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Research Article

Asymptotic Behaviour and Extinction of Delay Lotka-Volterra Model with Jump-Diffusion

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This paper studies the effect of jump-diffusion random environmental perturbations on the asymptotic behaviour and extinction of Lotka-Volterra population dynamics with delays. The contributions of this paper lie in the following: (a) to consider delay stochastic differential equation with jumps, we introduce a proper initial data space, in which the initial data may be discontinuous function with downward jumps; (b) we show that the delay stochastic differential equation with jumps associated with our model has a unique global positive solution and give sufficient conditions that ensure stochastically ultimate boundedness, moment average boundedness in time, and asymptotic polynomial growth of our model; (c) the sufficient conditions for the extinction of the system are obtained, which generalized the former results and showed that the sufficiently large random jump magnitudes and intensity (average rate of jump events arrival) may lead to extinction of the population.

1. Introduction

Populations of biological species in some regions are often subject to sudden environmental shocks, for example, earthquakes, floods, tsunami, hurricanes, and so forth. As it is well known, occurrence of these disasters has the properties of random unpredictability and great destruction. To illustrate our mathematical results in terms of their ecological implications, we consider the protection of wildlife rare species, in the southwest region Sichuan in China, best-preserved panda habitat on earth, which belongs to Longmen Shan active fault zone. Both the 2008 M8.0 Wenchuan and the 2013 M7.0 Ya'an earthquakes occurred in this region. Earthquakes and secondary disasters caused by the earthquake such as mudslides, landslides, barrier lake, and other geological disasters may destroy natural habitat of wildlife and even may lead to extinction of endangered wildlife. Thus, it is very interesting to reveal how these sudden environmental shocks have an effect on the populations through stochastic analysis of the underlying dynamic systems.

The classical deterministic Lotka-Volterra model with delays is generally described by the integrodifferential equation

$$\frac{dx_i(t)}{dt} = x_i(t)\left[b_i + \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}x_j\left(t-\tau_{ij}\right) + \sum_{j=1}^{n} c_{ij}\int_{-\infty}^{0} x_j(t+\theta)\,d\mu_{ij}(\theta)\right], \quad i = 1, 2, \dots, n, \tag{1}$$

which is used to describe the population dynamics of n-species with interactions, where $x_i(t)$ represents the population size of the ith species; b_i, a_{ij}, b_{ij}, and c_{ij} $(i, j = 1, 2, \dots, n)$ are constant parameters; b_i is the inherent net birth rate of the ith species; a_{ij}, b_{ij}, and c_{ij} represent the interaction rates; $\tau_{ij} \geq 0$ and $\mu_{ij}(\cdot)$ is a probability measure on $(-\infty, 0]$ that may be any function defined on $(-\infty, 0]$ of bounded variations. There is an extensive literature concerned with the dynamics of model (1) or systems similar to (1), regarding attractivity, persistence, global stabilities of equilibrium, and other dynamics, and we here only mention Gopalsamy [1], Kuang and Smith [2], Bereketoglu and Győri [3], He [4], Teng and Chen [5], Faria [6], and Chen [7] among many others. In particular, the books by Gopalsamy [8] and Kuang [9] are good references in this area.

The model mentioned above is a deterministic model, which assumes that parameters in the model are all deterministic irrespective of environmental fluctuations. However, population systems in the real word are often inevitably affected by environmental noises, which are important factors in an ecosystem (see, e.g., [10–12]). It is therefore useful to reveal how the noise affects the delay population systems. In most previously studied stochastic population models, it was assumed that populations change size continuously, as, for example, in diffusion processes which considered small environmental noise, namely, the white noise. The noise arises from a nearly continuous series of small or moderate perturbations that similarly affect the birth and death rates of all individuals (within each age or stage class) in a population [13]. Recall that b_i is the inherent net birth rate of the ith species. In practice we usually estimate it by an average value plus an error which follows a normal distribution. If we still use b_i to denote the inherent net birth rate, then

$$b_i \longrightarrow b_i + \beta_i \dot{w}_1(t), \tag{2}$$

where $\dot{w}_1(t)$ is a white noise and β_i is used to measure the intensity of the white noise imposed on the inherent net birth rate of the ith species. By the same way, every interaction parameter a_{ij} is stochastically perturbed, with

$$a_{ij} \longrightarrow a_{ij} + \sigma_{ij} \dot{w}_2(t), \tag{3}$$

where $\dot{w}_2(t)$ is another white noise and σ_{ij} measures the intensity of the noise imposed on the interaction rates a_{ij}. Then (1) takes the stochastic form

$$\begin{aligned} dx_i(t) = x_i(t) \Bigg[b_i + \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j\left(t - \tau_{ij}\right) \\ + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{0} x_j(t+\theta)\, d\mu_{ij}(\theta) \Bigg] dt \\ + \beta_i x_i(t)\, dw_1(t) + \sum_{j=1}^{n} \sigma_{ij} x_i(t) x_j(t)\, dw_2(t), \\ i = 1, 2, \ldots, n, \end{aligned} \tag{4}$$

where $w_1(t)$ and $w_2(t)$ are mutually independent Brownian motions with $w_i(0) = 0$, $i = 1, 2$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). In recent years, several authors introduced white noises into deterministic systems with delays to reveal the effect of environmental variability on the population dynamics (see, e.g., [14–18]). Particularly, it has also been revealed by Bahar and Mao [14] and Wu and Yin [17] that the environmental noise can suppress a potential population explosion of the Lotka-Volterra model with delays. These indicate clearly that environmental noise may change the properties of population systems significantly.

However, all real populations experience sudden catastrophic disasters of various magnitudes, ranging in size from very small to large. These adverse phenomena can have many causes: earthquakes, volcanoes, floods, tsunami, hurricanes, severe weather, fire, epidemics, and so forth. Stochastic dynamic system (4) is pure diffusion-type stochastic process and its population density change continuously, which cannot explain such large, occasional catastrophic disturbances. To explain these phenomena, introducing a jump process into the underlying population dynamics provides a feasible and more realistic model. Hanson and Tuckwell [19–21] have analyzed the impacts of such random disasters on persistence time of population by employing a stochastic differential equation which consists of a simple continuous deterministic model and a pure Poisson jump component. Bao et al. [22, 23] suggested that these phenomena can be described by a Lévy jump process and they considered stochastic Lotka-Volterra population systems with jumps for the first time. Campillo et al. [24] considered the stochastic model with jump perturbations in the chemostat circumstance. In [25], Liu and Wang studied the dynamics of a Leslie-Gower Holling-type II predator-prey system with Lévy jumps. We also refer the readers to [12, 21] and the references therein for more reasons why the disasters should be considered in population dynamics modeling.

To the best of our knowledge, no results related to delay Lotka-Volterra model with jumps have been reported. In this paper, we thus study possible superpositions of jump and diffusion processes, namely, what is called jump-diffusion processes. Jump-diffusion models have also some intuitive appeal in that they let population change continuously most of the time, but they also take into account the fact that from time to time larger jumps may occur that cannot be adequately modeled by pure diffusion-type processes of system (4). Motivated by these, let us now take a further step; in this paper we focus on stochastic population dynamics (4) which suffer occasional catastrophic disturbances; that is,

$$\begin{aligned} dx_i(t) = x_i(t) \Bigg[b_i + \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j\left(t - \tau_{ij}\right) \\ + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{0} x_j(t+\theta)\, d\mu_{ij}(\theta) \Bigg] dt \\ + \beta_i x_i(t)\, dw_1(t) + \sum_{j=1}^{n} \sigma_{ij} x_i(t) x_j(t)\, dw_2(t) \\ - \int_{\mathbb{Y}} \gamma_i(t,u)\, x_i(t^-)\, N(dt, du), \quad i = 1, 2, \ldots, n, \end{aligned} \tag{5}$$

where $x_i(t^-)$, $i = 1, 2, \ldots, n$, are the left limit of $x_i(t)$ and b_i, a_{ij}, b_{ij}, c_{ij}, τ_{ij}, μ_{ij}, β_i, and σ_{ij} are defined as in model (4). Let $w_i(t)$, $i = 1, 2$, be mutually independent Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\{\mathcal{F}_t\}_{\geq 0}$ satisfying the usual condition; let $N(dt, du)$ be the Poisson random measure and independent of $w_i(t)$, while compensated Poisson random measure is denoted by $\widetilde{N}(dt, du) := N(dt, du) - \nu(du)dt$, where ν is a characteristic measure on a measurable bounded below subset $\mathbb{Y}$ of $\mathbb{R} \setminus 0$ with $\nu(\mathbb{Y}) < \infty$. $\gamma_i : \mathbb{R}_+ \times \mathbb{Y} \rightarrow \mathbb{R}$ being a measurable function, and $0 < \gamma_i(t, u) < 1$, $\gamma_i(t, u)$, $u \in \mathbb{Y}$ is continuous periodic function of time $t \in [0, +\infty)$ with a period $\omega > 0$. The symbol $N(dt, du)$ counts the number of jumps of a compound Poisson process with amplitude $x_i(t^-)\gamma_i(t, u)$ of the ith species generated in the mark interval $(u, u + du)$ during the time interval $(t, t + dt)$.

When the effect of a disaster is proportional to population size, the disaster is said to be density independent, because the relative disaster size is independent of the population size [21]. The density independent disaster is usually due to abiotic causes which affect the population as a whole, whereas biological causes due to species interactions, such as epizootics, may lead to nonlinear or other density independent disasters [26]. In this paper, we only consider abiotic catastrophes such as earthquakes, tsunami, hurricanes, floods, and fire. Under this assumption, each catastrophe reduces instantaneously the population by a proportion $\gamma_i(t, u)$; that is, a population of size $x_i(t^-)$ just prior to a catastrophe is reduced to size $(1 - \gamma_i(t, u))x_i(t^-)$ just after the catastrophe. Note that any disaster greater than the population size has the same effect as a disaster wiping out the whole population. Consequently, our assumption that the loss magnitude of catastrophes in the last term of system (5) is represented by $x_i(t^-)\gamma_i(t, u)$ and $0 < \gamma_i(t, u) < 1$ is reasonable in abiotic catastrophic circumstances.

