Internat. J. Math. & Math. Sci.
Vol. 1 (1978) 203-208

A NOTE ON STRICTLY CYCLIC SHIFTS ON ℓ_p

GERD H. FRICKE

Department of Mathematics
Wright State University
Dayton, Ohio 45431

(Received June 28, 1977)

ABSTRACT. In this paper the author shows that a well known sufficient condition for strict cyclicity of a weighted shift on ℓ_p is not a necessary condition for any p with $1 < p < \infty$.

1. INTRODUCTION.

For $1 \le p < \infty$ let ℓ_p be the Banach space of absolutely p-summable sequences of complex numbers. Let S_α denote the weighted shift on ℓ_p with weight sequence $\alpha = \{\alpha_n\}_1^\infty$ defined by $S_\alpha\left[\sum_{n=0}^\infty x_n e_n\right] = \sum_{n=1}^\infty \alpha_n x_{n-1} e_n$. Let $\beta_0 = 1$ and $\beta_n = \alpha_1\alpha_2\ldots\alpha_n$ for all $n \ge 1$. (For more detail we refer the reader to [2] and [3]).

Mary Embry [1] showed that for p = 1 the weighted shift S_α is strictly cyclic if and only if

$$\sup_{n,m} \left|\frac{\beta_{n+m}}{\beta_n \beta_m}\right| < \infty. \tag{1.1}$$

Edward Kerlin and Alan Lambert [2] considered the natural extension to (1.1) for the case $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m \beta_{n-m}}\right|^q < \infty \tag{1.2}$$

and showed (1.2) implies S_α is strictly cyclic on ℓ_p. They also proved that (1.2) is necessary if the weight sequence α is eventually decreasing. This strongly suggested that (1.2) is a necessary and sufficient condition for strict cyclicity. However, we will show in this paper that (1.2) is not a necessary condition for S_α to be strictly cyclic for any p with $1 < p < \infty$.

2. PROOF.

To preserve the clarity of the proof we will consider the cases $1 < p \leq 2$ and $2 < p < \infty$ separately.

(a) Let $1 < p \leq 2$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\{n_k\}_1^\infty$ be a sequence of rapidly increasing positive integers, e.g., choose $n_1 = 10$ and $n_k = (10n_{k-1})^{10qn_{k-1}}$ for $k > 1$.

We now define the weight sequence $\alpha = \{\alpha_i\}_1^\infty$ by $\alpha_1 = \alpha_2 = \ldots = \alpha_{n_1} = 1$ and for $k > 1$

$$\alpha_i = \begin{cases} n_k^{-1} & \text{if } n_{k-1} < i \leq n_k - n_{k-1} \\ \\ n_{k-1}^{-1} & \text{if } n_k - n_{k-1} < i \leq n_k. \end{cases}$$

Clearly, $\alpha_{n_{k-1}+1} = \alpha_{n_{k-1}+2} = \ldots = \alpha_{n_k - n_{k-1}}$ and $\alpha_{n_k - \ell + 1} \leq \alpha_\ell$ for $1 \leq \ell \leq \frac{n_k}{2}$.

Thus, for $0 < m < n_k$

$$\frac{\beta_{n_k}}{\beta_m \beta_{n_k - m}} = \frac{\alpha_{n_k-m+1}\alpha_{n_k-m+2} \cdots \alpha_{n_k}}{\alpha_1 \alpha_2 \ldots \alpha_m} \geq \frac{\alpha_{n_k - n_{k-1}+1} \ldots \alpha_{n_k}}{\alpha_1 \alpha_2 \ldots \alpha_{n_{k-1}}} \geq (\alpha_{n_k})^{n_{k-1}}.$$

Therefore,

$$\sum_{m=1}^{n_k-1} \left|\frac{\beta_{n_k}}{\beta_m\beta_{n_k-m}}\right|^q \geq n_k(\alpha_{n_k})^{qn_{k-1}} = n_k n_{k-1}^{-qn_{k-1}} \to \infty \text{ as } k \to \infty.$$

Hence,

$$\sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|^q = \infty.$$

We now show that S_α is strictly cyclic. It is known that $S_\alpha[2]$ is strictly cyclic if and only if

$$\sum_{n=0}^{\infty} \left|\sum_{m=0}^{n} \frac{\beta_n}{\beta_m\beta_{n-m}} x_m y_{n-m}\right|^p < \infty \text{ for all } x,y \in \ell_p. \tag{1.3}$$

Obviously, for $0 < m < n$

$$\frac{\beta_n}{\beta_m\beta_{n-m}} = \frac{\alpha_{n-m+1} \cdots \alpha_n}{\alpha_1\alpha_2 \cdots \alpha_m} \leq \alpha_n$$

and $\frac{\beta_n}{\beta_m\beta_{n-m}} = 1$ for $m = 0$ or $m = n$.

Thus,

$$\sum_{n=0}^{\infty} \left|\sum_{m=0}^{n} \frac{\beta_n}{\beta_m\beta_{n-m}} x_m y_{n-m}\right|^p \leq \sum_{n=0}^{\infty} \{|x_0y_n| + |y_0x_n| + \sum_{m=1}^{n-1} \alpha_n|x_my_{n-m}|\}^p$$

$$\leq \sum_{n=0}^{\infty} \{|x_0y_n| + |y_0x_n| + \alpha_n\|x\|_2\|y\|_2\}^p < \infty$$

since $\alpha,x,y \in \ell_p$.

Hence (1.3) holds and S_α is strictly cyclic.

