) \neq \emptyset$. The set of vertices

$$\mathcal{N}(\mathbf{v}) := \{\mathbf{v}' : \mathbf{v}' \in \mathcal{V}_k \text{ is a neighbor of } \mathbf{v}\}$$

is called the neighborhood of $\mathbf{v}$ (with respect to Ω and $\{\mathcal{M}_k\}$). Note that $\mathbf{v} \in \mathcal{N}(\mathbf{v})$ by definition.

Next, we define an equivalence relation on $\mathbf{v}$.

Definition 3.2 Under the above assumptions, two vertices $\mathbf{v} \in \mathcal{V}_k$ and $\mathbf{v}' \in \mathcal{V}_k'$ are equivalent, denoted by $\mathbf{v} \sim \mathbf{v}'$ if, for $\tau := S_{\mathbf{v}'} S_{\mathbf{v}}^{-1} : \mathbf{R}^d \to \mathbf{R}^d$, the following conditions hold:

(i) $\{S_{\mathbf{u}'} : \mathbf{u}' \in \mathcal{N}(\mathbf{v}')\} = \{\tau S_{\mathbf{u}} : \mathbf{u} \in \mathcal{N}(\mathbf{v})\}$.
(ii) For $\mathbf{u} \in \mathcal{N}(\mathbf{v})$ and $\mathbf{u}' \in \mathcal{N}(\mathbf{v}')$ such that $S_{\mathbf{u}'} = \tau S_{\mathbf{u}}$, and for any positive integer $l \geq 1$, an index $I \in \Sigma^*$ satisfies $(S_{\mathbf{u}} S_I, k + l) \in \mathcal{V}_{k+l}$ if and only if it satisfies $(S_{\mathbf{u}'} S_I, k' + l) \in \mathcal{V}_{k'+l}$.

It is easy to see that $\sim$ is an equivalence relation. We denote the equivalence class containing $\mathbf{v}$ by $[\mathbf{v}]$ and call it the neighborhood type of $\mathbf{v}$ (with respect to Ω and $\{\mathcal{M}_k\}$). We define two important infinite directed graphs $\mathcal{G}$ and $\mathcal{G}_R$. The graph $\mathcal{G}$ has vertex set $\mathcal{V}$ and directed edges defined as follows. Let $\mathbf{v} \in \mathcal{V}_k$ and $\mathbf{u} \in \mathcal{V}_{k+1}$. Suppose there exist $I \in \mathcal{M}_k$, $J \in \mathcal{M}_{k+1}$, and $L \in \Sigma$ such that

$$\mathbf{v} = (S_I, k), \qquad \mathbf{u} = (S_J, k + 1), \qquad J = IL.$$

Then we connect a directed edge $\mathbf{L}$ from $\mathbf{v}$ to $\mathbf{u}$ and denote this by $\mathbf{v} \xrightarrow{\mathbf{L}} \mathbf{u}$. We call $\mathbf{v}$ a parent of $\mathbf{u}$ in $\mathcal{G}$ and $\mathbf{u}$ an offspring of $\mathbf{v}$ in $\mathcal{G}$. We write $\mathcal{G} = (\mathcal{V}, E)$, where E is the set of all directed edges defined above.

The reduced graph $\mathcal{G}_R$ is obtained from $\mathcal{G}$ by first removing all but the smallest (in the lexicographical order) directed edge going to a vertex. More precisely, let $\mathbf{v}_k \xrightarrow{\mathbf{L}_k} \mathbf{u}$, $k = 1, 2, \ldots, n, \ldots$, be all the directed edges going to the vertex $\mathbf{u} \in \mathcal{V}_{k+1}$, where $\mathbf{v}_k \in \mathcal{V}_k$ are distinct and thus $\mathbf{L}_k$ are distinct also. Suppose $\mathbf{L}_1 < \mathbf{L}_2 < \cdots < \mathbf{L}_n < \cdots$ in the lexicographical order. Then we keep only $\mathbf{L}_1$ in the reduced graph and remove all the edges $\mathbf{L}_k$ $(k \geq 2)$. Next, we denote the resulting graph by $\mathcal{G}'_R$. It is possible that a vertex in $\mathcal{V}$ does not have any offspring $\mathcal{G}_R$ (see the example in [10]). We remove all vertices that do not have any offspring in $\mathcal{G}'_R$, together with all vertices and edges leading only to them. The resulting graph is the reduced graph, denoted by $\mathcal{G}_R = (\mathcal{V}_R, \mathcal{E}_R)$, where $\mathcal{V}_R$ is the set of vertices and $\mathcal{E}_R$ is the set of all edges.

