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Research Article

A Note about the General Meromorphic Solutions of the Fisher Equation

Jian-ming Qi,[1] Qiu-hui Chen,[2] Wei-ling Xiong,[3] and Wen-jun Yuan[4,5]

[1] *Department of Mathematics and Physics, Shanghai Dianji University, Shanghai 201306, China*
[2] *Cisco School of Informatics, Guangdong University of Foreign Studies, Guangzhou 510420, China*
[3] *Department of Information and Computing Science, Guangxi University of Technology, Liuzhou 545006, China*
[4] *School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China*
[5] *Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong Higher Education Institutes, Guangzhou University, Guangzhou 510006, China*

Correspondence should be addressed to Wen-jun Yuan; wjyuan1957@126.com

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We employ the complex method to obtain the general meromorphic solutions of the Fisher equation, which improves the corresponding results obtained by Ablowitz and Zeppetella and other authors (Ablowitz and Zeppetella, 1979; Feng and Li, 2006; Guo and Chen, 1991), and $w_{g,i}(z)$ are new general meromorphic solutions of the Fisher equation for $c = \pm 5i/\sqrt{6}$. Our results show that the complex method provides a powerful mathematical tool for solving great many nonlinear partial differential equations in mathematical physics.

Dedicated to Professor Hongxun Yi 70th birthday

1. Introduction

Consider the Fisher equation

$$u_t = vu_{xx} + su(1-u), \tag{1}$$

which is a nonlinear diffusion equation as a model for the propagation of a mutant gene with an advantageous selection intensity s. It was suggested by Fisher as a deterministic version of a stochastic model for the spatial spread of a favored gene in a population in 1936.

Set $t' = st$ and $x' = (s/v)^{1/2}x$ and drop the primes; (1) becomes

$$u_t = u_{xx} + u(1-u). \tag{2}$$

Substituting the traveling wave transform $u(x,t) = w(z)$, $x - ct = z$ into (2) gives a nonlinear ordinary differential equation

$$w'' + cw' + w(1-w) = 0, \tag{3}$$

where c is a constant.

Finding solutions of nonlinear models is a difficult and challenging task. Recently, the complex method was introduced by Yuan et al. [1–3].

In this paper, we employ the complex method to obtain the general solutions of (3). The general traveling wave exact solutions of the Fisher equation can be deduced by the traveling wave transform $u(x,t) = w(z)$, $x - ct = z$. In order to state our results, we need some concepts and notations.

A meromorphic function $w(z)$ means that $w(z)$ is a holomorphic excepting for pole in the complex plane $\mathbb{C}$ except for poles. $\wp(z; g_2, g_3)$ is the Weierstrass elliptic function with invariants g_2 and g_3. We say that a meromorphic function f belongs to the class W if f is an elliptic function, or a rational function of $e^{\alpha z}$, $\alpha \in \mathbb{C}$, or a rational function of z.

Our main result is the following theorem.

Theorem 1. *Equation* (3) *is integrable if and only if* $c = 0$, $\pm 5/\sqrt{6}$, $\pm 5i/\sqrt{6}$. *Furthermore, the general solutions of* (3) *are of the forms below.*

(I) When $c = 0$, the elliptic general solutions of (3) *(see [1])*

$$w_d(z) = 6\left\{-\wp(z) + \frac{1}{4}\left[\frac{\wp'(z) + F}{\wp(z) - E}\right]^2\right\} - 6E + \frac{1}{2}, \quad (4)$$

where $g_2 = 1/12$, $F^2 = 4E^3 - g_2E - g_3$, g_3 and E are arbitrary, in particular, which degenerates the simply periodic solutions

$$w_s(z) = 6\alpha^2 \coth^2 \frac{\alpha}{2}(z - z_0) + \frac{1}{2}\alpha^2 + \frac{1}{2}, \quad (5)$$

where $\alpha^4 = 1$, $z_0 \in \mathbb{C}$.

(II) When $c = \pm 5/\sqrt{6}$, the general solutions of (3) *(see [4])*

$$w_g(z) = \exp\left\{\mp\frac{2z}{\sqrt{6}}\right\} \wp\left(\exp\left\{\mp\frac{z}{\sqrt{6}}\right\} - s_0; 0, g_3\right), \quad (6)$$

where both s_0 and g_3 are arbitrary constants. In particular, the degenerate one-parameter family of solutions is given by

$$w_f(z) = \frac{1}{\left\{1 - \exp\left\{\pm(z - z_0)/\sqrt{6}\right\}\right\}^2}, \quad (7)$$

where $z_0 \in \mathbb{C}$.

(III) When $c = \pm 5i/\sqrt{6}$, the general solutions of (3)

$$w_{g,i}(z) = \exp\left\{\mp\frac{2iz}{\sqrt{6}}\right\} \wp\left(i\exp\left\{\mp\frac{iz}{\sqrt{6}}\right\} - s_0; 0, g_3\right) + 1, \quad (8)$$

where both s_0 and g_3 are arbitrary constants. In particular, the degenerate one-parameter family of solutions is given by

$$w_{f,i}(z) = -\frac{1}{\left\{1 - \exp\left\{\pm i(z - z_0)/\sqrt{6}\right\}\right\}^2} + 1, \quad (9)$$

where $z_0 \in \mathbb{C}$.