(b) Let $2 < p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Let $\{n_k\}_1^\infty$ be a sequence of rapidly increasing integers, e.g., choose $n_1 = 10$ and n_k such that

$$\sum_{n=1}^{n_k} \frac{1}{n} \geq (10n_{k-1})^{10n_{k-1}} \text{ for } k > 1.$$

Define $\{d_i\}_1^\infty$ such that $\prod_{i=1}^{n} d_i = n^{-\frac{1}{q}}$ for $n = 1,2,\cdots$ and define $\{s_k\}_1^\infty$ by $s_1 = 10$ and $s_k = 2s_{k-1} + 2n_k$ for $k > 1$.

We now define the weight sequence $\alpha = \{\alpha_i\}_1^\infty$ by $\alpha_1 = \ldots = \alpha_{s_1} = 1$ and for $k > 1$,

$$\alpha_i = \begin{cases} n_k^{-2} & \text{if} \quad s_{k-1} < i \leq s_{k-1}+n_k \\ n_k^{-2}\, d_{s_k-s_{k-1}-i+1} & \text{if} \quad s_{k-1}+n_k < i \leq s_k - s_{k-1} \\ n_{k-1}^{-2} & \text{if} \quad s_k - s_{k-1} < i \leq s_k. \end{cases}$$

Now, for $s_{k-1} < m < \frac{s_k}{2}$,

$$\frac{\beta_{s_k}}{\beta_m \beta_{s_k-m}} = \frac{\alpha_{s_k-m+1} \cdots \alpha_{s_k}}{\alpha_1\alpha_2 \cdots \alpha_m} \geq \frac{\alpha_{s_k-m+1} \cdots \alpha_{s_k-s_{k-1}}}{\alpha_{s_{k-1}+1} \cdots \alpha_m}\, n_{k-1}^{-2s_{k-1}} \geq n_{k-1}^{-2s_{k-1}} \prod_{i=1}^{m-s_{k-1}} d_i$$

$$= n_{k-1}^{-2s_{k-1}} (m-s_{k-1})^{-\frac{1}{q}}.$$

Thus,

$$\sum_{m=0}^{s_k} \left|\frac{\beta_{s_k}}{\beta_m\beta_{s_k-m}}\right|^q \geq n_{k-1}^{-2qs_{k-1}} \sum_{i=1}^{n_k} i^{-1} \geq n_{k-1}^{-10n_{k-1}} \sum_{i=1}^{n_k} i^{-1} \to \infty \text{ as } k \to \infty.$$

Hence,
$$\sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|^q = \infty.$$

We now show that

$$\sum_{n=0}^{\infty} \left|\sum_{m=0}^{n} \frac{\beta_n}{\beta_m\beta_{n-m}} x_m y_{n-m}\right|^p < \infty \quad \text{for all } x,y \in \ell_p.$$

If $s_{k-1} < n < s_k - s_{k-1}$ and $0 < m < n$ then, $\frac{\beta_n}{\beta_m\beta_{n-m}} \leq n_k^{-2}$.

Let $h = \min\{m,n-m\}$ then, for $s_k - s_{k-1} \leq n \leq s_k$ and $0 < m < n$,

$$\frac{\beta_n}{\beta_m \beta_{n-m}} \leq n_{k-1}^{-2} \, h^{-\frac{1}{q}} .$$

Thus,

$$\sum_{k=2}^{\infty} \sum_{n=s_{k-1}+1}^{s_k - s_{k-1} - 1} \left| \sum_{m=1}^{n-1} \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m} \right|^p \leq \sum_{k=2}^{\infty} \sum_{n=s_{k-1}+1}^{s_k - s_{k-1} - 1} \left(n_k^{-2} \sum_{m=1}^{n-1} |x_m y_{n-m}| \right)^p < \infty. \tag{1.4}$$

Let $\delta = \frac{p}{p-2}$ then $\delta > q$ and $\frac{1}{\delta} + \frac{2}{p} = 1$. Let $M = \left(\sum_{m=1}^{\infty} 2m^{-\frac{\delta}{q}} \right)^{\frac{1}{\delta}} \left(|| x ||_p^p + || y ||_p^p \right)^{\frac{2}{p}} < \infty.$

Then,

$$\sum_{k=2}^{\infty} \sum_{n=s_k - s_{k-1}}^{s_k} \left| \sum_{m=1}^{n-1} \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m} \right|^p$$

$$\leq \sum_{k=2}^{\infty} \sum_{n=s_k - s_{k-1}}^{s_k} \left(\sum_{m=1}^{\infty} n_{k-1}^{-2} h^{-\frac{1}{q}} |x_m y_{n-m}| \right)^p, \text{ where } h = \min\{m,n-m\}$$

$$\leq \sum_{k=2}^{\infty} \sum_{n=s_k - s_{k-1}}^{s_k} n_{k-1}^{-2p} M^p < \infty.$$

Combining (1.4) and (1.5) we obtain that (1.3) is satisfied. QED

References

1. Embry, Mary. Strictly Cyclic Operator Algebras on a Banach Space, to appear in Pac. J. Math.

2. Kerlin, Edward and Alan Lambert. Strictly Cyclic Shifts on ℓ_p, Acta Sci. Math. 35 (1973) 87-94.

3. Lambert, Alan. Strictly Cyclic Weighted Shifts, Proc. Amer. Math. Soc. 29 (1971) 331-336.

KEY WORDS AND PHRASES. *Strictly cyclic shifts, absolutely p-summable sequence, ℓ_p-space, cyclic operator algebras.*

AMS(MOS) SUBJECT CLASSIFICATION (1970) CODES. *40H05.*


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization