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*Research Article*

# Nontrivial Solutions for Time Fractional Nonlinear Schrödinger-Kirchhoff Type Equations

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We study the existence of solutions for time fractional Schrödinger-Kirchhoff type equation involving left and right Liouville-Weyl fractional derivatives via variational methods.

## 1. Introduction

In recent years, there has been a great interest in studying problems involving fractional Schrödinger equations [1–5], Kirchhoff type equations [6–8], fractional Navier-Stokes equations [9, 10], and fractional ordinary differential equations and Hamiltonian systems [11–17], and so forth. For further details and applications, we refer the reader to [18, 19] and the references cited therein.

On the other hand, the integer-order Schrödinger-Kirchhoff type equations have also been investigated by many authors; for example, see [20–23]. In fact, Schrödinger-Kirchhoff type equations play an important role in modelling several physical and biological systems. However, to the best of our knowledge, the existence of solutions to the time fractional Schrödinger-Kirchhoff type equations has yet to be addressed.

The objective of the present paper is to study time fractional Schrödinger-Kirchhoff type equation of the form

$$\left(a+b\int_{\mathbb{R}}\left|{}_{-\infty}D_t^{\alpha}u(t)\right|^2dt\right)^{\theta-1}{}_tD_{\infty}^{\alpha}\left({}_{-\infty}D_t^{\alpha}u(t)\right)+\mu V(t)u=f(t,u),\quad t\in\mathbb{R},\ u\in H^{\alpha}(\mathbb{R}),\tag{1}$$

where $\alpha\in(1/2,1]$, ${}_{-\infty}D_t^{\alpha}$ and ${}_tD_{\infty}^{\alpha}$, respectively, denote left and right Liouville-Weyl fractional derivatives of order $\alpha$ on $\mathbb{R}$, $a,b>0$ are constants, $\mu>0$ is parameter, $\theta>1$, $f\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, and $V:\mathbb{R}\to\mathbb{R}^+$ is a potential function.

The rest of the paper is organized as follows. Section 2 contains preliminary concepts of fractional calculus and fractional Sobolev space, while some important lemmas, which are needed in the proof of main results, are obtained in Section 3. We present our main results in Section 4.

## 2. Preliminaries

In this section, we recall important definitions and concepts of fractional calculus and then prove certain results about fractional Sobolev space $H^{\alpha}(\mathbb{R})$ related to our study of the problem at hand.

*Definition 1* (see [24]). The left and right Liouville-Weyl fractional integrals of order $\alpha\in(0,1)$ on $\mathbb{R}$ are defined by

$$\begin{aligned}{}_{-\infty}I_x^{\alpha}\phi(x)&=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-\xi)^{\alpha-1}\phi(\xi)\,d\xi,\\ {}_xI_{\infty}^{\alpha}\phi(x)&=\frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(\xi-x)^{\alpha-1}\phi(\xi)\,d\xi,\end{aligned}\tag{2}$$

respectively, where $x\in\mathbb{R}$.

The left and right Liouville-Weyl fractional derivatives of order $\alpha \in (0,1)$ on $\mathbb{R}$ are defined by

$$ {}_{-\infty}D_x^{\alpha}\phi(x) = \frac{d}{dx}\,{}_{-\infty}I_x^{1-\alpha}\phi(x), \qquad {}_xD_{\infty}^{\alpha}\phi(x) = -\frac{d}{dx}\,{}_xI_{\infty}^{1-\alpha}\phi(x), \tag{3} $$

respectively, where $x \in \mathbb{R}$.

The definitions (3) may be written in an alternative form as follows:

$$ {}_{-\infty}D_x^{\alpha}\phi(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{\phi(x)-\phi(x-\xi)}{\xi^{\alpha+1}}d\xi, \qquad {}_xD_{\infty}^{\alpha}\phi(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{\phi(x)-\phi(x+\xi)}{\xi^{\alpha+1}}d\xi. \tag{4} $$

Also, we define the Fourier transform $\mathcal{F}(u)(\xi)$ of $u(x)$ as

$$ \mathcal{F}(u)(\xi) = \int_{-\infty}^{\infty} e^{-ix\cdot\xi}u(x)\,dx. \tag{5} $$

For any $\alpha > 0$, we define the seminorm and norm, respectively, as [16]

$$ |u|_{I_{-\infty}^{\alpha}} = \left\| {}_{-\infty}D_x^{\alpha}u \right\|_{L^2}, \qquad \|u\|_{I_{-\infty}^{\alpha}} = \left(\|u\|_{L^2}^2 + |u|_{I_{-\infty}^{\alpha}}^2\right)^{1/2}, \tag{6} $$

and let the space $I_{-\infty}^{\alpha}(\mathbb{R})$ denote the completion of $C_0^{\infty}(\mathbb{R})$ with respect to the norm $\|\cdot\|_{I_{-\infty}^{\alpha}}$.

Next, for $0 < \alpha < 1$, we give the relationship between classical fractional Sobolev space $H^{\alpha}(\mathbb{R})$ and $I_{-\infty}^{\alpha}(\mathbb{R})$, where $H^{\alpha}(\mathbb{R})$ is defined by

$$ H^{\alpha}(\mathbb{R}) = \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{\alpha}}, \tag{7} $$

with the norm

$$ \|u\|_{\alpha} = \left(\|u\|_{L^2}^2 + |u|_{\alpha}^2\right)^{1/2}, \tag{8} $$

and seminorm

$$ |u|_{\alpha} = \left\| |\xi|^{\alpha}\mathcal{F}(u) \right\|_{L^2}. \tag{9} $$

Observe that the spaces $H^{\alpha}(\mathbb{R})$ and $I_{-\infty}^{\alpha}(\mathbb{R})$ are equal and have equivalent norms (see [16]).

Therefore, we define

$$ H^{\alpha}(\mathbb{R}) = \left\{u \in L^2(\mathbb{R}) \mid |\xi|^{\alpha}\mathcal{F}(u) \in L^2(\mathbb{R})\right\}. \tag{10} $$

Let

$$ X^{\alpha} = \left\{u \in H^{\alpha}(\mathbb{R}) \mid \int_{\mathbb{R}}\left(\left|{}_{-\infty}D_t^{\alpha}u(t)\right|^2 + |u(t)|^2\right)dt < \infty\right\}. \tag{11} $$

The space $X^{\alpha}$ is a reflexive and separable Hilbert space with the inner product

$$ \langle u, v\rangle_{X^{\alpha}} = \int_{\mathbb{R}}\left({}_{-\infty}D_t^{\alpha}u(t)\cdot{}_{-\infty}D_t^{\alpha}v(t) + u(t)v(t)\right)dt \tag{12} $$

and the corresponding norm

$$ \|u\|_{X^{\alpha}}^2 = \langle u, u\rangle_{X^{\alpha}}. \tag{13} $$

Define the space

$$ X_{\mu}^{\alpha} = \left\{u \in X^{\alpha} : \int_{\mathbb{R}}\mu V(t)|u|^2\,dt < +\infty\right\}, \tag{14} $$

with the norm

$$ \|u\|_{X_{\mu}^{\alpha}} = \left(\int_{\mathbb{R}} a^{\theta-1}\left(\left|{}_{-\infty}D_t^{\alpha}u(t)\right|^2\right)dt + \int_{\mathbb{R}}\mu V(t)|u|^2\,dt\right)^{1/2}. \tag{15} $$