Remark 2. The Fisher equation is classic and simplest case of the nonlinear reaction-diffusion equation, but there are many applications about it and many authors researched it [5]. The first explicit form of a traveling wave solution for the Fisher equation was obtained by Ablowitz and Zeppetella [4] using the Painlevé analysis. Many authors obtained only $w_{f,i}(z)$ by using other methods [5–7]; $w_{g,i}(z)$ are new general meromorphic solutions of the Fisher equation for $c = \pm 5i/\sqrt{6}$.

This paper is organized as follows. In the next section, the preliminary lemmas and the complex method are given. The proof of Theorem 1 is given and the general meromorphic solutions of (3) are derived by complex method in Section 3. Some conclusions and discussions are given in the final section.

2. Preliminary Lemmas and the Complex Method

In order to give complex method and the proof of Theorem 1, we need some notations and results.

Set $m \in \mathbb{N} := \{1, 2, 3, \ldots\}$, $r_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $r = (r_0, r_1, \ldots, r_m)$, and $j = 0, 1, \ldots, m$. We define a differential monomial denoted by

$$M_r[w](z) := [w(z)]^{r_0}\left[w'(z)\right]^{r_1}\left[w''(z)\right]^{r_2} \cdots \left[w^{(m)}(z)\right]^{r_m}. \quad (10)$$

$p(r) := r_0 + r_1 + \cdots + r_m$ is called the degree of $M_r[w]$. A differential polynomial $P(w, w', \ldots, w^{(m)})$ is defined as follows

$$P\left(w, w', \ldots, w^{(m)}\right) := \sum_{r \in I} a_r M_r[w], \quad (11)$$

where a_r are constants, and I is a finite index set. The total degree of $P(w, w', \ldots, w^{(m)})$ is defined by $\deg P(w, w', \ldots, w^{(m)}) := \max_{r \in I}\{p(r)\}$.

We will consider the following complex ordinary differential equations

$$P\left(w, w', \ldots, w^{(m)}\right) = bw^n + c, \quad (12)$$

where $b \neq 0$, c are constants, $n \in \mathbb{N}$.

Let $p, q \in \mathbb{N}$. Suppose that (12) has a meromorphic solution w with at least one pole, we say that (12) satisfies weak $\langle p, q\rangle$ condition if substituting Laurent series

$$w(z) = \sum_{k=-q}^{\infty} c_k z^k, \quad q > 0, \ c_{-q} \neq 0 \quad (13)$$

into (12) we can determine p distinct Laurent singular parts below

$$\sum_{k=-q}^{-1} c_k z^k. \quad (14)$$

In order to give the representations of elliptic solutions, we need some notations and results concerning elliptic function [8].

Let ω_1, ω_2 be two given complex numbers such that $\operatorname{Im}(\omega_1/\omega_2) > 0$, $L = L[2\omega_1, 2\omega_2]$ be discrete subset $L[2\omega_1, 2\omega_2] = \{\omega \mid \omega = 2n\omega_1 + 2m\omega_2,\ n, m \in \mathbb{Z}\}$, which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. The discriminant $\Delta = \Delta(c_1, c_2) := c_1^3 - 27c_2^2$ and

$$s_n = s_n(L) := \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^n}. \quad (15)$$

Weierstrass elliptic function $\wp(z) := \wp(z, g_2, g_3)$ is a meromorphic function with double periods $2\omega_1$, $2\omega_2$ and satisfying the equation

$$\left(\wp'(z)\right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \quad (16)$$

where $g_2 = 60s_4$, $g_3 = 140s_6$, and $\Delta(g_2, g_3) \neq 0$.

By changing (16) to the form

$$\left(\wp'(z)\right)^2 = 4\left(\wp(z) - e_1\right)\left(\wp(z) - e_2\right)\left(\wp(z) - e_2\right), \quad (17)$$

we have $e_1 = \wp(\omega_1)$, $e_2 = \wp(\omega_2)$, $e_3 = \wp(\omega_1 + \omega_2)$.

Inversely, given two complex numbers g_2 and g_3 such that $\Delta(g_2, g_3) \neq 0$, then there exists double periods $2\omega_1, 2\omega_2$ Weierstrass elliptic function $\wp(z)$ such that above results hold.

Lemma 3 (see [8, 9]). *Weierstrass elliptic functions* $\wp(z) := \wp(z, g_2, g_3)$ *have two successive degeneracies and addition formula:*

(I) degeneracy to simply periodic functions (i.e., rational functions of one exponential e^{kz}*) according to*

$$\wp\left(z, 3d^2, -d^3\right) = 2d - \frac{3d}{2}\coth^2\sqrt{\frac{3d}{2}}z, \tag{18}$$

if one root e_j *is double* ($\Delta(g_2, g_3) = 0$),

(II) degeneracy to rational functions of z according to

$$\wp(z, 0, 0) = \frac{1}{z^2}, \tag{19}$$

if one root e_j *is triple* ($g_2 = g_3 = 0$),

(III) addition formula

$$\wp(z - z_0) = -\wp(z) - \wp(z_0) + \frac{1}{4}\left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}\right]^2. \tag{20}$$

By above lemma and results, we can give a new method below, say *complex method*, to find exact solutions of some PDEs.