In reference to the existing results, our contributions are as follows.

(i) We use jump-diffusion process to model Lotka-Volterra systems with delays which suffer sudden catastrophic disturbances and introduce a proper initial data space, in which the initial data may be discontinuous function with downward jumps.

(ii) Using the Khasminskii-Mao theorem and appropriate Lyapunov functions, we show that the delay stochastic differential equation with jumps associated with the model has a unique global positive solution.

(iii) We give sufficient conditions that ensure stochastically ultimate boundedness, moment average boundedness in time, and asymptotic polynomial growth of our model.

(iv) We show that the sufficiently large random jump magnitudes and intensity (average rate of jump events arrival) may lead to extinction of the population.

This paper is organized as follows. In the next section, we provide some notations and show that there exists a unique positive global solution with any initial positive data. In Sections 3 and 4, we discuss the asymptotic moment estimation and asymptotic pathwise estimation, respectively. We obtain the sufficient conditions for the extinction of the population in Section 5, and we try to interpret our mathematical results in terms of their ecological implications compared with the former results in the final section.

2. Global Positive Solution

As the dynamics of system (5) are associated with the biological species, it should be nonnegative. Moreover, in order to guarantee that (5) has a unique global (i.e., no explosion in a finite time) solution for any given initial data, the coefficients of the equation are generally required to satisfy the linear growth and local Lipschitz conditions (cf. [27]). However, the drift coefficient of (5) does not satisfy the linear growth condition, though it is locally Lipschitz continuous, so the solution of (5) may explode in a finite time. It is therefore necessary to provide some conditions under which the solution of (5) not only is positive but also does not explode to infinity in any finite time. Khasminskii [28, Theorem 4.1] and Mao [29] gave the Lyapunov function argument, which is a powerful test for nonexplosion of solutions without the linear growth condition and is referred to as the Khasminskii-Mao theorem. In this section, we will apply the Khasminskii-Mao approach to show that white noise of the interactions among biological species can suppress the explosion to our new model.

Throughout this paper, unless otherwise specified, we use the following notations. x denotes a vector of Euclidean space $\mathbb{R}^n$; that is, $x = (x_1, x_2, \ldots, x_n)$. Let $b = (b_1, b_2, \ldots, b_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, $\sigma = [\sigma_{ij}]_{n\times n}$, and $|\cdot|$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A'. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A'A)}$. Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0$ for all $1 \leq i \leq n\}$. For any $x \in \mathbb{R}$, denote $x^+ = x \vee 0$. If $x(t)$ is an $\mathbb{R}^n$-valued stochastic process on $t \in \mathbb{R}$, we let $x_t = \{x(t + \theta) : \theta \in (-\infty, 0]\}$ for $t \geq 0$.

To consider delay stochastic differential equation with jumps, we here also need to introduce an initial data space. Let $\mathscr{A}$ denote the family of functions that map $(-\infty, 0]$ into $\mathbb{R}^n_+$ satisfying the following.

(i) For every $\phi(t) \in \mathscr{A}$, $\phi_i(t)$ is right-continuous on $(-\infty, 0]$ with finite left-hand limits and has only finitely many downward jumps over any finite time interval, $i = 1, 2, \ldots, n$.

(ii) For every $\phi(t) \in \mathscr{A}$,

$$\sup_{s\leq 0} \phi_i(s) < \infty, \quad i = 1, 2, \ldots, n. \tag{6}$$

In addition, we impose the following assumption on the probability measure μ_{ij}.

Assumption 1. There exists a sufficiently small constant $\lambda > 0$ such that

$$\overline{\mu}_{ij} := \int_{-\infty}^{0} e^{-\lambda\theta} d\mu_{ij}(\theta) < \infty \quad i, j = 1, 2, \ldots, n. \tag{7}$$

Clearly, the above assumption may be satisfied when $\mu_{ij}(\theta) = e^{k\lambda\theta}$ $(k > 1)$ for $\theta \le 0$, so there exist a large number of these probability measures. Clearly, the smaller λ makes Assumption 1 easier to satisfy since $\int_{-\infty}^{0} d\mu_{ij} = 1 < \infty$.

To carry out the analysis, we also need the following simple assumptions on the interaction noise intensity and the jump-diffusion coefficient.

Assumption 2. (H1) $\sigma_{ii} > 0$ if $1 \le i \le n$ whilst $\sigma_{ij} \ge 0$ if $i \ne j$.

(H2) There exists a function $\delta : \mathbb{R}_+ \times \mathbb{Y} \to (0,1)$ such that for each $t \in [0, +\infty)$

$$\gamma_i(t,u) \le \delta(t,u), \quad u \in \mathbb{Y},\ i = 1,2,\dots,n,$$
$$\sup_{t\ge 0}\left\{\int_{\mathbb{Y}} |\ln(1-\delta(t,u))|\,\nu(du)\right\} < \infty. \tag{8}$$

For convenience of reference, we recall two fundamental inequalities stated as a lemma.

Lemma 3. *The following inequalities hold:*

$$x^r \le 1 + r(x-1), \quad x \ge 0,\ 1 \ge r \ge 0, \tag{9}$$

$$n^{(1-(p/2))\wedge 0}|x|^p \le \sum_{i=1}^{n} x_i^p \le n^{(1-p/2)\vee 0}|x|^p, \qquad \forall p > 0, \quad x \in \mathbb{R}_+^n. \tag{10}$$

Then the following theorem on the global positive solution follows.

Theorem 4. *Under Assumption 2, for any initial data $\xi \in \mathscr{A}$, there is a unique positive solution $x(t,\xi)$ of system* (5)*, and the solution will remain in $\mathbb{R}_+^n$ with probability 1; namely, $x(t,\xi) \in \mathbb{R}_+^n$ for all $t \ge 0$ a.s.*

Proof. Since the drift coefficient does not satisfy the linear growth condition, the general theorems of existence and uniqueness cannot be implemented for system (5). However, it is locally Lipschitz continuous; therefore, for any given initial data $\xi(\theta) \in \mathscr{A}$, there is a unique maximal local solution $x(t,\xi)$ for $t \in [0,\tau_e)$, where τ_e is the explosion time. For the sake of simplicity, we write $x(t,\xi) = x(t)$ thereafter. To show that the solution is global, we only need to prove that $\tau_e = \infty$ a.s. Note that $|\xi(0)| < \infty$. Let k_0 be a sufficiently large positive number such that $1/k_0 < \xi_i(0) < k_0$ for all $i = 1,2,\dots,n$. For each integer $k \ge k_0$, define the stopping time

$$\tau_k = \inf\left\{t \in [0,\tau_e) : x_i(t) \notin \left(\frac{1}{k}, k\right) \text{ for some } i = 1,2,\dots,n\right\} \tag{11}$$

with the traditional convention $\inf\emptyset = \infty$, where $\emptyset$ denotes the empty set. Clearly, τ_k is increasing as $k \to \infty$ and $\tau_k \to \tau_\infty \le \tau_e$ a.s. If we can show that $\tau_\infty = \infty$, then $\tau_e = \infty$ a.s., which implies the desired result. This is also equivalent to proving that, for any $t > 0$, $\mathbb{P}(\tau_k \le t) \to 0$ as $k \to \infty$. To prove this statement, let us define a C^2-function $U : \mathbb{R}_+^n \to \mathbb{R}_+$ by

$$U(x) = \sum_{i=1}^{n}\left[\sqrt{x_i} - 1 - 0.5\ln x_i\right]. \tag{12}$$

It is easy to see that $U(x) \ge 0$ for all $x \in \mathbb{R}_+^n$. Applying Itô's formula to (5) yields

$$\begin{aligned}
dU(x) = {} & 0.5\sum_{i=1}^{n}\left(x_i^{0.5} - 1\right) \times \Bigg[b_i + \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} b_{ij}x_j\left(t-\tau_{ij}\right) \\
& + \sum_{j=1}^{n} c_{ij}\int_{-\infty}^{0} x_j(t+\theta)\,d\mu_{ij}(\theta)\Bigg]dt \\
& + 0.25\sum_{i=1}^{n}\left(-0.5x_i^{0.5} + 1\right)\left[\beta_i^2 + \left(\sum_{j=1}^{n}\sigma_{ij}x_j\right)^2\right]dt \\
& + 0.5\sum_{i=1}^{n}\beta_i\left(x_i^{0.5} - 1\right)dw_1(t) \\
& + 0.5\sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i^{0.5} - 1\right)\sigma_{ij}x_j\,dw_2(t) \\
& + \sum_{i=1}^{n}\int_{\mathbb{Y}}\left[\left(1-\gamma_i(t,u)\right)^{0.5} - 1\right]x_i^{0.5}N(dt,du) \\
& - \sum_{i=1}^{n}\int_{\mathbb{Y}} 0.5\ln\left(1-\gamma_i(t,u)\right)N(dt,du).
\end{aligned} \tag{13}$$