**Lemma 2.** *$(X_{\mu}^{\alpha}, \|\cdot\|_{X_{\mu}^{\alpha}})$ is a uniformly convex Banach space.*

*Proof.* $X_{\mu}^{\alpha}$ is obviously Banach space. Now, we can prove that $(X_{\mu}^{\alpha}, \|\cdot\|_{X_{\mu}^{\alpha}})$ is uniformly convex. To this end, let $0 < \varepsilon < 2$ and $u, v \in X_{\mu}^{\alpha}$ with $\|u\|_{X_{\lambda}^{\alpha}} = \|v\|_{X_{\mu}^{\alpha}} = 1$ and $\|u - v\|_{X_{\mu}^{\alpha}} \geq \varepsilon$. Using the following inequality:

$$ \left|\frac{a+b}{2}\right|^2 + \left|\frac{a-b}{2}\right|^2 \leq \frac{1}{2}\left(|a|^2 + |b|^2\right), \quad \forall a, b \in \mathbb{R}, \tag{16} $$

we get

$$ \begin{aligned} &\left\|\frac{u+v}{2}\right\|_{X_{\mu}^{\alpha}}^2 + \left\|\frac{u-v}{2}\right\|_{X_{\mu}^{\alpha}}^2 \\ &= \int_{\mathbb{R}} a^{\theta-1}\left(\left|{}_{-\infty}D_t^{\alpha}\left(\frac{u+v}{2}\right)(t)\right|^2\right)dt + \int_{\mathbb{R}}\mu V(t)\left|\frac{u+v}{2}\right|^2 dt \\ &+ \int_{\mathbb{R}} a^{\theta-1}\left(\left|{}_{-\infty}D_t^{\alpha}\left(\frac{u-v}{2}\right)(t)\right|^2\right)dt + \int_{\mathbb{R}}\mu V(t)\left|\frac{u-v}{2}\right|^2 dt \\ &\leq \frac{1}{2}\left(\int_{\mathbb{R}} a^{\theta-1}\left(\left|{}_{-\infty}D_t^{\alpha}u(t)\right|^2\right)dt + \int_{\mathbb{R}} a^{\theta-1}\left(\left|{}_{-\infty}D_t^{\alpha}v(t)\right|^2\right)dt + \int_{\mathbb{R}}\mu V(t)|u|^2 dt + \int_{\mathbb{R}}\mu V(t)|v|^2 dt\right) = \frac{1}{2}\left(\|u\|_{X_{\mu}^{\alpha}}^2 + \|v\|_{X_{\mu}^{\alpha}}^2\right) = 1, \end{aligned} \tag{17} $$

which implies that $\|(u+v)/2\|_{X_{\mu}^{\alpha}}^2 \leq 1 - \varepsilon/2$. Hence, taking $\delta = \delta(\varepsilon)$ such that $1-\varepsilon/2 = 1-\delta$, we have $\|(u+v)/2\|_{X_{\mu}^{\alpha}}^2 \leq 1-\delta$. Therefore, $(X_{\mu}^{\alpha}, \|\cdot\|_{X_{\mu}^{\alpha}})$ is uniformly convex. □

In the sequel, we need the following assumptions.

(V1) $V(t) \in C(\mathbb{R}, \mathbb{R})$, $V_0 := \inf_{t\in\mathbb{R}} V(t) > 0$;

(V2) there exists $r > 0$ such that, for any $M > 0$,

$$\text{meas}\left(\{t \in (y-r, y+r) : V(t) \le M\}\right) \longrightarrow 0 \quad \text{as } |y| \longrightarrow \infty; \tag{18}$$

(V3) there exists $l_0 > 0$ such that $\int_{|t|\ge l_0} V(t)^{-1} dt < \infty$;

(F1) $f \in C(\mathbb{R}\times\mathbb{R}, \mathbb{R})$ and there exist constants $c_0, c_1, \ldots, c_l > 0$ and $q_j \in (2, 2\theta)$ such that

$$|f(t,u)| \le c_0 |u| + \sum_{j=1}^{l} c_j |u|^{q_j - 1}, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}; \tag{19}$$

(F2) $f(t,u) = o(|u|)$ as $|u| \to 0$ uniformly in $t \in \mathbb{R}^N$;

(F3) there exist $\lambda \in (2\theta, \infty)$ such that

$$\lambda F(t,u) \le f(t,u)u, \quad \forall t \in \mathbb{R},\ u \in \mathbb{R}; \tag{20}$$

(F4) $F(t,u)/|u|^{2\theta} \to +\infty$ as $|u| \to +\infty$ uniformly in $t \in \mathbb{R}$;

(F5) $f(t,-u) = -f(t,u)$ for all $(t,u) \in \mathbb{R}\times\mathbb{R}$;

(F6) $f \in C(\mathbb{R}\times\mathbb{R}, \mathbb{R})$ and there exists $1 < p < 2$ such that

$$|f(t,u)| \le |u|^{p-1}, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}; \tag{21}$$

(F7) there exist $\sigma_1 > 0$, $0 < \sigma_2 < 1/8D_2^2$ ($D_2$ is defined in Remark 6), $1 \le \gamma < 2$, and small constants $0 < r_0 < r_1$ such that

$$\sigma_1 |u|^{\gamma} < F(t,u) \le \sigma_2 |u|^2, \quad r_0 \le |u| \le r_1, \text{ a.e. } t \in \mathbb{R}. \tag{22}$$

**Lemma 3.** *Assume that (V1) holds. Then the embeddings $X_\mu^\alpha \hookrightarrow X^\alpha \hookrightarrow L^2(\mathbb{R})$ are continuous. In particular, there exists a constant $C_2 > 0$ such that*

$$\|u\|_{L^2(\mathbb{R})} \le C_2 \|u\|_{X_\mu^\alpha} \quad \forall u \in X_\mu^\alpha. \tag{23}$$

*Moreover, if (V1) and (V2) hold, then the embedding $X_\mu^\alpha \hookrightarrow L^2(\mathbb{R})$ is compact.*

*Proof.* Clearly, the chain of embeddings $X_\mu^\alpha \hookrightarrow X^\alpha \hookrightarrow L^2(\mathbb{R})$ is continuous and consequently one can obtain (23). Also in view of (V1), (V2), and following the method of proof similar to that of Lemma 2.2 in [15], the embedding $X_\mu^\alpha \hookrightarrow L^2(\mathbb{R})$ is compact. □