It follows from the invariance of Ω under $\{S_i\}_{i=1}^{\infty}$ that only vertices in $\Omega(\mathbf{v})$ can be parents of any offspring of $\mathbf{v}$ in $\mathcal{G}$. In fact, if $\mathbf{u} = (S_{\mathbf{v}} S_I, k+1) \in \mathcal{V}_{k+1}$ is an offspring of $\mathbf{v}$ in $\mathcal{G}$ and if $\mathbf{w} \in \mathcal{V}_k \backslash \mathcal{N}(\mathbf{v})$, then for any index $I \in \Sigma^*$,

$$S_{\mathbf{w}} S_I(\Omega) \cap S_{\mathbf{u}}(\Omega) \subseteq S_{\mathbf{w}}(\Omega) \cap S_{\mathbf{v}}(\Omega) = \emptyset.$$

Hence $\mathbf{w}$ cannot be a parent of $\mathbf{u}$.

Proposition 3.3 *Let* $\mathbf{v}, \mathbf{v}' \in \mathcal{V}_k$ *and* $\mathbf{u}$, $\mathbf{u}'$ *be their offspring. If* $\mathbf{v}$ *and* $\mathbf{v}'$ *are not neighbors, then neither are* $\mathbf{u}$ *and* $\mathbf{u}'$.

Proof Let $S_{\mathbf{u}} = S_{\mathbf{v}} S_{\mathbf{w}}$ and $S_{\mathbf{u}'} = S_{\mathbf{v}'} S_{\mathbf{w}'}$ for some $\mathbf{w}, \mathbf{w}' \in \Sigma^*$. Since $S_{\mathbf{w}}(\Omega) \subseteq \Omega$ and $S_{\mathbf{w}'}(\Omega) \subseteq \Omega$, we have

$$S_{\mathbf{u}}(\Omega) \cap S_{\mathbf{u}'}(\Omega) \subseteq S_{\mathbf{v}}(\Omega) \cap S_{\mathbf{v}'}(\Omega) = \emptyset.$$

This leads to the conclusion. □

Proposition 3.4 *Let* Ω *be a bounded invariant open set for the IFS* $\{S_i\}_{i=1}^{\infty}$ *and let* $\mathcal{G}_R$ *be the corresponding reduced graph. Then there exists a unique path in* $\mathcal{G}_R$ *from the root vertex* $\mathbf{v}_{\text{root}}$ *to any given vertex.*

Proof The existence of a path is obvious. Next, we prove the uniqueness. Suppose $\mathbf{v} \in \mathcal{V}$, if there are two different paths in $\mathcal{G}_R$ from $\mathbf{v}_{\text{root}}$ to $\mathbf{v}$, then the vertex at which the two paths cross will have two parents in $\mathcal{G}_R$, it is a contradiction. □

Proposition 3.5 *Suppose* $\mathbf{v} \in \mathcal{V}_k$ *and* $\mathbf{v}' \in \mathcal{V}'_k$ *be two vertices with offspring* $\mathbf{u}_i$ *and* $\mathbf{u}'_j$ $(i, j \geq 0)$ *in* $\mathcal{G}_R$. *Suppose* $[\mathbf{v}] = [\mathbf{v}']$ *and let*

$$\mathcal{N}(\mathbf{v}) = \{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots\}, \qquad \mathcal{N}(\mathbf{v}') = \{\mathbf{v}', \mathbf{v}'_1, \mathbf{v}'_2, \ldots\}.$$