Compute

$$\begin{aligned}
& \sum_{i=1}^{n}\left(x_i^{0.5} - 1\right)\left(b_i + \sum_{j=1}^{n} a_{ij}x_j\right) \\
& \le \sum_{i=1}^{n}|b_i|\left(x_i^{0.5} + 1\right) + \sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|x_j + \sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|x_i^{0.5}x_j \\
& \le \sum_{i=1}^{n}|b_i|\left(x_i^{0.5} + 1\right) + \sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|x_j \\
& \quad + \sum_{i=1}^{n}\sum_{j=1}^{n} 0.5\left|a_{ij}\right|\left(x_i + x_j^2\right) \\
& = \sum_{i=1}^{n}\Bigg[|b_i|\left(x_i^{0.5} + 1\right) + \sum_{j=1}^{n}\left(\left|a_{ji}\right| + 0.5\left|a_{ij}\right|\right)x_i \\
& \quad + 0.5\sum_{j=1}^{n}\left|a_{ji}\right|x_i^2\Bigg],
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{n}\left(x_i^{0.5}-1\right)\sum_{j=1}^{n}b_{ij}x_j\left(t-\tau_{ij}\right)\\
&\quad\le 0.5\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ij}\right|^2\left(x_i^{0.5}-1\right)^2+0.5\sum_{i=1}^{n}\sum_{j=1}^{n}x_j^2\left(t-\tau_{ij}\right)\\
&\quad\le \sum_{i=1}^{n}\left[0.5\sum_{j=1}^{n}\left|b_{ij}\right|^2x_i+\sum_{j=1}^{n}\left|b_{ij}\right|^2x_i^{0.5}+0.5\sum_{j=1}^{n}\left|b_{ij}\right|^2\right]\\
&\qquad+0.5\sum_{i=1}^{n}\sum_{j=1}^{n}x_j^2\left(t-\tau_{ij}\right),\\
&\sum_{i=1}^{n}\left(x_i^{0.5}-1\right)\sum_{j=1}^{n}c_{ij}\int_{-\infty}^{0}x_j\left(t+\theta\right)d\mu_{ij}\left(\theta\right)\\
&\quad\le 0.5\sum_{i=1}^{n}\sum_{j=1}^{n}\left|c_{ij}\right|^2\left(x_i^{0.5}-1\right)^2\\
&\qquad+0.5\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\int_{-\infty}^{0}x_j\left(t+\theta\right)d\mu_{ij}\left(\theta\right)\right|^2\\
&\quad\le \sum_{i=1}^{n}\left[0.5\sum_{j=1}^{n}\left|c_{ij}\right|^2x_i+\sum_{j=1}^{n}\left|c_{ij}\right|^2x_i^{0.5}\right.\\
&\qquad\left.+0.5\sum_{j=1}^{n}\left|c_{ij}\right|^2\right]\\
&\qquad+0.5\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{0}x_j^2\left(t+\theta\right)d\mu_{ij}\left(\theta\right).
\end{aligned}\tag{14}$$

Moreover, by assumption (H1),

$$\begin{aligned}
&\sum_{i=1}^{n}\left(-0.5x_i^{0.5}+1\right)\left[\beta_i^2+\left(\sum_{j=1}^{n}\sigma_{ij}x_j\right)^2\right]\\
&\quad=\sum_{i=1}^{n}\left[-0.5\beta_i^2x_i^{0.5}-0.5x_i^{0.5}\left(\sum_{j=1}^{n}\sigma_{ij}x_j\right)^2\right.\\
&\qquad\left.+\beta_i^2+\left(\sum_{j=1}^{n}\sigma_{ij}x_j\right)^2\right]\\
&\quad\le -0.5\sum_{i=1}^{n}x_i^{0.5}\left(\sum_{j=1}^{n}\sigma_{ij}x_j\right)^2+\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\sigma_{ij}x_j\right)^2+\sum_{i=1}^{n}\beta_i^2\\
&\quad\le -0.5\sum_{i=1}^{n}\sigma_{ii}^2x_i^{2.5}+\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\sigma_{ij}^2\sum_{j=1}^{n}x_j^2\right)+\sum_{i=1}^{n}\beta_i^2\\
&\quad=\sum_{i=1}^{n}\left[-0.5\sigma_{ii}^2x_i^{2.5}+|\sigma|^2x_i^2+\beta_i^2\right].
\end{aligned}\tag{15}$$

The above inequalities imply

$$\begin{aligned}
&dU\left(x\right)\\
&\quad\le \sum_{i=1}^{n}\left\{-0.125\sigma_{ii}^2x_i^{2.5}+\left[0.25\sum_{j=1}^{n}\left|a_{ji}\right|+0.25|\sigma|^2\right]x_i^2\right.\\
&\qquad+\left[0.5\sum_{j=1}^{n}\left|a_{ji}\right|+0.25\sum_{j=1}^{n}\left|a_{ij}\right|\right.\\
&\qquad\qquad\left.+0.25\sum_{j=1}^{n}\left|b_{ij}\right|^2+0.25\sum_{j=1}^{n}\left|c_{ij}\right|^2\right]x_i\\
&\qquad+\left[0.5\left|b_i\right|+0.5\sum_{j=1}^{n}\left|b_{ij}\right|^2+0.5\sum_{j=1}^{n}\left|c_{ij}\right|^2\right]x_i^{0.5}\\
&\qquad+\left[0.5\left|b_i\right|+0.25\sum_{j=1}^{n}\left|b_{ij}\right|^2\right.\\
&\qquad\qquad\left.\left.+0.25\sum_{j=1}^{n}\left|c_{ij}\right|^2+\beta_i^2\right]\right\}dt\\
&\qquad+0.25\sum_{i=1}^{n}\sum_{j=1}^{n}x_j^2\left(t-\tau_{ij}\right)dt\\
&\qquad+0.25\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{0}x_j^2\left(t+\theta\right)d\mu_{ij}\left(\theta\right)dt\\
&\qquad+0.5\sum_{i=1}^{n}\beta_i\left(x_i^{0.5}-1\right)dw_1\left(t\right)\\
&\qquad+0.5\sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i^{0.5}-1\right)\sigma_{ij}x_jdw_2\left(t\right)\\
&\qquad-\sum_{i=1}^{n}\int_{\mathbb{Y}}0.5\ln\left(1-\gamma_i\left(t,u\right)\right)N\left(dt,du\right),
\end{aligned}\tag{16}$$

since $(1-\gamma_i(t,u))^{0.5}-1\le 0$. Define

$$\begin{aligned}
U_1\left(x,t\right)=U\left(x\right)&+0.25\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_{ij}}^{t}x_j^2\left(s\right)ds\\
&+0.25\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{0}\int_{t+\theta}^{t}x_j^2\left(s\right)ds\,d\mu_{ij}\left(\theta\right).
\end{aligned}\tag{17}$$

Note that

$$\begin{aligned}
&\sum_{i=1}^{n}\int_{\mathbb{Y}}0.5\ln\left(1-\gamma_i\left(t,u\right)\right)N\left(dt,du\right)\\
&\quad=\sum_{i=1}^{n}\int_{\mathbb{Y}}0.5\ln\left(1-\gamma_i\left(t,u\right)\right)\widetilde{N}\left(dt,du\right)\\
&\qquad+\sum_{i=1}^{n}\int_{\mathbb{Y}}0.5\ln\left(1-\gamma_i\left(t,u\right)\right)\nu\left(du\right)dt.
\end{aligned}\tag{18}$$

By assumption (H2), we have

$$\begin{aligned}
&-\sum_{i=1}^{n}\int_{\mathbb{Y}}0.5\ln\left(1-\gamma_i(t,u)\right)\nu(du)\\
&\leq\sum_{i=1}^{n}\int_{\mathbb{Y}}0.5\left|\ln\left(1-\gamma_i(t,u)\right)\right|\nu(du)\\
&\leq\sum_{i=1}^{n}\int_{\mathbb{Y}}0.5\left|\ln\left(1-\delta(t,u)\right)\right|\nu(du)\\
&<\infty.
\end{aligned}\tag{19}$$

Then

$$\begin{aligned}
&dU_1(x,t)\\
&\leq\sum_{i=1}^{n}\Bigg\{-0.125\sigma_{ii}^2x_i^{2.5}\\
&\quad+\left[0.25\sum_{j=1}^{n}\left|a_{ji}\right|+0.25|\sigma|^2+0.5n\right]x_i^2\\
&\quad+\Bigg[0.5\sum_{j=1}^{n}\left|a_{ji}\right|+0.25\sum_{j=1}^{n}\left|a_{ij}\right|\\
&\qquad+0.25\sum_{j=1}^{n}\left|b_{ij}\right|^2+0.25\sum_{j=1}^{n}\left|c_{ij}\right|^2\Bigg]x_i\\
&\quad+\left[0.5\left|b_i\right|+0.5\sum_{j=1}^{n}\left|b_{ij}\right|^2+0.5\sum_{j=1}^{n}\left|c_{ij}\right|^2\right]x_i^{0.5}\\
&\quad+\Bigg[0.5\left|b_i\right|+0.25\sum_{j=1}^{n}\left|b_{ij}\right|^2\\
&\qquad+0.25\sum_{j=1}^{n}\left|c_{ij}\right|^2+\beta_i^2\Bigg]\Bigg\}dt\\
&\quad+0.5\sum_{i=1}^{n}\beta_i\left(x_i^{0.5}-1\right)dw_1(t)\\
&\quad+0.5\sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i^{0.5}-1\right)\sigma_{ij}x_jdw_2(t)\\
&\quad-\sum_{i=1}^{n}\int_{\mathbb{Y}}0.5\ln\left(1-\gamma_i(t,u)\right)N(dt,du)\\
&\leq Kdt+0.5\sum_{i=1}^{n}\beta_i\left(x_i^{0.5}-1\right)dw_1(t)\\
&\quad+0.5\sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i^{0.5}-1\right)\sigma_{ij}x_jdw_2(t)\\
&\quad-\sum_{i=1}^{n}\int_{\mathbb{Y}}0.5\ln\left(1-\gamma_i(t,u)\right)\widetilde{N}(dt,du),
\end{aligned}\tag{20}$$

where K is a positive constant. We therefore have

$$\begin{aligned}
\mathbb{E}U\left(x\left(t\wedge\tau_k\right)\right)&\leq\mathbb{E}U_1\left(x\left(t\wedge\tau_k\right),t\wedge\tau_k\right)\\
&\leq U_1\left(\xi(0),0\right)+\mathbb{E}\int_0^{t\wedge\tau_k}Kds=:K_t.
\end{aligned}\tag{21}$$

Define for each $u>1$

$$\rho(u):=\inf\left\{U(x):x_i\notin\left(\frac{1}{u},u\right)\ \text{for some } i=1,2,\ldots,n\right\}.\tag{22}$$

Due to the property of the function $h(x):=\sqrt{x}-1-0.5\ln x$, $x>0$, we see that

$$\lim_{x\to+\infty}h(x)=\infty,\qquad\lim_{x\to0}h(x)=\infty,\tag{23}$$

and hence

$$\lim_{u\to\infty}\rho(u)=\infty.\tag{24}$$

By the definition of τ_k, $x_i(\tau_k)\geq k$ or $x_i(\tau_k)\leq 1/k$ for some $i=1,2,\ldots,n$, so

$$\begin{aligned}
\mathbb{P}\left(\tau_k\leq t\right)\rho(k)&\leq\mathbb{P}\left(\tau_k\leq t\right)U\left(x\left(t\wedge\tau_k\right)\right)\\
&\leq\mathbb{E}U\left(x\left(t\wedge\tau_k\right)\right)\\
&\leq K_t,
\end{aligned}\tag{25}$$

which implies that

$$\limsup_{k\to\infty}\mathbb{P}\left(\tau_k\leq t\right)\leq\lim_{k\to\infty}\frac{K_t}{\rho(k)}=0,\tag{26}$$

as required. This completes the proof of Theorem 4. □

3. Asymptotic Moment Properties

Theorem 4 shows that the solution of system (5) remains in the positive cone $\mathbb{R}_+^n$ with probability 1. This nice property permits us to further examine how the solutions vary in $\mathbb{R}_+^n$ in more detail. Compared with the nonexplosion property of the solution, the moment properties are more interesting from the biological point of view. In this section, we will show that the solution of the system (5) is stochastically ultimate boundedness, and the average in time of the moment of the solution to (5) is also bounded. To discuss stochastically ultimate boundedness, we first examine the pth moment boundedness, which is also interesting.

Definition 5. Equation (5) is said to be stochastically ultimately bounded, if for any $\varepsilon\in(0,1)$, there exists a positive constant $\chi(=\chi(\varepsilon))$, such that, for any initial data $\xi\in\mathscr{A}$, the solution of (5) has the property that

$$\limsup_{t\to\infty}P\left\{|x(t)|>\chi\right\}<\varepsilon,\tag{27}$$

where $x(t)$ is the solution of (5) with any initial data $\xi\in\mathscr{A}$.

Theorem 6. *Let Assumption 1 hold. Under Assumption 2, for any $p \in (0,1)$, there exists a positive constant K_p independent of the initial data $\xi \in \mathscr{A}$ such that the global positive solution $x(t,\xi)$ of system (5) has the property that*

$$\limsup_{t\to\infty} \mathbb{E}|x(t)|^p \le K_p. \tag{28}$$

Proof. By Theorem 4, the solution of system (5) remains in $\mathbb{R}_+^n$ a.s. for all t. For any $p \in (0,1)$, define a C^2-function $V_p : \mathbb{R}_+^n \to \mathbb{R}_+$ by

$$V_p(x) = \sum_{i=1}^n x_i^p. \tag{29}$$

For any $\varepsilon \in (0,\lambda)$, applying Itô's formula to $e^{\varepsilon t}V_p(x)$ and taking expectation yield

$$\begin{aligned}\mathbb{E}V_p(x(t)) &= e^{-\varepsilon t}\mathbb{E}V_p(\xi(0)) + e^{-\varepsilon t}\\ &\quad\times \mathbb{E}\int_0^t e^{\varepsilon s}\left[\mathscr{L}V_p(x_s) + \varepsilon V_p(x(s))\right]ds,\end{aligned} \tag{30}$$

where $\mathscr{L}V_p$ is defined as

$$\begin{aligned}&\mathscr{L}V_p(x_t)\\ &\quad= p\sum_{i=1}^n x_i^p\left[b_i + \sum_{j=1}^n a_{ij}x_j + \sum_{j=1}^n b_{ij}x_j(t-\tau_{ij})\right.\\ &\qquad\left.+\sum_{j=1}^n c_{ij}\int_{-\infty}^0 x_j(t+\theta)\,d\mu_{ij}(\theta)\right]\\ &\qquad+\frac{p(p-1)}{2}\sum_{i=1}^n x_i^p\left[\beta_i^2 + \left(\sum_{j=1}^n \sigma_{ij}x_j\right)^2\right]\\ &\qquad+\sum_{i=1}^n\int_{\mathbb{Y}}\left[(1-\gamma_i(t,u))^p - 1\right]x_i^p\,\nu(du).\end{aligned} \tag{31}$$

By Young's inequality,

$$\begin{aligned}&\sum_{i=1}^n x_i^p\left[\sum_{j=1}^n a_{ij}x_j + \sum_{j=1}^n b_{ij}x_j(t-\tau_{ij})\right.\\ &\qquad\left.+\sum_{j=1}^n c_{ij}\int_{-\infty}^0 x_j(t+\theta)\,d\mu_{ij}(\theta)\right]\\ &\quad\le \sum_{i=1}^n x_i^p\left[\sum_{j=1}^n a_{ij}^+x_j + \sum_{j=1}^n b_{ij}^+x_j(t-\tau_{ij})\right.\\ &\qquad\left.+\sum_{j=1}^n c_{ij}^+\int_{-\infty}^0 x_j(t+\theta)\,d\mu_{ij}(\theta)\right]\\ &\quad\le \frac{1}{1+p}\sum_{i=1}^n\sum_{j=1}^n\left[p\left(a_{ij}^+ + b_{ij}^+ + c_{ij}^+\right) + a_{ji}^+\right.\\ &\qquad\left.+e^{\varepsilon\tau_{ji}}b_{ji}^+ + \overline{\mu}_{ji}c_{ji}^+\right]x_i^{1+p}\\ &\qquad+\frac{1}{1+p}\sum_{i=1}^n\sum_{j=1}^n b_{ji}^+\left(x_i^{1+p}(t-\tau_{ji}) - e^{\varepsilon\tau_{ji}}x_i^{1+p}\right)\\ &\qquad+\frac{1}{1+p}\sum_{i=1}^n\sum_{j=1}^n c_{ji}^+\left(\int_{-\infty}^0 x_i^{1+p}(t+\theta)\,d\mu_{ji}(\theta) - \overline{\mu}_{ji}x_i^{1+p}\right).\end{aligned} \tag{32}$$

Moreover, it is easy to show that

$$\begin{aligned}&\sum_{i=1}^n\int_{\mathbb{Y}}\left[(1-\gamma_i(t,u))^p - 1\right]x_i^p\,\nu(du) \le 0,\\ &\sum_{i=1}^n x_i^p\left[\beta_i^2 + \left(\sum_{j=1}^n\sigma_{ij}x_j\right)^2\right]\\ &\qquad= \sum_{i=1}^n\beta_i^2x_i^p + \sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sigma_{ij}\sigma_{ik}x_i^px_jx_k\\ &\qquad\ge \sum_{i=1}^n\beta_i^2x_i^p + \sum_{i=1}^n\sigma_{ii}^2x_i^{2+p}.\end{aligned} \tag{33}$$

As a result, we obtain

$$\begin{aligned}&\mathscr{L}V_p(x_t) + \varepsilon V_p(x(t))\\ &\quad\le \sum_{i=1}^n R_i(x_i) + \frac{p}{1+p}\sum_{i=1}^n\sum_{j=1}^n b_{ji}^+\left(x_i^{1+p}(t-\tau_{ji}) - e^{\varepsilon\tau_{ji}}x_i^{1+p}\right)\\ &\qquad+\frac{p}{1+p}\sum_{i=1}^n\sum_{j=1}^n c_{ji}^+\left(\int_{-\infty}^0 x_i^{1+p}(t+\theta)\,d\mu_{ji}(\theta) - \overline{\mu}_{ji}x_i^{1+p}\right),\end{aligned} \tag{34}$$

where

$$\begin{aligned}R_i(x) &= -\frac{p(1-p)}{2}\sigma_{ii}^2x^{2+p}\\ &\quad+\frac{p}{1+p}\sum_{j=1}^n\left[p\left(a_{ij}^+ + b_{ij}^+ + c_{ij}^+\right) + a_{ji}^+ + e^{\varepsilon\tau_{ji}}b_{ji}^+\right.\\ &\qquad\left.+\overline{\mu}_{ji}c_{ji}^+\right]x^{1+p}\\ &\quad+\left[pb_i + \frac{p(p-1)}{2}\beta_i^2 + \varepsilon\right]x^p.\end{aligned} \tag{35}$$

Noting that $\sigma_{ii} > 0$ and $p \in (0,1)$, by the boundedness property of polynomial functions, there exists a positive constant $K(= K(p,\varepsilon))$ such that $\sum_{i=1}^n R_i(x_i) \le K$, which implies that

$$\begin{aligned}&\mathbb{E}V_p(x(t))\\ &\quad\le e^{-\varepsilon t}\mathbb{E}V_p(\xi(0)) + \varepsilon^{-1}K\left(1-e^{-\varepsilon t}\right)\end{aligned}$$

$$+e^{-\varepsilon t}\frac{p}{1+p}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ji}^{+}\mathbb{E}\left[\int_0^t e^{\varepsilon s}\left(x_i^{1+p}\left(s-\tau_{ji}\right)-e^{\varepsilon\tau_{ji}}x_i^{1+p}(s)\right)ds\right]$$
$$+e^{-\varepsilon t}\frac{p}{1+p}\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ji}^{+}\mathbb{E}\left[\int_0^t e^{\varepsilon s}\left(\int_{-\infty}^0 x_i^{1+p}(s+\theta)\,d\mu_{ji}(\theta)-\overline{\mu}_{ji}x_i^{1+p}(s)\right)ds\right]. \tag{36}$$

It is easy to show that

$$\begin{aligned}\int_0^t e^{\varepsilon s}\left(x_i^{1+p}\left(s-\tau_{ji}\right)-e^{\varepsilon\tau_{ji}}x_i^{1+p}(s)\right)ds &= \int_{-\tau_{ji}}^{t-\tau_{ji}} e^{\varepsilon(s+\tau_{ji})}x_i^{1+p}(s)\,ds-\int_0^t e^{\varepsilon(s+\tau_{ji})}x_i^{1+p}(s)\,ds\\ &\le \varepsilon^{-1}e^{\varepsilon\tau_{ji}}\sup_{u\le 0}\xi_i^{1+p}(u).\end{aligned} \tag{37}$$

Using a similar argument as Theorem 3.1 in [16], by Fubini's Theorem, we may estimate that

$$\begin{aligned}&\int_0^t e^{\varepsilon s}\int_{-\infty}^0 x_i^{1+p}(s+\theta)\,d\mu_{ji}(\theta)\,ds\\ &= \int_0^t e^{\varepsilon s}ds\left[\int_{-\infty}^{-s} x_i^{1+p}(s+\theta)\,d\mu_{ji}(\theta)+\int_{-s}^{0} x_i^{1+p}(s+\theta)\,d\mu_{ji}(\theta)\right]\\ &\le \int_0^t e^{\varepsilon s}ds\int_{-\infty}^{-s} x_i^{1+p}(s+\theta)\,e^{-\lambda(s+\theta)}d\mu_{ji}(\theta)+\int_{-t}^{0}d\mu_{ji}(\theta)\int_{-\theta}^{t}e^{\varepsilon s}x_i^{1+p}(s+\theta)\,ds\\ &\le \sup_{u\le 0}\xi_i^{1+p}(u)\int_0^t e^{-(\lambda-\varepsilon)s}ds\int_{-\infty}^0 e^{-\lambda\theta}d\mu_{ji}(\theta)+\int_{-\infty}^{0}d\mu_{ji}(\theta)\int_0^t e^{\varepsilon(s-\theta)}x_i^{1+p}(s)\,ds\\ &\le \frac{\overline{\mu}_{ji}}{\lambda-\varepsilon}\sup_{u\le 0}\xi_i^{1+p}(u)+\overline{\mu}_{ji}\int_0^t e^{\varepsilon s}x_i^{1+p}(s)\,ds,\end{aligned} \tag{38}$$

which implies

$$\begin{aligned}&\int_0^t e^{\varepsilon s}\left[\int_{-\infty}^0 x_i^{1+p}(s+\theta)\,d\mu_{ji}(\theta)-\overline{\mu}_{ji}x_i^{1+p}(s)\right]ds\\ &\le \frac{\overline{\mu}_{ji}}{\lambda-\varepsilon}\sup_{u\le 0}\xi_i^{1+p}(u).\end{aligned} \tag{39}$$

Hence

$$\begin{aligned}&\mathbb{E}V_p(x(t))\\ &\le e^{-\varepsilon t}\mathbb{E}V_p(\xi(0))+\varepsilon^{-1}K\left(1-e^{-\varepsilon t}\right)\\ &\quad+e^{-\varepsilon t}\frac{p}{1+p}\sum_{i=1}^{n}\sum_{j=1}^{n}\sup_{u\le 0}\xi_i^{1+p}(u)\left[\varepsilon^{-1}b_{ji}^{+}e^{\varepsilon\tau_{ji}}+c_{ji}^{+}\frac{\overline{\mu}_{ji}}{\lambda-\varepsilon}\right],\end{aligned} \tag{40}$$

which implies

$$\limsup_{t\to\infty}\mathbb{E}V_p(x(t))\le\varepsilon^{-1}K. \tag{41}$$

Since $p\in(0,1)$, by inequality (10), we obtain that

$$\limsup_{t\to\infty}\mathbb{E}|x(t)|^p\le\limsup_{t\to\infty}\mathbb{E}V_p(x(t))\le\varepsilon^{-1}K, \tag{42}$$

and the assertion (28) follows by setting $K_p=\varepsilon^{-1}K$. □

Theorem 7. *Let Assumption 1 hold. Under Assumption 2, the solution of* (5) *is stochastically ultimately bounded.*

The proof of Theorem 7 is a simple application of the Chebyshev inequality and Theorem 6.

The following result shows that the average in time of the moment of the solution to system (5) will be bounded.

Theorem 8. *Let Assumption 1 hold. Under Assumption 2, for any $p\in(0,1)$, there exists a positive constant K_p^* independent of the initial data $\xi\in\mathscr{A}$ such that the global positive solution $x(t,\xi)$ of* (5) *has the property that*

$$\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\int_0^t\sum_{i=1}^{n}x_i^{2+p}(s)\,ds\right]\le K_p^*. \tag{43}$$

Proof. Applying Itô's formula to $V_p(x)$ defined by (29) yields

$$\mathbb{E}V_p(x(t))=\mathbb{E}V_p(\xi(0))+\mathbb{E}\int_0^t\mathscr{L}V_p(x_s)\,ds, \tag{44}$$

where $\mathscr{L}V_p$ is defined as

$$\begin{aligned}&\mathscr{L}V_p(x_t)\\ &= p\sum_{i=1}^{n}x_i^p\left[b_i+\sum_{j=1}^{n}a_{ij}x_j+\sum_{j=1}^{n}b_{ij}x_j\left(t-\tau_{ij}\right)+\sum_{j=1}^{n}c_{ij}\int_{-\infty}^0 x_j(t+\theta)\,d\mu_{ij}(\theta)\right]\\ &\quad+\frac{p(p-1)}{2}\sum_{i=1}^{n}x_i^p\left[\beta_i^2+\left(\sum_{j=1}^{n}\sigma_{ij}x_j\right)^2\right]\\ &\quad+\sum_{i=1}^{n}\int_{\mathbb{Y}}\left[\left(1-\gamma_i(t,u)\right)^p-1\right]x_i^p\,\nu(du).\end{aligned} \tag{45}$$

By a similar computation to (34), we have

$$\begin{aligned}\mathscr{L}V_p(x_t)&+\frac{p(1-p)}{4}\sum_{i=1}^{n}\sigma_{ii}^2x_i^{2+p}\\&\le\sum_{i=1}^{n}\Psi_i(x_i)+\frac{p}{1+p}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ji}^{+}\left(x_i^{1+p}\left(t-\tau_{ji}\right)-x_i^{1+p}\right)\\&\quad+\frac{p}{1+p}\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ji}^{+}\left(\int_{-\infty}^{0}x_i^{1+p}(t+\theta)\,d\mu_{ji}(\theta)-x_i^{1+p}\right),\end{aligned}\tag{46}$$

where

$$\begin{aligned}\Psi_i(x)=&-\frac{p(1-p)}{4}\sigma_{ii}^2x^{2+p}+\frac{p}{1+p}\\&\times\sum_{j=1}^{n}\left[p\left(a_{ij}^{+}+b_{ij}^{+}+c_{ij}^{+}\right)+a_{ji}^{+}+b_{ji}^{+}+c_{ji}^{+}\right]x^{1+p}\\&+\left[pb_i+\frac{p(p-1)}{2}\beta_i^2\right]x^p.\end{aligned}\tag{47}$$

Noting that $p\in(0,1)$, the polynomial $\sum_{i=1}^{n}\Psi_i(x_i)$ has an upper positive bound, say R_p; inequality (46) yields

$$\begin{aligned}\mathbb{E}V_p(x(t))&+\frac{p(1-p)}{4}\sum_{i=1}^{n}\sigma_{ii}^2\mathbb{E}\left[\int_0^t x_i^{2+p}(s)\,ds\right]\\&\le\mathbb{E}V_p(\xi(0))+\int_0^t R_p\,ds\\&\quad+\frac{p}{1+p}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ji}^{+}\mathbb{E}\left[\int_0^t\left(x_i^{1+p}\left(s-\tau_{ji}\right)-x_i^{1+p}(s)\right)ds\right]\\&\quad+\frac{p}{1+p}\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ji}^{+}\mathbb{E}\left[\int_0^t\left(\int_{-\infty}^{0}x_i^{1+p}(s+\theta)\,d\mu_{ji}(\theta)\right.\right.\\&\qquad\left.\left.-x_i^{1+p}(s)\right)ds\right],\end{aligned}\tag{48}$$

since $(1-\gamma_i(t,u))^p-1\le 0$, for $0<\gamma_i(t,u)<1$. Compute

$$\begin{aligned}\int_0^t&\left(x_i^{1+p}\left(s-\tau_{ji}\right)-x_i^{1+p}(s)\right)ds\\&=\int_{-\tau_{ji}}^{t-\tau_{ji}}x_i^{1+p}(s)\,ds-\int_0^t x_i^{1+p}(s)\,ds\\&\le\tau_{ji}\sup_{u\le 0}\xi_i^{1+p}(u),\end{aligned}\tag{49}$$

$$\begin{aligned}\int_0^t&\int_{-\infty}^{0}x_i^{1+p}(s+\theta)\,d\mu_{ji}(\theta)\,ds\\&=\int_0^t ds\left[\int_{-\infty}^{-s}x_i^{1+p}(s+\theta)\,d\mu_{ji}(\theta)\right.\\&\qquad\left.+\int_{-s}^{0}x_i^{1+p}(s+\theta)\,d\mu_{ji}(\theta)\right]\\&\le\int_0^t ds\int_{-\infty}^{-s}x_i^{1+p}(s+\theta)\,e^{-\lambda(s+\theta)}d\mu_{ji}(\theta)\\&\quad+\int_{-t}^{0}d\mu_{ji}(\theta)\int_{-\theta}^{t}x_i^{1+p}(s+\theta)\,ds\\&\le\sup_{u\le 0}\xi_i^{1+p}(u)\int_0^t e^{-\lambda s}ds\int_{-\infty}^{0}e^{-\lambda\theta}d\mu_{ji}(\theta)\\&\quad+\int_{-\infty}^{0}d\mu_{ji}(\theta)\int_0^t x_i^{1+p}(s)\,ds\\&\le\frac{\overline{\mu}_{ji}}{\lambda}\sup_{u\le 0}\xi_i^{1+p}(u)+\int_0^t x_i^{1+p}(s)\,ds,\end{aligned}\tag{50}$$

which implies

$$\begin{aligned}\int_0^t&\left(\int_{-\infty}^{0}x_i^{1+p}(s+\theta)\,d\mu_{ji}(\theta)-x_i^{1+p}(s)\right)ds\\&\le\frac{\overline{\mu}_{ji}}{\lambda}\sup_{u\le 0}\xi_i^{1+p}(u).\end{aligned}\tag{51}$$

Hence

$$\begin{aligned}\mathbb{E}V_p(x(t))&+\frac{p(1-p)}{4}\sum_{i=1}^{n}\sigma_{ii}^2\mathbb{E}\left[\int_0^t x_i^{2+p}(s)\,ds\right]\\&\le\mathbb{E}V_p(\xi(0))+\int_0^t R_p\,ds\\&\quad+\frac{p}{1+p}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ji}^{+}\tau_{ji}\sup_{u\le 0}\xi_i^{1+p}(u)\\&\quad+\frac{p}{\lambda(1+p)}\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ji}^{+}\overline{\mu}_{ji}\sup_{u\le 0}\xi_i^{1+p}(u);\end{aligned}\tag{52}$$

this implies

$$\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\int_0^t\sum_{i=1}^{n}x_i^{2+p}(s)\,ds\right]\le\frac{4R_p}{\widehat{\sigma}p(1-p)},\tag{53}$$

where

$$\widehat{\sigma}=\min\left\{\sigma_{ii}^2,1\le i\le n\right\}.\tag{54}$$

The required assertion (43) follows immediately. □

4. Asymptotic Pathwise Estimation

In Section 3 we have discussed how the solutions vary in the $\mathbb{R}_+^n$ in probability or in moment. In this section, we continue to examine the pathwise properties of the solutions to system (5). The following result shows that this stochastic population system will grow at most polynomially.

Theorem 9. *Let Assumption 1 hold. Under Assumption 2, for any initial data $\xi\in\mathscr{A}$, the solution of* (5) *satisfies*

$$\limsup_{t\to\infty}\frac{\ln|x(t)|}{\ln t}\le 1\quad a.s.\tag{55}$$

That is, $x(t)$ grows with at most polynomial speed.

Proof. By Theorem 4, the solution of (5) remains in $\mathbb{R}_+^n$ a.s. for all t. Define a C^2-function $V_1 : \mathbb{R}_+^n \to \mathbb{R}_+$ by

$$V_1(x(t)) = \sum_{i=1}^{n} x_i. \tag{56}$$

Applying Itô's formula yields

$$\begin{aligned} dV_1(x(t)) = & \sum_{i=1}^{n} x_i \left[b_i + \sum_{j=1}^{n} a_{ij} x_j + \sum_{j=1}^{n} b_{ij} x_j (t - \tau_{ij}) \right. \\ & \left. + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{0} x_j (t+\theta) \, d\mu_{ij}(\theta) \right] dt \\ & + \sum_{i=1}^{n} \beta_i x_i dw_1(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j dw_2(t) \\ & - \sum_{i=1}^{n} \int_{\mathbb{Y}} x_i(t^-) \gamma_i(t,u) N(dt,du). \end{aligned} \tag{57}$$

Then, applying Itô's formula to $\ln V_1(x)$ yields

$$\begin{aligned} & d \ln V_1(x) \\ & = \left[\frac{1}{V_1(x)} \sum_{i=1}^{n} x_i \left(b_i + \sum_{j=1}^{n} a_{ij} x_j + \sum_{j=1}^{n} b_{ij} x_j (t - \tau_{ij}) \right. \right. \\ & \qquad \left. + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{0} x_j (t+\theta) \, d\mu_{ij}(\theta) \right) \\ & \qquad \left. - \frac{1}{2} Z_1^2(t) - \frac{1}{2} Z_2^2(t) \right] dt \\ & \quad + Z_1(t) dw_1(t) + Z_2(t) dw_2(t) \\ & \quad + \int_{\mathbb{Y}} \ln \left(1 - \frac{\sum_{i=1}^{n} x_i(t^-) \gamma_i(t,u)}{V_1(x(t^-))} \right) N(dt,du), \end{aligned} \tag{58}$$

where

$$\begin{aligned} Z_1(t) &= \frac{1}{V_1(x)} \sum_{i=1}^{n} \beta_i x_i(t), \\ Z_2(t) &= \frac{1}{V_1(x)} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i(t) x_j(t). \end{aligned} \tag{59}$$

Hence, for any $\varepsilon \in (0, \lambda)$, applying Itô's product formula to $e^{\varepsilon t} \ln V_1(x)$ gives

$$\begin{aligned} & \ln V_1(x(t)) \\ & = e^{-\varepsilon t} \ln V_1(\xi(0)) \\ & \quad + e^{-\varepsilon t} \int_0^t e^{\varepsilon s} \left[\varepsilon \ln V_1(x(s)) - \frac{1}{2} Z_1^2(s) - \frac{1}{2} Z_2^2(s) + \frac{1}{V_1(x)} \right. \\ & \qquad \times \sum_{i=1}^{n} x_i \left(b_i + \sum_{j=1}^{n} a_{ij} x_j + \sum_{j=1}^{n} b_{ij} x_j (s - \tau_{ij}) \right. \\ & \qquad \left. \left. + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{0} x_j (s+\theta) \, d\mu_{ij}(\theta) \right) \right] ds \\ & \quad + e^{-\varepsilon t} \int_0^t e^{\varepsilon s} Z_1(s) dw_1(s) + e^{-\varepsilon t} \int_0^t e^{\varepsilon s} Z_2(s) dw_2(s) \\ & \quad + e^{-\varepsilon t} \int_0^t e^{\varepsilon s} \int_{\mathbb{Y}} \ln \left(1 - \frac{\sum_{i=1}^{n} x_i(s^-) \gamma_i(s,u)}{V_1(x(s^-))} \right) N(ds,du). \end{aligned} \tag{60}$$

By virtue of the exponential martingale inequality, for any T, α, $\beta > 0$, we have

$$\begin{aligned} \mathbb{P} & \left\{ \sup_{0 \le t \le T} \left[\int_0^t e^{\varepsilon s} Z_1(s) dw_1(s) + \int_0^t e^{\varepsilon s} Z_2(s) dw_2(s) \right. \right. \\ & \qquad \left. \left. - \frac{\alpha}{2} \int_0^t e^{2\varepsilon s} \left(Z_1^2(s) + Z_1^2(s) \right) ds \right] \ge \beta \right\} \\ & \le e^{-\alpha\beta}. \end{aligned} \tag{61}$$

Choose $T = k\varepsilon$, $\alpha = pe^{-k\varepsilon}$, and $\beta = p^{-1}\theta e^{k\varepsilon} \ln k$, when $k \in \mathbb{N}$, $0 < p < 1$, and $\theta > 1$ in the above equation. Since $\sum_{k=1}^{\infty} k^{-\theta} < \infty$, we can deduce from the Borel-Cantelli lemma [30, 2.2.4] that there exists an $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that, for any $\omega \in \Omega_0$, an integer $\bar{k} = \bar{k}(\omega, p)$ can be found such that

$$\begin{aligned} & \int_0^t e^{\varepsilon s} Z_1(s) dw_1(s) + \int_0^t e^{\varepsilon s} Z_2(s) dw_2(s) \\ & \quad \le p^{-1}\theta e^{k\varepsilon} \ln k + \frac{pe^{-k\varepsilon}}{2} \int_0^t e^{2\varepsilon s} \left(Z_1^2(s) + Z_2^2(s) \right) ds \\ & \quad \le p^{-1}\theta e^{k\varepsilon} \ln k + \frac{p}{2} \int_0^t e^{\varepsilon s} \left(Z_1^2(s) + Z_2^2(s) \right) ds \end{aligned} \tag{62}$$

whenever $k \ge \bar{k}$, $0 \le t \le k\varepsilon$. Thus, for $\omega \in \Omega_0$ and $(k-1)\varepsilon \le t \le k\varepsilon$ with $k \ge \bar{k} + 1$, we have

$$\begin{aligned} & \frac{\ln V_1(x(t))}{\ln t} \\ & \le \frac{\ln V_1(\xi(0))}{e^{\varepsilon t} \ln t} + \frac{p^{-1}\theta e^{k\varepsilon} \ln k}{e^{(k-1)\varepsilon} \ln((k-1)\varepsilon)} + \frac{1}{\ln t} \int_0^t e^{\varepsilon(s-t)} \\ & \quad \times \left[\varepsilon \ln V_1(x(s)) + \frac{p-1}{2} Z_2^2(s) \right. \\ & \qquad + \frac{1}{V_1(x)} \sum_{i=1}^{n} x_i \left(b_i + \sum_{j=1}^{n} a_{ij} x_j + \sum_{j=1}^{n} b_{ij} x_j (s - \tau_{ij}) \right. \\ & \qquad \left. \left. + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{0} x_j (s+\theta) \, d\mu_{ij}(\theta) \right) \right] ds \end{aligned}$$

$$+\frac{1}{\ln t}\int_0^t e^{\varepsilon(s-t)}\int_{\mathbb{Y}}\ln\left(1-\frac{\sum_{i=1}^n x_i(s^-)\gamma_i(s,u)}{V_1(x(s^-))}\right)\times N(ds,du). \tag{63}$$