**Lemma 4.** *Let $\alpha > 1/2$. Then $H^\alpha(\mathbb{R}) \subset C(\mathbb{R})$ and there exists a constant $C = C_\alpha$ such that*

$$\sup_{x\in\mathbb{R}} |u(x)| \le C \|u\|_{X_\mu^\alpha}. \tag{24}$$

*Proof.* The proof is similar to that of Theorem 2.1 in [16], so we omit it. □

Also by Lemma 4, there is a constant $C_\alpha > 0$ such that

$$\|u\|_\infty \le C_\alpha \|u\|_{X_\mu^\alpha}. \tag{25}$$

*Remark 5.* If $u \in H^\alpha(\mathbb{R})$ with $1/2 < \alpha < 1$, then it follows by Lemma 4 that $u \in L^q(\mathbb{R})$ for all $q \in [2, \infty)$ as

$$\int_{\mathbb{R}} |u(x)|^q dx \le \|u\|_\infty^{q-2} \|u\|_{L^2(\mathbb{R})}^2. \tag{26}$$

*Remark 6.* From Remark 5 and Lemma 3, it is easy to verify that the imbedding of $X_\mu^\alpha$ in $L^q(\mathbb{R})$ is also compact for $q \in (2, \infty)$. Hence, for all $2 \le q < \infty$, the imbedding of $X_\mu^\alpha$ in $L^q(\mathbb{R})$ is continuous and compact, which together with Lemma 4 implies that there exists $D_q > 0$ such that

$$\|u\|_{L^q(\mathbb{R})} \le D_q \|u\|_{X_\mu^\alpha}. \tag{27}$$

**Lemma 7.** *Assume that (V1) and (V3) hold. Then the embedding $X_\mu^\alpha \hookrightarrow L^p(\mathbb{R})$ is continuous and compact for $p \in [1, +\infty)$.*

*Proof.* By (V3) and Hölder's inequality, we have

$$\begin{aligned}\int_{|t|\ge l_0} |u(t)| dt &\le \left(\int_{|t|\ge l_0} V(t)|u(t)|^2 dt\right)^{1/2} \left(\int_{|t|\ge l_0} V(t)^{-1/2} dt\right)^{1/2} \\ &\le c_1 \|u\|_{X_\mu^\alpha},\end{aligned} \tag{28}$$

for some positive constant $c_1$. So Lemma 4 implies that

$$\begin{aligned}\|u\|_1 &= \int_{-l_0}^{l_0} |u(t)| dt + \int_{|t|\ge l_0} |u(t)| dt \\ &\le 2l_0 \|u\|_\infty + c_1 \|u\|_{X_\mu^\alpha} \le c_2 \|u\|_{X_\mu^\alpha},\end{aligned} \tag{29}$$

for some positive constant $c_2$. Hence, by Remark 6, we can get continuous embeddings $X_\mu^\alpha$ into $L^p(\mathbb{R})$ for $p \in [1, +\infty)$. Now, we will show that the embedding is compact for $p \in [1, +\infty)$. Let $\{u_n\} \subset X_\mu^\alpha$ such that $u_n \rightharpoonup 0$ and $M > 0$ such that $\|u\|_{X_\mu^\alpha} \le M$. In view of (V3), given $\varepsilon > 0$, for $l > 0$ large enough, one can obtain

$$\int_{|t|\ge l_0} V(t)^{-1/2} dt < \left(\frac{\varepsilon}{2M}\right)^2. \tag{30}$$

Then,

$$\begin{aligned}\int_{|t|\ge l} |u(t)| dt &\le \left(\int_{|t|\ge l} V(t)|u(t)|^2 dt\right)^{1/2} \left(\int_{|t|\ge l} V(t)^{-1/2} dt\right)^{1/2} \\ &\le \frac{\varepsilon}{2M} \|u\|_{X_\mu^\alpha} \le \frac{\varepsilon}{2}.\end{aligned} \tag{31}$$

On the other hand, by Sobolev's theorem (see, e.g., [25]) which implies that $u_n \to 0$ uniformly on $[-l, l]$, there is $n_0$ such that $\int_{-l}^{l} |u(t)| dt < \varepsilon/2$ for all $n \geq n_0$. Thus $u_n \to 0$ in $L^1(\mathbb{R})$. So, for $1 < p < \infty$, we have

$$\int_{\mathbb{R}} |u(x)|^p \, dx \leq \|u\|_\infty^{p-1} \int_{\mathbb{R}} |u(t)| \, dt \leq c_3 \|u\|_1 \longrightarrow 0, \quad (32)$$

and consequently, $u_n \to 0$ in $L^p(\mathbb{R})$ for $p \in [1, +\infty)$. □

*Definition 8.* Let $X$ be a Banach space, $I \in C^1(X, \mathbb{R})$. One says that $I$ satisfies the Palais-Smale (PS) condition if any sequence $(u_n) \in X$ for which $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

In order to establish the main results, we need the following known Theorems.

**Theorem 9** (see [26, Theorem 2.2]). *Let $X$ be a real Banach space and $I \in C^1(X, \mathbb{R})$ satisfies (PS) condition. Suppose $I(0) = 0$ and*

(i) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq 0$;*

(ii) *there is an $e \in X \setminus \overline{B_\rho(0)}$ such that $I(e) \leq 0$.*

*Then $I$ possesses a critical value $c \geq \alpha$. Moreover $c$ can be characterized as*

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s)), \quad (33)$$

*where*

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \ \gamma(1) = e\}. \quad (34)$$

**Theorem 10** (see [26, Theorem 9.12]). *Let $X$ be an infinite dimensional Banach space and let $I \in C^1(X, \mathbb{R})$ be even, satisfying (PS) condition, and $I(0) = 0$. If $X = Y \oplus Z$, where $Y$ is finite dimensional and $I$ satisfies the following conditions:*

(I1) *there exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap Z} \geq \alpha$;*

(I2) *for any finite dimensional subspace $\widetilde{X} \subset X$, there is $R = R(\widetilde{X}) > 0$ such that $I(u) \leq 0$ on $\widetilde{X} \setminus B_R$,*

*then $I$ possesses an unbounded sequence of critical values.*

## 3. Some Lemmas

Recall that $u \in X_\mu^\alpha$ is said to be a weak solution of problem (1) if

$$\left(a + b \int_{\mathbb{R}} \left|{}_{-\infty}D_t^\alpha u(t)\right|^2 dt\right)^{\theta-1} \cdot \int_{\mathbb{R}} {}_{-\infty}D_t^\alpha u(t) \cdot {}_{-\infty}D_t^\alpha \varphi(t) \, dt + \int_{\mathbb{R}} \mu V(t) u(t) \varphi(t) \, dt = \int_{\mathbb{R}} f(t, u(t)) \varphi(t) \, dt, \quad \forall \varphi \in X_\mu^\alpha, \quad (35)$$

and the energy functional $I_{\mu,\theta} : X_\mu^\alpha \to \mathbb{R}$ is given by the formula

$$I_{\mu,\theta}(u) = \frac{1}{2b\theta}\left(a + b\int_{\mathbb{R}} \left|{}_{-\infty}D_t^\alpha u(t)\right|^2 dt\right)^\theta + \frac{1}{2}\int_{\mathbb{R}} \mu V(x) |u(t)|^2 dt - \int_{\mathbb{R}} F(t, u(t)) \, dt, \quad (36)$$

where $F(x, u) = \int_0^u f(t, s) ds$.