Suppose that in the graph $\mathcal{G}$ we have edges k_1, k_2 such that

$$\mathbf{v}_i \xrightarrow{k_1} \mathbf{u}_1, \qquad \mathbf{v}_j \xrightarrow{k_2} \mathbf{u}_2,$$
$$\mathbf{v}_i' \xrightarrow{k_1} \mathbf{u}_1', \qquad \mathbf{v}_j' \xrightarrow{k_2} \mathbf{u}_2'.$$

Then $\mathbf{u}_1 = \mathbf{u}_2$ if and only if $\mathbf{u}_1' = \mathbf{u}_2'$ and $\mathbf{u}_1$, $\mathbf{u}_2$ are neighbors if and only if $\mathbf{u}_1'$, $\mathbf{u}_2'$ are.

Proof Observe that

$$S_{\mathbf{u}_1'} = S_{\mathbf{v}_i'} S_{k_1} = \tau S_{\mathbf{v}_i} S_{k_1} = \tau S_{\mathbf{u}_1}. \tag{1}$$

Similarly we have $S_{\mathbf{u}_2'} = \tau S_{\mathbf{u}_2}$. Hence $S_{\mathbf{u}_1} = S_{\mathbf{u}_2}$ if and only if $S_{\mathbf{u}_1'} = S_{\mathbf{u}_2'}$, and so $\mathbf{u}_1 = \mathbf{u}_2$ if and only if $\mathbf{u}_1' = \mathbf{u}_2'$.

Furthermore,

$$S_{\mathbf{u}_1}(\Omega) \cap S_{\mathbf{u}_2}(\Omega) = \emptyset \quad \text{if and only if} \quad \tau S_{\mathbf{u}_1}(\Omega) \cap \tau S_{\mathbf{u}_2}(\Omega) = \emptyset.$$

This proves the second part of the proposition. □

Proposition 3.6 *Suppose $\mathbf{v} \in \mathcal{V}_k$ and $\mathbf{v}' \in \mathcal{V}_k'$ be two vertices with offspring $\mathbf{u}_i$ and $\mathbf{u}_j'$ $(i,j \geq 0)$ in $\mathcal{G}_R$. Suppose that $[\mathbf{v}] = [\mathbf{v}']$, and let*

$$\mathcal{N}(\mathbf{v}) = \{v, \mathbf{v}_1, \mathbf{v}_2, \ldots\}, \qquad \mathcal{N}(\mathbf{v}') = \{\mathbf{v}', \mathbf{v}_1', \mathbf{v}_2', \ldots\}.$$

Then

$$\{[\mathbf{u}_i] | i \geq 1\} = \{[\mathbf{u}_j'] | j \geq 1\}. \tag{2}$$

Proof The proposition says roughly that two vertices of the same neighborhood type have equivalent offspring. Let W and W' be the set of offspring of the vertices in $\mathcal{N}(\mathbf{v})$ and $\mathcal{N}(\mathbf{v}')$. We define a map $\chi : W \to W'$ as follows: Suppose that $\mathbf{u}$ is an offspring of $\mathbf{v}_i$ in $\mathcal{G}$ by an edge k. We let $\chi(\mathbf{u})$ be the offspring of $\mathbf{v}_i'$ by an edge k. Propositions 3.4 and 3.5 show that χ is a one-to-one correspondence. Furthermore, by (1) we have

$$S_{\chi}(\mathbf{u}) = \tau S_{\mathbf{u}}.$$

By Proposition 3.4 only vertices in $\mathcal{N}(\mathbf{v})$ can be parents of any offspring of $\mathbf{v}$ in $\mathcal{G}$. Again by Propositions 3.4 and 3.5, $\mathbf{u}$ is an offspring of $\mathbf{v}$ in $\mathcal{G}_R$ if and only if $\chi(\mathbf{u})$ is an offspring of $\mathbf{v}'$ in $\mathcal{G}_R$. This yields (2). □

Definition 3.7 Let $\{S_i\}_{i=1}^{\infty}$ be an self-map IFS on a subset $X \subset \mathbf{R}^d$. We say that $\{S_i\}_{i=1}^{\infty}$ is of generalized finite type if there exist a sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^{\infty}$ and a nonempty invariant open set $\Omega \subseteq X$ such that, with respect to Ω and $\{\mathcal{M}_k\}_{k=0}^{\infty}$, $\mathcal{V}/\sim = \{[\mathbf{v}] | \mathbf{v} \in \mathcal{V}\}$ is a finite set. In this case, we say that Ω is a generalized finite type set.

In the rest of this section, we establish classes of infinite IFSs of generalized finite type condition.

Proposition 3.8 *If* $\{S_i\}_{i=1}^{\infty}$ *is of OSC, then it is of the generalized finite type condition.*

Proof Let Ω be the open set of OSC. Suppose $\mathcal{M}_k = \Sigma^k$. For each $\mathbf{v} \in \mathcal{V}_1 = \{(S_i, 1) : i \geq 1\}$, the OSC implies that $\mathcal{N}(\mathbf{v}) = \{v\}$. Let $\tau = IS_{\mathbf{v}}^{-1}$, *i.e.* $\tau S_{\mathbf{v}} = I$, we have $\mathbf{v} \sim (I, 0) = \mathbf{v}_{\text{root}}$. By Proposition 3.4, $\mathcal{V}/\sim = \{[\mathbf{v}_{\text{root}}]\}$. So $\{S_i\}_{i=1}^{\infty}$ satisfies the generalized finite type condition. □

Example 3.9 $S_1(x) = \frac{1}{3}x$, $S_2(x) = \frac{1}{4}x + \frac{1}{4}$, $S_3(x) = \frac{1}{4}x + \frac{3}{4}$, $S_i(x) = 3^{-i}x - 2^{-i+1} + \frac{3}{4}$ $(i = 4, 5, \ldots)$, this IFS satisfies the generalized finite type condition.

Proof Let $\Omega = (0, 1)$. For each $k \geq 0$ let $\mathcal{M}_k = \Sigma^k$. Upon iterating the IFS once, the root vertex generates the following vertices:

$$\mathbf{v}_1 = (S_1, 1), \qquad \mathbf{v}_2 = (S_2, 1), \qquad \ldots, \qquad \mathbf{v}_n = (S_n, 1), \ldots.$$

Obviously, $\mathcal{N}(\mathbf{v}_k) = \{\mathbf{v}_k\}$ $(k \geq 3)$. So $\mathbf{v}_k \sim \mathbf{v}_{\text{root}}$ $(k \geq 3)$ with $\tau = IS_{\mathbf{v}_k}^{-1}$. Upon one more iteration, there is no new neighborhood type generated (see the example 2.8 in [10]). So $\mathcal{V}/\sim = \{[\mathbf{v}_{\text{root}}], [\mathbf{v}_1], [\mathbf{v}_2]\}$ and the result follows. □

4 Proof of the main theorem

Proof of Theorem 1.1 Assume that $\{S_i\}_{i=1}^{\infty}$ is a finite type similar IFS on X and let $\{\mathcal{M}_k\}_{k=0}^{\infty}$ and Ω be defined as above. We will show that there exists $\gamma \in \mathbf{N}$ such that, for all $0 < b \leq 1$ and $x \in X$,