Note that $\ln u \le 2\sqrt{u}$ for any $u>0$. Hence, by inequality (10) we have

$$\ln V_1(x) \le 2\sqrt{V_1(x)} \le 2n^{1/4}|x|^{1/2}. \tag{64}$$

Clearly,

$$\frac{1}{V_1(x)}\sum_{i=1}^n b_i x_i \le \max_{1\le i\le n}\{b_i^+\}. \tag{65}$$

It is easy to show that

$$\begin{aligned}
&\frac{1}{V_1(x)}\sum_{i=1}^n\sum_{j=1}^n a_{ij}x_ix_j \\
&\quad\le \frac{1}{V_1(x)}\sum_{i=1}^n\sum_{j=1}^n a_{ij}^+x_ix_j \le \max_{1\le i,j\le n}\{a_{ij}^+\}V_1(x) \qquad (66)\\
&\quad\le \max_{1\le i,j\le n}\{a_{ij}^+\}\sqrt{n}|x|, \\
&\frac{1}{V_1(x)}\sum_{i=1}^n\sum_{j=1}^n b_{ij}x_ix_j(t-\tau_{ij}) \\
&\quad\le \frac{1}{V_1(x)}\sum_{i=1}^n\sum_{j=1}^n b_{ij}^+x_ix_j(t-\tau_{ij}) \qquad (67)\\
&\quad\le \max_{1\le i,j\le n}\{b_{ij}^+\}\sum_{j=1}^n x_j(t-\tau_{ij}), \\
&\frac{1}{V_1(x)}\sum_{i=1}^n\sum_{j=1}^n c_{ij}x_i\int_{-\infty}^0 x_j(t+\theta)\,d\mu_{ij}(\theta) \\
&\quad\le \frac{1}{V_1(x)}\sum_{i=1}^n\sum_{j=1}^n c_{ij}^+x_i\int_{-\infty}^0 x_j(t+\theta)\,d\mu_{ij}(\theta) \qquad (68)\\
&\quad\le \max_{1\le i,j\le n}\{c_{ij}^+\}\sum_{j=1}^n\int_{-\infty}^0 x_j(t+\theta)\,d\mu_{ij}(\theta).
\end{aligned}$$

By inequality (10), it is also easy to see that

$$Z_2^2(t) \ge \left[\frac{1}{V_1(x)}\sum_{i=1}^n \sigma_{ii}x_i^2(t)\right]^2 \ge \frac{\min_{1\le i\le n}\{\sigma_{ii}^2\}}{n}|x(t)|^2. \tag{69}$$

Note that

$$\frac{1}{\ln t}\int_0^t e^{\varepsilon(s-t)}\int_{\mathbb{Y}}\ln\left(1-\frac{\sum_{i=1}^n x_i(s^-)\gamma_i(s,u)}{V_1(x(s^-))}\right)\times N(ds,du)\le 0; \tag{70}$$

this, together with (64)–(69), implies

$$\begin{aligned}
&\frac{\ln V_1(x(t))}{\ln t} \\
&\le \frac{\ln V_1(\xi(0))}{e^{\varepsilon t}\ln t} + \frac{p^{-1}\theta e^{k\varepsilon}\ln k}{e^{(k-1)\varepsilon}\ln((k-1)\varepsilon)} \\
&\quad + \frac{1}{\ln t}\int_0^t e^{\varepsilon(s-t)}\Psi(x(s))\,ds + \frac{1}{\ln t}\max_{1\le i,j\le n}\{b_{ij}^+\} \\
&\quad\times\sum_{j=1}^n\int_0^t e^{\varepsilon(s-t)}\left[x_j(s-\tau_{ij}) - e^{\varepsilon\tau_{ij}}x_j(s)\right]ds \\
&\quad + \frac{1}{\ln t}\max_{1\le i,j\le n}\{c_{ij}^+\} \\
&\quad\times\sum_{j=1}^n\int_0^t e^{\varepsilon(s-t)}\left[\int_{-\infty}^0 x_j(s+\theta)\,d\mu_{ij}(\theta) - \overline{\mu}_{ij}x_j(s)\right]ds,
\end{aligned} \tag{71}$$

where

$$\begin{aligned}
\Psi(x) = &-\frac{1-p}{2n}\min_{1\le i\le n}\{\sigma_{ii}^2\}|x|^2 + \max_{1\le i,j\le n}\{a_{ij}^+\}\sqrt{n}|x| \\
&+ \max_{1\le i,j\le n}\{b_{ij}^+\}\sum_{j=1}^n e^{\varepsilon\tau_{ij}}x_j + \max_{1\le i,j\le n}\{c_{ij}^+\}\sum_{j=1}^n\overline{\mu}_{ij}x_j \\
&+ 2n^{1/4}|x|^{1/2} + \max_{1\le i\le n}\{b_i^+\}.
\end{aligned} \tag{72}$$

Recalling that $p\in(0,1)$, by the boundedness of polynomial functions, there exists a positive constant $\overline{K}$ such that $\Psi(x)\le\overline{K}$. In addition, noting that $\varepsilon\in(0,\lambda)$, using a similar argument as (37) and (39), we may estimate that

$$\begin{aligned}
&\int_0^t e^{\varepsilon s}\left[x_j(s-\tau_{ij}) - e^{\varepsilon\tau_{ij}}x_j(s)\right]ds \le \varepsilon^{-1}e^{\varepsilon\tau_{ij}}\sup_{u\le 0}\xi_j(u), \\
&\int_0^t e^{\varepsilon s}\left[\int_{-\infty}^0 x_j(s+\theta)\,d\mu_{ij}(\theta) - \overline{\mu}_{ij}x_j(s)\right]ds \\
&\quad\le \frac{\overline{\mu}_{ij}}{\lambda-\varepsilon}\sup_{u\le 0}\xi_j(u).
\end{aligned} \tag{73}$$

Hence,

$$\begin{aligned}
\frac{\ln V_1(x(t))}{\ln t} &\le \frac{\ln V_1(\xi(0))}{e^{\varepsilon t}\ln t} + \frac{p^{-1}\theta e^{k\varepsilon}\ln k}{e^{(k-1)\varepsilon}\ln((k-1)\varepsilon)} \\
&\quad + \frac{1}{\ln t}\varepsilon^{-1}\overline{K}\left(1-e^{-\varepsilon t}\right) \\
&\quad + \frac{1}{e^{\varepsilon t}\ln t}\max_{1\le i,j\le n}\{b_{ij}^+\}\sum_{j=1}^n\varepsilon^{-1}e^{\varepsilon\tau_{ij}}\sup_{u\le 0}\xi_j(u) \\
&\quad + \frac{1}{e^{\varepsilon t}\ln t}\max_{1\le i,j\le n}\{c_{ij}^+\}\sum_{j=1}^n\frac{\overline{\mu}_{ij}}{\lambda-\varepsilon}\sup_{u\le 0}\xi_j(u),
\end{aligned} \tag{74}$$

which implies that by inequality (10)

$$\limsup_{t\to\infty}\frac{\ln|x(t)|}{\ln t}\le\frac{\theta e^{\varepsilon}}{p}. \tag{75}$$

Letting $p\to 1,\theta\to 1$, and $\varepsilon\to 0$ gives the desired result. □

5. Extinction

Understanding the impacts of white noises and random catastrophes on the extinction of the population is important in both pure and applied ecology, and in the formulation of effective conservation plans for threatened and endangered species (Lande [31]; Gilpin and Hanski [32]). In this section, the sufficient conditions for the extinction of the system (5) are established. We show that if the intensity of white noises, the magnitude of catastrophes, or the frequency of disasters is sufficiently large, the solution to system (5) will become extinct with probability 1, although the solution to the original deterministic model (1) may be persistent.

For later applications, let us cite a strong law of large number for local martingales (see, e.g., [33]) as the following lemma.