In view of assumptions (V1) and (F1), the functional $I_{\mu,\theta}$ is of class $C^1(X_\mu^\alpha, \mathbb{R})$ and by similar method in Theorem 4.1 in [27] and the definition of Gâteaux derivative, one can get

$$\left\langle I'_{\mu,\theta}(u), \varphi \right\rangle = \left(a + b\int_{\mathbb{R}} \left|{}_{-\infty}D_t^\alpha u(t)\right|^2 dt\right)^{\theta-1} \cdot \int_{\mathbb{R}} {}_{-\infty}D_t^\alpha u(t) \cdot {}_{-\infty}D_t^\alpha \varphi(t) \, dt + \int_{\mathbb{R}} \mu(t) u(t) \varphi(t) \, dt - \int_{\mathbb{R}} f(t, u(t)) \varphi(t) \, dt, \quad \forall u, \varphi \in X_\mu^\alpha. \quad (37)$$

**Lemma 11.** *Assume that (V) and (F1)–(F3) hold. Then $I_{\mu,\theta}$ satisfies the (PS) condition.*

*Proof.* Let $\{u_n\}_{n\in\mathbb{N}} \subset X_\mu^\alpha$ be a sequence such that $\{I_{\mu,\theta}(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'_{\mu,\theta}(u_n) \to 0$ as $n \to \infty$. Then there exits $D > 0$ such that $|\langle I'_{\mu,\theta}(u_n), u_n\rangle| \leq D\|u\|_{X_\mu^\alpha}$ and $|I_{\mu,\theta}(u_n)| \leq D$. So, by (F3), (23), and the fact that $\lambda > 2\theta > 1$, we get

$$\begin{aligned}
\lambda D + D\|u\|_{X_\mu^\alpha} &\geq \lambda I_{\mu,\theta}(u_n) - \left\langle I'_{\mu,\theta}(u_n), u_n \right\rangle \\
&= \frac{\lambda}{2b\theta}\left(a + b\int_{\mathbb{R}} \left|{}_{-\infty}D_t^\alpha u_n(t)\right|^2 dt\right)^\theta + \frac{\lambda}{2} \cdot \int_{\mathbb{R}} \mu V(t) |u_n(t)|^2 dt - \lambda \int_{\mathbb{R}} F(t, u_n(t)) \, dt \\
&\quad - \left(a + b\int_{\mathbb{R}} \left|{}_{-\infty}D_t^\alpha u_n(t)\right|^2 dt\right)^{\theta-1} \cdot \int_{\mathbb{R}} \left|{}_{-\infty}D_t^\alpha u_n(t)\right|^2 dt - \int_{\mathbb{R}} \mu V(t) |u_n(t)|^2 dt \\
&\quad + \int_{\mathbb{R}} f(t, u_n(t)) u_n(t) \, dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a\lambda}{2b\theta}\left(a + b\int_{\mathbb{R}} \left|{}_{-\infty}D_t^{\alpha} u_n(t)\right|^2 dt\right)^{\theta-1} \\
&+ \frac{\lambda - 2\theta}{2\theta}\left(a + b\int_{\mathbb{R}} \left|{}_{-\infty}D_t^{\alpha} u_n(t)\right|^2 dt\right)^{\theta-1} \\
&\cdot \int_{\mathbb{R}} \left|{}_{-\infty}D_t^{\alpha} u_n(t)\right|^2 dt + \frac{\lambda - 2}{2} \\
&\cdot \int_{\mathbb{R}} \mu V(t) \left|u_n(t)\right|^2 dt \\
&+ \int_{\mathbb{R}} \left(f(t, u_n(t)) u_n(t) - \lambda F(t, u_n(t))\right) dt \\
&\geq \frac{\lambda - 2\theta}{2\theta} a^{\theta-1} \int_{\mathbb{R}} \left|{}_{-\infty}D_t^{\alpha} u_n(t)\right|^2 dt + \frac{\lambda - 2}{2} \\
&\cdot \int_{\mathbb{R}} \mu V(t) \left|u_n(t)\right|^2 dt \geq \frac{\lambda - 2\theta}{2\theta} \|u\|_{X_\mu^\alpha}^2.
\end{aligned}
\tag{38}
$$

Hence, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $X_\mu^\alpha$.

So, passing onto subsequence if necessary, thanks to Lemma 3, we have

$$
\begin{aligned}
&u_n \rightharpoonup u, \quad \text{weakly in } X_\mu^\alpha, \\
&u_n \longrightarrow u, \quad \text{strongly a.e. in } \mathbb{R}, \\
&u_n \longrightarrow u, \\
&\text{strongly a.e. in } L^s\left(\mathbb{R}^N\right),\ 2 \leq s < +\infty,
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}} \left|{}_{-\infty}D_t^{\alpha} u_n(t)\right|^2 dt \longrightarrow \rho_1 \geq 0, \\
&\int_{\mathbb{R}} \mu V(t) \left|u_n\right|^2 dt \longrightarrow \rho_2 \geq 0.
\end{aligned}
\tag{40}
$$

We will prove that

$$
\begin{aligned}
&\int_{\mathbb{R}} \left|{}_{-\infty}D_t^{\alpha} u(t)\right|^2 dt = \rho_1, \\
&\int_{\mathbb{R}} \mu V(t) |u|^2 dx = \rho_2.
\end{aligned}
\tag{41}
$$