Step 1. Substituting the transform $T : u(x,t) \to w(z)$, $(x,t) \to z$ into a given PDE gives a nonlinear ordinary differential equations (12).

Step 2. Substitute (13) into (12) to determine that weak $\langle p, q\rangle$ condition holds, and pass the Painlevé test for (12).

Step 3. Find the meromorphic solutions $w(z)$ of (12) with pole at $z = 0$, which have $m - 1$ integral constants.

Step 4. By Lemma 3, we obtain the general meromorphic solutions $w(z - z_0)$.

Step 5. Substituting the inverse transform T^{-1} into these meromorphic solutions $w(z - z_0)$, then we get all exact solutions $u(x,t)$ of the original given PDE.

3. Proof of Theorem 1

Substituting (13) into (3), we have $q = 2$, $p = 1$, $c_{-2} = 6$, $c_{-1} = -6c/5$, $c_0 = -c^2/50 + 1/2$, $c_1 = -c^3/250$, $c_2 = 1/40 - 189c^4/135000$, $c_3 = 891c/48600 - 6399c^5/6075000$, and

$$0 \times c_4 + \frac{5}{2}\left(\frac{c}{5}\right)^2 - 90\left(\frac{c}{5}\right)^6 = 0. \tag{21}$$

For the Laurent expansion (13) to be valid c satisfies this equation and c_4 is an arbitrary constant. Therefore, $c = 0$ or $c = \pm 5/\sqrt{6}$ or $c = \pm 5i/\sqrt{6}$, where $i^2 = -1$. For other c it would be necessary to add logarithmic terms to the expansion, thus giving a branch point rather than a pole. For $c = 0$, the solution of (3) has been given in Theorem 1 and can be also found by direct integration.

For $c = \pm 5/\sqrt{6}$, (3) is integrable by Ablowitz and Zeppetella [4] using the Painlevé analysis and the general solutions were given. That is, when $c = \pm 5/\sqrt{6}$, the general solutions of (3)

$$w_g(z) = \exp\left\{\mp\frac{2z}{\sqrt{6}}\right\}\wp\left(\exp\left\{\mp\frac{z}{\sqrt{6}}\right\} - s_0; 0, g_3\right), \tag{22}$$

where both s_0 and g_3 are arbitrary constants, in particular, which degenerates the one-parameter family of solutions

$$w_f(z) = \frac{1}{\left\{1 - \exp\left\{\pm(z - z_0)/\sqrt{6}\right\}\right\}^2}, \tag{23}$$

where $z_0 \in \mathbb{C}$.

Thus we consider only for cases $c = \pm 5i/\sqrt{6}$, where $i^2 = -1$. By the same arguments of Ablowitz and Zeppetella, we transform (3) with $c = \pm 5i/\sqrt{6}$ into the first Painlevé type equation. In this way we find the general solutions.

Setting $w(z) = f(z)u(s) + 1$, $s = g(z)$, and substituting in Fisher's equation (3), we obtain that the equation for $u(s)$ is

$$u'' = 6u^2, \tag{24}$$

where

$$f(z) = \exp\left\{\mp\frac{2iz}{\sqrt{6}}\right\}, \qquad g(z) = i\exp\left\{\mp\frac{iz}{\sqrt{6}}\right\}. \tag{25}$$

The general solutions of (24) are the Weierstrass elliptic functions $u(s) = \wp(s - s_0; 0, g_3)$, where s_0 and g_3 are two arbitrary constants.

Therefore, when $c = \pm 5i/\sqrt{6}$, the general solutions of (3)

$$w_{g,i}(z) = \exp\left\{\mp\frac{2iz}{\sqrt{6}}\right\}\wp\left(i\exp\left\{\mp\frac{iz}{\sqrt{6}}\right\} - s_0; 0, g_3\right) + 1, \tag{26}$$

where both s_0 and g_3 are arbitrary constants. In particular, by Lemma 3 and $g_3 = 0$, $w_{g,i}(z)$ degenerate the one-parameter family of solutions

$$w_{f,i}(z) = -\frac{1}{\left\{1 - \exp\left\{\pm i(z - z_0)/\sqrt{6}\right\}\right\}^2} + 1, \tag{27}$$

where $z_0 \in \mathbb{C}$.

4. Conclusions

Complex method is a very important tool in finding the exact solutions of nonlinear evolution equations, and the Fisher equation is classic and simplest case of the nonlinear reaction-diffusion equation. In this paper, we employ the complex method to obtain the general meromorphic solutions of the Fisher equation, which improves the corresponding result obtained by Ablowitz and Zeppetella and other authors [4–6], and $w_{g,i}(z)$ are new general meromorphic solutions of the Fisher equation for $c = \pm 5i/\sqrt{6}$. Our results show that the complex method provides a powerful mathematical tool for solving great many nonlinear partial differential equations in mathematical physics.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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