$$\#\bigl\{S \in \mathcal{A}_b : x \in S(\Omega)\bigr\} \leq \gamma.$$

Let $\Psi = \{S \in \mathcal{A}_b : x \in S(\Omega)\}$. List all elements of Ψ as $S_{I_1}, S_{I_2}, \ldots$. For I_j there exists a unique $I_j' \in \mathcal{M}_{k_j}$ such that $I_j' \preceq I_j$. The choice of the particular I_j does not affect the following proof. We assume that I_j' is chosen such that k_j is maximum, *i.e.*, if $I_j'' \preceq I_j$ and $I_j'' \in \mathcal{M}_l$ for some l, then $l \leq k_j$ and $I_j'' \preceq I_j'$. We assume without loss of generality that

$$k_1 = \min\{k_j : S_{I_j} \in \Psi\} = k \quad \text{and} \quad I_1' \in \mathcal{M}_k.$$

For each j, here $S_{I_j} \in \Psi$, let I_j'' be the initial segment of I_j such that $I_j'' \in \mathcal{M}_{k_1}$. In particular, $I_1'' = I_1'$. Since $x \in S(\Omega)$ for all $S \in \Psi$, it follows that

$$\mathbf{v}_2 = (S_{I_2''}, k_1), \qquad \ldots, \qquad \mathbf{v}_m = (S_{I_m''}, k_1), \ldots,$$

are neighbors of $\mathbf{v}_1 = (S_{I_1''}, k_1)$. The finite type condition implies that the number of vertices in each neighborhood is uniformly bounded by some constant M independent of x, b, and the choice of I_j. That is,

$$\#\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \ldots\} \leq M,$$

i.e.

$$\#\bigl\{S_{I_j''} : S_{I_j} \in \Psi, I_j'' \preceq I_j\bigr\} \leq M.$$

Since each S_{I_j} belongs to $\mathcal{A}_b$, we have

$$\rho b|\Omega| < \left|S_{I_j}(\Omega)\right| \leq b|\Omega|, \quad S_{I_j} \in \Psi.$$

Also, by the definition of $\{\mathcal{M}_k\}_{k=0}^{\infty}$, there exists a constant n, independent of x and b, such that $|I_j| - |I_j'| \leq n$ for all I_j, here $S_{I_j} \in \Psi$. Hence,

$$\rho b|\Omega| < \left|S_{I_j'}(\Omega)\right| \leq b\rho^{-n}|\Omega|, \quad S_{I_j} \in \Psi, I_j' \preceq I_j.$$

It yields

$$\rho \leq \frac{|S_{I_1'}(\Omega)|}{|S_{I_j}(\Omega)|} \leq \rho^{-(n+1)}. \tag{3}$$

It also implies that there exists a constant $c_1 > 0$, independent of x and b, such that

$$c_1^{-1} \leq \frac{|S_{I_j''}(\Omega)|}{|S_{I_1'}(\Omega)|} \leq c_1, \quad S_{I_j} \in \Psi. \tag{4}$$

Combining (3) and (4) yields

$$c_2^{-1} \leq \frac{|S_{I_j''}(\Omega)|}{|S_{I_j}(\Omega)|} \leq c_2, \quad S_{I_j} \in \Psi. \tag{5}$$

Here $c_2 := \rho^{-(n+1)}c_1$.

We write $I_j = (I_j'', L_j'')$, for any $j \geq 1$. For each $S_{I_j} \in \Psi$, (5) implies that

$$\rho_{I_j''} \leq \rho_{I_j} c_2 = \rho_{I_j''}\rho_{L_j''} c_2.$$

Hence

$$\rho_{\max}^{|L_j''|} \geq \rho_{I_j''} \geq c_2^{-1}.$$

If we let $l := [-\log(c_2)/\log\rho_{\max}] + 1$, then $|L_j''| \leq l$. The finite type condition implies that

$$\#\left\{S_{I_j} : I'' \preceq I_j\right\} \leq M^l.$$

Thus, $\#\{S \in \mathcal{A}_b : x \in S(\Omega)\} \leq \gamma$ follows by taking $\gamma = M^{l+1}$. Lemma 2.3 implies that the $\{S_i\}_{i=1}^{\infty}$ satisfies the WSC. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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