Let $\varphi \in X_\mu^\alpha$ be fixed and denote by $B_\varphi$ the linear functional on $X_\mu^\alpha$ defined by

$$
B_\varphi(v) := \int_{\mathbb{R}} {}_{-\infty}D_t^{\alpha} \varphi(t) \cdot {}_{-\infty}D_t^{\alpha} v(t)\, dt, \tag{42}
$$

and set

$$
\Delta_\alpha(u) := \int_{\mathbb{R}} \left|{}_{-\infty}D_t^{\alpha} u(t)\right|^2 dt, \tag{43}
$$

for all $v \in X_\mu^\alpha$. In view of the Hölder inequality and definition of $B_\varphi$, we have

$$
\begin{aligned}
&\left\langle I'_{\mu,\theta}(u_n) - I'_{\mu,\theta}(u), u_n - u\right\rangle = \left(a + b\Delta_\alpha(u_n)\right)^{\theta-1} \\
&\cdot B_{u_n}(u_n - u) - \left(a + b\Delta_\alpha(u_n)\right)^{\theta-1} B_u(u_n - u) \\
&+ \int_{\mathbb{R}} \mu V(t)(u_n - u)(u_n - u)\, dt \\
&- \int_{\mathbb{R}} \left(f(t, u_n) - f(t, u)\right)(u_n - u)\, dt \geq (a \\
&+ b\Delta_\alpha(u_n))^{\theta-1} \Delta_\alpha(u_n) - \left(a + b\Delta_\alpha(u_n)\right)^{\theta-1} \\
&\cdot \left(\Delta_\alpha(u_n)\right)^{(\theta-1)/2} \left(\Delta_\alpha(u_n)\right)^{1/2} + \left(a + b\Delta_\alpha(u)\right)^{\theta-1} \\
&\cdot \Delta_\alpha(u) - \left(a + b\Delta_\alpha(u)\right)^{\theta-1} \left(\Delta_\alpha(u)\right)^{(\theta-1)/2} \\
&\cdot \left(\Delta_\alpha(u_n)\right)^{1/2} + \int_{\mathbb{R}} \mu V(t) \left|u_n\right|^2 dt \\
&- \left(\int_{\mathbb{R}} \mu V(t) \left|u_n\right|^2 dt\right)^{1/2} \left(\int_{\mathbb{R}} \mu V(t) |u|^2 dt\right)^{1/2} \\
&+ \int_{\mathbb{R}} \mu V(t) |u|^2 dt - \left(\int_{\mathbb{R}} \mu V(t) |u|^2 dt\right)^{1/2} \\
&\cdot \left(\int_{\mathbb{R}} \mu V(t) \left|u_n\right|^2 dt\right)^{1/2} \\
&- \int_{\mathbb{R}} \left(f(t, u_n) - f(t, u)\right)(u_n - u)\, dt = (a \\
&+ b\Delta_\alpha(u_n))^{\theta-1} \left(\Delta_\alpha(u_n)\right)^{(\theta-1)/2} \left[\left(\Delta_\alpha(u_n)\right)^{1/2}\right. \\
&\left. - \left(\Delta_\alpha(u)\right)^{1/2}\right] + \left(a + b\Delta_\alpha(u)\right)^{\theta-1} \left(\Delta_\alpha(u)\right)^{(\theta-1)/2} \\
&\cdot \left[\left(\Delta_\alpha(u)\right)^{1/2} - \left(\Delta_\alpha(u_n)\right)^{1/2}\right] \\
&+ \left(\int_{\mathbb{R}} \mu V(t) \left|u_n\right|^2 dt\right)^{1/2} \left[\left(\int_{\mathbb{R}} \mu V(t) \left|u_n\right|^2 dt\right)^{1/2}\right. \\
&\left. - \left(\int_{\mathbb{R}} \mu V(t) |u|^2 dt\right)^{1/2}\right] + \left(\int_{\mathbb{R}} \mu V(t) |u|^2 dt\right)^{1/2} \\
&\cdot \left[\left(\int_{\mathbb{R}} \mu V(t) |u|^2 dt\right)^{1/2}\right. \\
&\left. - \left(\int_{\mathbb{R}} \mu V(t) \left|u_n\right|^2 dt\right)^{1/2}\right] \\
&- \int_{\mathbb{R}} \left(f(t, u_n) - f(t, u)\right)(u_n - u)\, dt \\
&= \left[\left(\Delta_\alpha(u_n)\right)^{1/2} - \left(\Delta_\alpha(u)\right)^{1/2}\right] \\
&\cdot \left[\left(a + b\Delta_\alpha(u_n)\right)^{\theta-1} \left(\Delta_\alpha(u_n)\right)^{(\theta-1)/2}\right. \\
&\left. - \left(a + b\Delta_\alpha(u)\right)^{\theta-1} \left(\Delta_\alpha(u)\right)^{(\theta-1)/2}\right]
\end{aligned}
$$

$$+\left[\left(\int_{\mathbb{R}}\mu V(t)\left|u_n\right|^2 dt\right)^{1/2} - \left(\int_{\mathbb{R}}\mu V(t)|u|^2 dt\right)^{1/2}\right]^2 - \int_{\mathbb{R}}\left(f(t,u_n)-f(t,u)\right)(u_n-u)\,dt. \quad (44)$$

Since $u_n \rightharpoonup u$ in $X_\mu^\alpha$ and $I'_{\mu,\theta}(u_n) \to 0$ as $n \to \infty$ in $(X_\mu^\alpha)^*$, therefore $\langle I'_{\mu,\theta}(u_n) - I'_{\mu,\theta}(u), u_n - u\rangle \to 0$ as $n \to \infty$. Now, using (F1) and Hölder inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}}\left(f(t,u_n)-f(t,u)\right)(u_n-u)\,dt &\le \int_{\mathbb{R}}\left|c_0\left(|u_n|+|u|\right)+\sum_{j=1}^{l}c_j\left(|u_n|^{q_j-1}+|u|^{q_j-1}\right)\right| \\ &\cdot |u_n-u|\,dx \le c_0\left(\|u_n\|_{L^2(\mathbb{R})}+\|u\|_{L^2(\mathbb{R})}\right)\|u_n \\ &- u\|_{L^2(\mathbb{R})} + \sum_{j=1}^{l}c_j\left(\|u_n\|_{L^{q_j}(\mathbb{R})}^{q_j-1}+\|u_n\|_{L^{q_j}(\mathbb{R})}^{q_j-1}\right) \\ &\cdot \|u_n-u\|_{L^{q_j}(\mathbb{R})},\end{aligned} \quad (45)$$

which, in view of (39), yields

$$\lim_{n\to\infty}\int_{\mathbb{R}}\left(f(t,u_n)-f(t,u)\right)(u_n-u)\,dt = 0. \quad (46)$$

Since $u_n \to u$ a.e. in $\mathbb{R}$, it follows by Fatou's lemma that

$$\begin{aligned}\Delta_\alpha(u) &\le \liminf_{n\to\infty}\Delta_\alpha(u_n) = \rho_1, \\ \int_{\mathbb{R}}\mu V(t)|u|^2 dt &\le \liminf_{n\to\infty}\int_{\mathbb{R}}\mu V(t)\left|u_n\right|^2 dt = \rho_2.\end{aligned} \quad (47)$$

Noting that $\Pi(s) = (a+bs)^{\theta-1}s^{(\theta-1)/2}$ is a nondecreasing function for $s \ge 0$, we get

$$\begin{aligned}&\left[(\rho_1)^{1/2}-(\Delta_\alpha(u))^{1/2}\right]\left[(a+b\rho_1)^{\theta-1}(\rho_1)^{(\theta-1)/2} - (a+b\Delta_\alpha(u))^{\theta-1}(\Delta_\alpha(u))^{(\theta-1)/2}\right], \\ &\left[(\rho_2)^{1/2}-\left(\int_{\mathbb{R}}\mu V(t)|u|^2 dt\right)^{1/2}\right]^2 \ge 0.\end{aligned} \quad (48)$$

Now, in view of $\langle I'_{\mu,\theta}(u_n) - I'_{\mu,\theta}(u), u_n - u\rangle \to 0$ as $n \to \infty$, (46), and (47), one has

$$\begin{aligned}0 &\ge \liminf_{n\to\infty}\left\{\left[(\Delta_\alpha(u_n))^{1/2}-(\Delta_\alpha(u))^{1/2}\right]\cdot\left[(a+b\Delta_\alpha(u_n))^{\theta-1}(\Delta_\alpha(u_n))^{(\theta-1)/2} - (a+b\Delta_\alpha(u))^{\theta-1}(\Delta_\alpha(u))^{(\theta-1)/2}\right]\right. \\ &+ \left[\left(\int_{\mathbb{R}}\mu V(t)\left|u_n\right|^2 dt\right)^{1/2} - \left(\int_{\mathbb{R}}\mu V(t)|u|^2 dt\right)^{1/2}\right]^2 \\ &\left. - \int_{\mathbb{R}}\left(f(t,u_n)-f(t,u)\right)(u_n-u)\,dt\right\} \\ &\ge \lim_{n\to\infty}\left\{\left[(\Delta_\alpha(u_n))^{1/2}-(\Delta_\alpha(u))^{1/2}\right]\cdot\left[(a+b\Delta_\alpha(u_n))^{\theta-1}(\Delta_\alpha(u_n))^{(\theta-1)/2} - (a+b\Delta_\alpha(u))^{\theta-1}(\Delta_\alpha(u))^{(\theta-1)/2}\right]\right\} \\ &+ \lim_{n\to\infty}\left[\left(\int_{\mathbb{R}}\mu V(t)\left|u_n\right|^2 dt\right)^{1/2} - \left(\int_{\mathbb{R}}\mu V(t)|u|^2 dt\right)^{1/2}\right]^2 \\ &- \lim_{n\to\infty}\left\{\int_{\mathbb{R}}\left(f(t,u_n)-f(t,u)\right)(u_n-u)\,dt\right\} \\ &\ge \left[(\rho_1)^{1/2}-(\Delta_\alpha(u))^{1/2}\right]\left[(a+b\rho_1)^{\theta-1}(\rho_1)^{(\theta-1)/2} - (a+b\Delta_\alpha(u))^{\theta-1}(\Delta_\alpha(u))^{(\theta-1)/2}\right] + \left[(\rho_2)^{1/2} - \left(\int_{\mathbb{R}}\mu V(t)|u|^2 dt\right)^{1/2}\right]^2.\end{aligned} \quad (49)$$

Then, from (48)-(49), we get

$$\begin{aligned}\Delta_\alpha(u) &= \int_{\mathbb{R}}\left|{}_{-\infty}D_t^\alpha u(t)\right|^2 dt = \rho_1, \\ \int_{\mathbb{R}}\mu V(t)|u|^2 dt &= \rho_2.\end{aligned} \quad (50)$$

Hence, we obtain $\|u_n\|_{X_\mu^\alpha} \to \|u\|_{X_\mu^\alpha}$. As $X_\mu^\alpha$ is a reflexive Banach space (see Lemma 2), it is isomorphic to a locally uniformly convex space. So the weak convergence and norm convergence imply strong convergence. This completes the proof. □

Let $\{e_j\}$ be a total orthonormal basis of $L^2(\mathbb{R})$ and define $X_j = \mathbb{R}e_j$, $j \in \mathbb{N}$,

$$Y_k = \oplus_{j=1}^{k} X_j, \quad Z_k = \oplus_{j=k+1}^{\infty} X_j, \quad k \in \mathbb{N}. \tag{51}$$

**Lemma 12.** *Assume that (V1) holds. Then, for $2 < p < +\infty$,*

$$\beta_k := \sup_{u\in Z_k, \|u\|_{X_\mu^\alpha}=1} \|u\|_{L^p(\mathbb{R})} \longrightarrow 0, \quad k \longrightarrow \infty. \tag{52}$$

*Proof.* The proof is similar to that of Lemma 3.8 in [28]. So it is omitted. □

In view of Lemma 12, we can choose an integer $k \geq 1$ such that

$$\int_{\mathbb{R}} |u|^2 dt \leq \frac{1}{2c_0} \left( \int_{\mathbb{R}} a\left(\left|{}_{-\infty}D_t^\alpha u(t)\right|^2\right) dt + \int_{\mathbb{R}} \mu V(t) |u|^2 dt \right) \quad \forall u \in Z_m \cap X_\mu^\alpha, \tag{53}$$

where $c_1$ is a constant given in condition (F1). Let

$$\mathfrak{R}(t) = \begin{cases} 1, & |t| > r, \\ 0, & |t| \leq r, \end{cases} \tag{54}$$

and set $Y = \{(1-\mathfrak{R})u : u \in X_\mu^\alpha,\ (1-\mathfrak{R})u \in Y_k\}$ and $Z = \{(1-\mathfrak{R})u : u \in X_\mu^\alpha,\ (1-\mathfrak{R})u \in Z_k\} + \{\mathfrak{R}v : v \in X_\mu^\alpha\}$. Hence $Y$ and $Z$ are subspaces of $X_\mu^\alpha$, and $X_\mu^\alpha = Y \oplus Z$.

**Lemma 13.** *Suppose that (V1), (V2), and (F1) are satisfied. Then there exist constants $\varrho, \beta > 0$ such that $I_{\mu,\theta}|_{\partial B_\varrho \cap Z} \geq \alpha$.*

*Proof.* In view of (V2), (53), and definition of the space $Z$, we have

$$\begin{aligned} \|u\|_{L^2(\mathbb{R})}^2 &= \int_{|t|<r} |u(t)|^2 dt + \int_{|t|\geq r} |u(t)|^2 dt \\ &\leq \frac{1}{2c_0} \|u\|_{X_\mu^\alpha}^2 + \frac{1}{\mu\omega} \int_{\{t\in\mathbb{R}, V(t)>\omega\}} \mu V(t) |u(t)|^2 dt \\ &\leq \frac{1}{2c_0} \|u\|_{X_\mu^\alpha}^2 + \frac{1}{\mu\omega} \|u\|_{X_\mu^\alpha}^2 \quad \forall u \in Z. \end{aligned} \tag{55}$$

Therefore, from (23), (55), and (F1) and for large enough value of $\mu$, we get

$$\begin{aligned} I_{\mu,\theta}(u) &= \frac{1}{2b\theta} \left(a + b\Delta_\alpha(u)\right)^\theta + \frac{1}{2} \int_{\mathbb{R}} \mu V(t) |u|^2 dt - \int_{\mathbb{R}} F(t,u) dt \\ &\geq \frac{a^{\theta-1}}{2} \Delta_\alpha(u) + \frac{1}{2} \int_{\mathbb{R}} \mu V(t) |u|^2 dt - \int_{\mathbb{R}} F(t,u) dt \\ &\geq \frac{1}{2} \|u\|_{X_\mu^\alpha}^2 - \frac{c_0}{2} \|u\|_{L^2(\mathbb{R})}^2 - \sum_{j=1}^{l} \frac{c_j}{q_j} \|u\|_{L^{q_j}(\mathbb{R})}^{q_j} \\ &\geq \frac{1}{4} \|u\|_{X_\mu^\alpha}^2 - \frac{c_0}{\mu\omega 2} \|u\|_{X_\mu^\alpha}^2 - \sum_{j=1}^{l} \frac{c_j D_{q_j}^{q_j}}{q_j} \|u\|_{X_\mu^\alpha}^{q_j} \\ &\geq \frac{1}{8} \|u\|_{X_\mu^\alpha}^2 - \sum_{j=1}^{l} \frac{c_j D_{q_j}^{q_j}}{q_j} \|u\|_{X_\mu^\alpha}^{q_j}. \end{aligned} \tag{56}$$

Since $2 < q_j$ $(j = 1, \ldots, l)$, there exist constants $\varrho, \beta > 0$ such that $I_{\mu,\theta}|_{\partial B_\varrho \cap Z} \geq \beta$. □

**Lemma 14.** *Assume that (F1) and (F4) are satisfied. Then, for any finite dimensional subspace $\widetilde{X}_\mu^\alpha \subset X_\mu^\alpha$, there is $R = R(\widetilde{X}_\mu^\alpha) > 0$ such that $I_{\mu,\theta}(u) \leq 0$ on $\widetilde{X}_\mu^\alpha \setminus B_R$.*

*Proof.* Since all the norms in the finite dimensional space are equivalent, there exists a constant $\Upsilon$ such that

$$\|u\|_{L^{2\theta}(\mathbb{R})} \geq \Upsilon \|u\|_{X_\mu^\alpha}, \quad \forall u \in \widetilde{X}_\mu^\alpha. \tag{57}$$

From (F1) and (F4), for any $L > b^{\theta-1}/2\theta\Upsilon^{2\theta} a^{\theta(\theta-1)}$, there exists a constant $C_L > 0$ such that

$$F(t,u) \geq L|u|^{2\theta} - C_L |u|^2, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}. \tag{58}$$

Thus

$$\begin{aligned} I_{\mu,\theta}(u) &= \frac{1}{2b\theta} \left(a + b\Delta_\alpha(u)\right)^\theta + \frac{1}{2} \int_{\mathbb{R}} \mu V(t) |u|^2 dt - \int_{\mathbb{R}} F(t,u) dt \\ &\leq \frac{1}{2b\theta} \left(a + \frac{b}{a^{\theta-1}} \|u\|_{X_\mu^\alpha}^2\right)^\theta + \frac{1}{2} \|u\|_{X_\mu^\alpha}^2 + C_L \|u\|_{L^2(\mathbb{R})}^2 - L \|u\|_{L^{2\theta}(\mathbb{R})}^{2\theta} \\ &\leq \frac{1}{2b\theta} \left(a + \frac{b}{a^{\theta-1}} \|u\|_{X_\mu^\alpha}^2\right)^\theta + \left(\frac{1}{2} + C_L D_2^2\right) \|u\|_{X_\mu^\alpha}^2 - L\Upsilon^{2\theta} \|u\|_{X_\mu^\alpha}^{2\theta} \end{aligned} \tag{59}$$

for all $u \in \widetilde{X}_\mu^\alpha$. Consequently, there is a large $R > 0$ such that $I_{\mu,\theta}(u) \leq 0$ on $\widetilde{X}_\mu^\alpha \setminus B_R$. Therefore, the proof is completed. □

## 4. Existence of Weak Solutions

In this section, we present our main results.

**Theorem 15.** *Assume that (V1), (V2), (F1), (F3), (F4), and (F5) hold. Then problem (1) has infinitely many nontrivial weak solutions whenever $\mu > 0$ is sufficiently large.*

*Proof.* We know that $I_{\mu,\theta}(0) = 0$, and it is even by (F5). Let $X = X_\mu^\alpha$ and $Y$ and $Z$ be as defined in Section 2. By Lemmas 11, 13, and 14, it follows that $I_{\mu,\theta}$ satisfies all the condition of the Theorem 10. Therefore, problem (1) has infinitely many nontrivial weak solutions whenever $\mu > 0$ is sufficiently large. □

**Theorem 16.** *Assume that (V1), (V2), (F1), (F2), (F3), and (F4) hold. Then problem (1) has at least one nontrivial weak solution when $\mu > 0$.*

*Proof.* We complete the proof in three steps.

*Step 1.* Clearly $I_{\mu,\theta}(0) = 0$ and $I_{\mu,\theta} \in C^1(X_\mu^\alpha, \mathbb{R})$ satisfies the (PS) condition by Lemma 11.

*Step 2.* It will be shown that there exist constants $\varrho, \beta > 0$ such that $I_{\mu,\theta}$ satisfies condition (i) of Theorem 9. For any $\varepsilon > 0$, by (F1) and (F2), there exists a constant $c_\varepsilon > 0$ such that

$$|F(t,u)| \le \frac{\epsilon}{2}|u|^2 + \sum_{j=1}^{l}\frac{c_j^\varepsilon}{q_j}|u|^{q_j}. \tag{60}$$

Thus, by (23) and (60), for small $\rho > 0$, we get

$$\begin{aligned} I_{\mu,\theta}(u) &= \frac{1}{2b\theta}\left(a + b\Delta_\alpha(u)\right)^\theta + \frac{1}{2}\int_{\mathbb{R}}\mu V(t)|u|^2\,dt \\ &\quad - \int_{\mathbb{R}} F(t,u)\,dt \\ &\ge \frac{a^{\theta-1}}{2}\Delta_\alpha(u) + \frac{1}{2}\int_{\mathbb{R}}\mu V(t)|u|^2\,dt \\ &\quad - \int_{\mathbb{R}} F(t,u)\,dt \\ &\ge \frac{1}{2}\left(\|u\|_{X_\mu^\alpha}^2 - \varepsilon D_2^2\|u\|_{X_\mu^\alpha}^2\right) - \sum_{j=1}^{l}\frac{c_j^\varepsilon}{q_j}D_{q_j}^{q_j}\|u\|_{X_\mu^\alpha}^{q_j} \\ &\ge \frac{1}{8}\left(1 - \varepsilon D_2^2\right)\varrho^2, \end{aligned} \tag{61}$$

for all $u \in \overline{B}_\varrho$, where $B_\varrho = \{u \in X_\mu^\alpha : \|u\|_{X_\mu^\alpha} < \varrho\}$. So it suffices to choose $\varepsilon = 1/2D_2^2$ so that

$$I_{\mu,\theta}\big|_{\partial B_\varrho} \ge \frac{1}{16}\varrho^2 := \beta > 0. \tag{62}$$

*Step 3.* It remains to prove that there exists an $e \in X_\mu^\alpha$ such that $\|u\|_{X_\mu^\alpha} > \varrho$ and $I_{\mu,\theta}(e) \le 0$, where $\rho$ is defined in Step 2. Let us consider

$$\begin{aligned} I_{\mu,\theta}(\sigma u) &= \frac{1}{2b\theta}\left(a + b\sigma^2\Delta_\alpha(u)\right)^\theta \\ &\quad + \frac{\sigma^2}{2}\int_{\mathbb{R}}\mu V(t)|u|^2\,dt - \int_{\mathbb{R}} F(t,\sigma u)\,dt, \end{aligned} \tag{63}$$

for all $\sigma \in \mathbb{R}$. Take $0 \ne u \in X_\mu^\alpha$. By (F1) and (F4), for any $\kappa > b^{\theta-1}(\Delta_\alpha(u))^\theta/2\theta\int_{\mathbb{R}}|u|^{2\theta}dt$, there is a constant $C_\kappa > 0$ such that

$$F(t,u) \ge \kappa|u|^{2\theta} - C_\kappa|u|^2. \tag{64}$$

So we have

$$\begin{aligned} I_{\mu,\theta}(\sigma u) &\le \frac{1}{2b\theta}\left(a + b\sigma^2\Delta_\alpha(u)\right)^\theta \\ &\quad + \frac{\sigma^2}{2}\int_{\mathbb{R}}\mu V(t)|u|^2\,dt + C_\kappa\sigma^2\int_{\mathbb{R}}|u|^2\,dt \\ &\quad - \kappa\sigma^{2\theta}\int_{\mathbb{R}}|u|^{2\theta}\,dt \longrightarrow -\infty, \end{aligned} \tag{65}$$

as $\sigma \to +\infty$. Thus, there is a point $e \in X_\mu^\alpha \setminus \overline{B}\varrho$ such that $I_{\mu,\theta}(e) \le 0$. By Theorem 9, $I_{\mu,\theta}$ possesses a critical value $c \ge \alpha > 0$ given by

$$c = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} I_{\mu,\theta}(\gamma(s)), \tag{66}$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = e\}. \tag{67}$$

Hence there is $u \in X_\mu^\alpha$ such that $I_{\mu,\theta}(u) = c$ and $I'_{\mu,\theta}(u) = 0$; that is, problem (1) has a nontrivial weak solution in $X_\mu^\alpha$. □

**Theorem 17.** *Assume that (V1), (V3), (F5), (F6), and (F7) hold. Then problem (1) has infinitely many nontrivial weak solutions for $\mu > 0$.*

*Proof.* One can obtain the proof by employing the method of proof for Theorem 15 and using Lemma 7. □

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.


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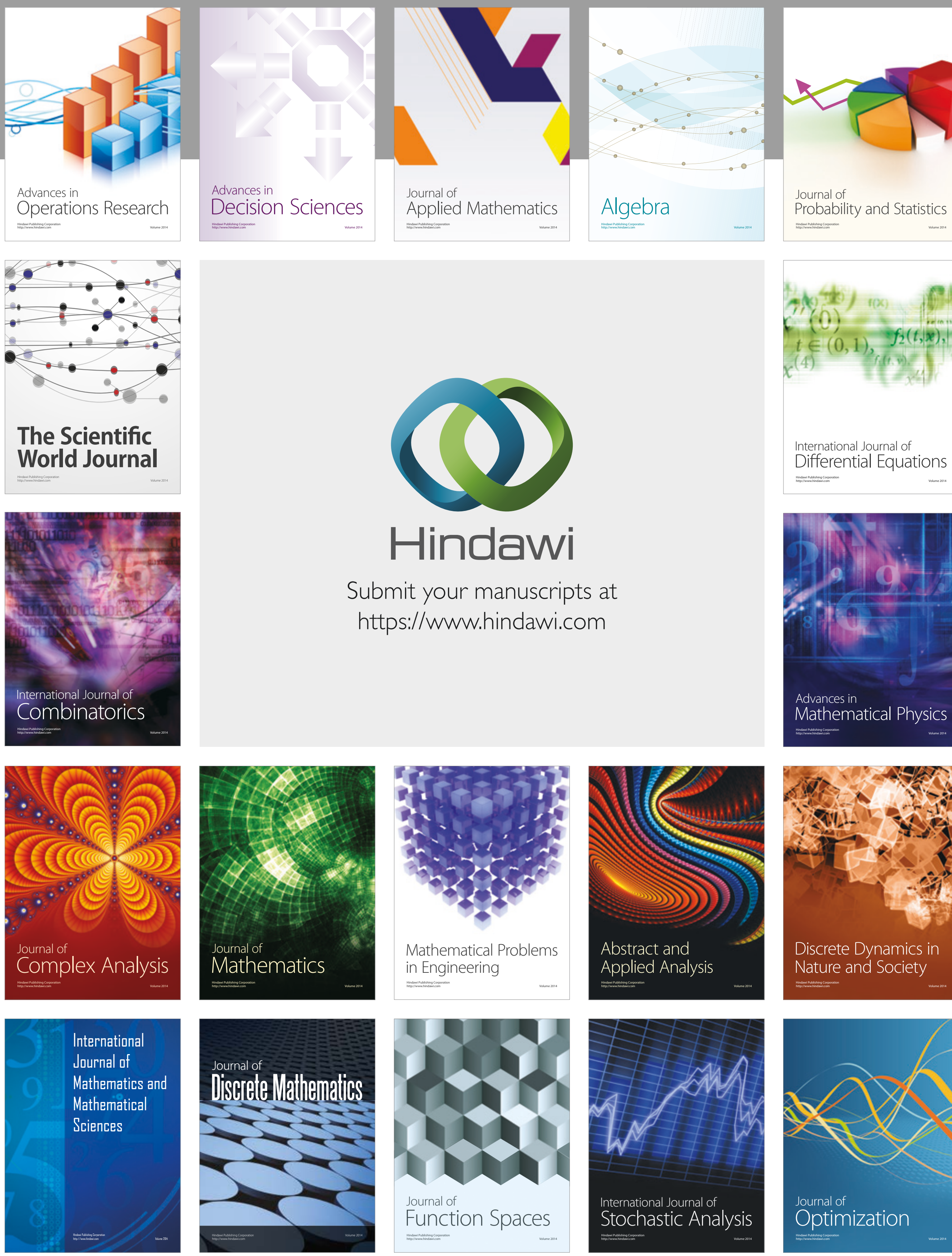

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