Lemma 10. *Let $M(t)$, $t\ge 0$, be a local martingale vanishing at time 0 and define*

$$\rho_M(t):=\int_0^t\frac{d\langle M\rangle(s)}{(1+s)^2},\quad t\ge 0, \tag{76}$$

where $\langle M\rangle(t):=\langle M,M\rangle(t)$ is Meyer's angle bracket process. Then

$$\lim_{t\to\infty}\frac{M(t)}{t}=0\quad a.s.\qquad \text{provided that }\lim_{t\to\infty}\rho_M(t)<\infty\quad a.s. \tag{77}$$

Remark 11. Suppose that

$$\Psi^2_{\text{loc}}:=\left\{\Psi(t,z)\mid\Psi(t,z)\text{ is predictable},\ \int_0^t\int_{\mathbb{Y}}|\Psi(s,z)|^2\nu(du)\,ds<\infty\right\} \tag{78}$$

and for $\Psi\in\Psi^2_{\text{loc}}$,

$$M(t):=\int_0^t\int_{\mathbb{Y}}\Psi(s,z)\widetilde{N}(ds,du). \tag{79}$$

Then, by, for example, Kunita [34, Proposition 2.4],

$$\langle M\rangle(t)=\int_0^t\int_{\mathbb{Y}}|\Psi(s,z)|^2\nu(du)\,ds. \tag{80}$$

Theorem 12. *Let Assumptions 1 and 2 hold. Assume moreover that the following conditions are satisfied:*

(i) *there exists a function $\gamma_d:\mathbb{R}_+\times\mathbb{Y}\to(0,1)$ such that for each $t\in[0,+\infty)$*

$$\gamma_d(t,u)\le\gamma_i(t,u)\le\delta(t,u),\quad u\in\mathbb{Y},\quad i=1,2,\dots,n, \tag{81}$$

(ii) *there exists a constant $0<p<1$ such that*

$$\frac{\kappa}{2n}+\Gamma>\frac{n^2\left(\max_{1\le i,j\le n}\{a^+_{ij}\}+\max_{1\le i,j\le n}\{b^+_{ij}\}+\max_{1\le i,j\le n}\{c^+_{ij}\}\right)^2}{2(1-p)\min_{1\le i\le n}\{\sigma^2_{ii}\}}. \tag{82}$$

Then, for any initial data $\xi\in\mathcal{A}$, the solution of (5) satisfies

$$\limsup_{t\to\infty}\frac{\ln|x(t)|}{t}\le-\eta\quad a.s., \tag{83}$$

where η is a positive constant, and

$$\kappa=\min\left\{\beta_i\beta_j-b_i-b_j,\ 1\le i,\ j\le n\right\}\ge 0, \tag{84}$$

$$\Gamma=-\max_{t\in[0,\omega]}\left\{\int_{\mathbb{Y}}\ln(1-\gamma_d(t,u))\nu(du)\right\}; \tag{85}$$

namely, the population will decay exponentially with probability 1.

Proof. Recall from (58) that

$$\begin{aligned}d\ln V_1(x)&=\Bigg[\frac{1}{V_1(x)}\sum_{i=1}^n x_i\Bigg(b_i+\sum_{j=1}^n a_{ij}x_j+\sum_{j=1}^n b_{ij}x_j(t-\tau_{ij})+\sum_{j=1}^n c_{ij}\int_{-\infty}^0 x_j(t+\theta)\,d\mu_{ij}(\theta)\Bigg)\\&\quad-\frac{1}{2}Z_1^2(t)-\frac{1}{2}Z_2^2(t)+\int_{\mathbb{Y}}\ln\left(1-\frac{\sum_{i=1}^n x_i\gamma_i(t,u)}{V_1(x)}\right)\nu(du)\Bigg]dt\\&\quad+Z_1(t)\,dw_1(t)+Z_2(t)\,dw_2(t)\\&\quad+\int_{\mathbb{Y}}\ln\left(1-\frac{\sum_{i=1}^n x_i(t^-)\gamma_i(t,u)}{V_1(x(t^-))}\right)\widetilde{N}(dt,du),\end{aligned} \tag{86}$$

where

$$Z_1(t)=\frac{1}{V_1(x)}\sum_{i=1}^n\beta_i x_i(t),\qquad Z_2(t)=\frac{1}{V_1(x)}\sum_{i=1}^n\sum_{j=1}^n\sigma_{ij}x_i(t)x_j(t). \tag{87}$$

Integrating both sides of this equality yields

$$\begin{aligned}
&\ln V_1(x)\\
&= \ln V_1(\xi(0))\\
&\quad + \int_0^t \left[\frac{1}{V_1(x)} \sum_{i=1}^n x_i \left(b_i + \sum_{j=1}^n a_{ij} x_j + \sum_{j=1}^n b_{ij} x_j (s - \tau_{ij}) \right.\right.\\
&\qquad\qquad \left. + \sum_{j=1}^n c_{ij} \int_{-\infty}^0 x_j(s+\theta)\, d\mu_{ij}(\theta) \right)\\
&\qquad + \int_{\mathbb{Y}} \ln \left(1 - \frac{\sum_{i=1}^n x_i \gamma_i(s,u)}{V_1(x)} \right) \nu(du)\\
&\qquad \left. - \frac{1}{2} Z_1^2(s) - \frac{1}{2} Z_2^2(s) \right] ds\\
&\quad + M_1(t) + M_2(t) + M_3(t),
\end{aligned} \tag{88}$$

where $M_i(t)$, $i = 1, 2, 3$, are local martingales defined, respectively, as follows:

$$\begin{aligned}
M_1(t) &= \int_0^t Z_1(s)\, dw_1(s) = \int_0^t \frac{1}{V_1(x)} \sum_{i=1}^n \beta_i x_i(s)\, dw_1(s),\\
M_2(t) &= \int_0^t Z_2(s)\, dw_2(s),\\
M_3(t) &= \int_0^t \int_{\mathbb{Y}} \ln \left(1 - \frac{\sum_{i=1}^n x_i(s^-) \gamma_i(s,u)}{V_1(x(s^-))} \right) \widetilde{N}(ds, du)\\
&= \int_0^t \int_{\mathbb{Y}} \ln (1 - H(x(s^-), s, u))\, \widetilde{N}(ds, du)
\end{aligned} \tag{89}$$

with $M_i(0) = 0$, $i = 1, 2, 3$, where H is defined by

$$H(x, t, u) = \frac{\sum_{i=1}^n x_i \gamma_i(t,u)}{V_1(x)}. \tag{90}$$

The quadratic variation of the continuous local martingale $M_2(t)$ is

$$\langle M_2, M_2 \rangle (t) = \int_0^t Z_2^2(s)\, ds. \tag{91}$$

For any $\theta > 1$ and each integer $k \geq 1$, $0 < p < 1$, the exponential martingale inequality yields

$$\mathbb{P} \left\{ \sup_{1 \leq t \leq k} \left[M_2(t) - \frac{p}{2} \langle M_2, M_2 \rangle (t) \right] \geq \frac{\theta \ln k}{p} \right\} \leq \frac{1}{k^\theta}. \tag{92}$$

Since $\sum_{k=1}^\infty k^{-\theta} < \infty$, by the Borel-Cantelli lemma [30, 2.2.4], there exists an $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that, for any $\omega \in \Omega_0$, there exists an integer $\overline{k}(\omega)$, when $k > \overline{k}(\omega)$, and $k - 1 \leq t \leq k$,

$$M_2(t) \leq \frac{p}{2} \int_0^t Z_2^2(s)\, ds + \frac{\theta \ln(t+1)}{p}. \tag{93}$$

Clearly,

$$\begin{aligned}
&\frac{1}{V_1(x)} \sum_{i=1}^n x_i b_i - \frac{1}{2V_1^2(x)} \left(\sum_{i=1}^n \beta_i x_i \right)^2\\
&\quad = \frac{1}{2V_1^2(x)} \left(2x' b \vec{1} x - x' \beta \beta' x \right)\\
&\quad = \frac{1}{2V_1^2(x)} \left(x' \left(b \vec{1} + \vec{1}' b' \right) x - x' \beta \beta' x \right)\\
&\quad =: -\frac{1}{2V_1^2(x)} x' K x,
\end{aligned} \tag{94}$$

where $\vec{1} = (1, 1, \ldots, 1)$ and $K = \beta\beta' - b\vec{1} - \vec{1}' b'$. Note that the ijth element of the matrix K is $\beta_i \beta_j - b_i - b_j \geq \kappa \geq 0$ by (84). By the inequality (10), for $x \in \mathbb{R}_+^n$, it is easy to find that

$$x' K x \geq \kappa |x|^2 \geq \frac{\kappa}{n} V_1^2(x). \tag{95}$$

Hence

$$\frac{1}{V_1(x)} \sum_{i=1}^n x_i b_i - \frac{1}{2V_1^2(x)} \left(\sum_{i=1}^n \beta_i x_i \right)^2 \leq -\frac{\kappa}{2n}. \tag{96}$$

By (81) and Jensen's inequality, we may obtain that

$$\begin{aligned}
&\int_{\mathbb{Y}} \ln \left(1 - \frac{\sum_{i=1}^n x_i \gamma_i(s,u)}{V_1(x)} \right) \nu(du)\\
&\qquad \leq \int_{\mathbb{Y}} \ln (1 - \gamma_d(s,u))\, \nu(du).
\end{aligned} \tag{97}$$

This, together with (93), (96), and (66)–(69), gives from (88) that

$$\begin{aligned}
&\ln V_1(x)\\
&\leq \ln V_1(\xi(0)) + \int_0^t \Phi(x(s))\, ds\\
&\quad + \frac{\theta \ln(t+1)}{p} + M_1(t) + M_3(t)\\
&\quad + \max_{1 \leq i,j \leq n} \{b_{ij}^+\} \sum_{j=1}^n \int_0^t \left[x_j(s - \tau_{ij}) - x_j(s) \right] ds\\
&\quad + \max_{1 \leq i,j \leq n} \{c_{ij}^+\} \sum_{j=1}^n \int_0^t \left[\int_{-\infty}^0 x_j(s+\theta)\, d\mu_{ij}(\theta) - x_j(s) \right] ds,
\end{aligned} \tag{98}$$

where

$$\begin{aligned}
\Phi(x) = &-\frac{1-p}{2n} \min_{1 \leq i \leq n} \{\sigma_{ii}^2\} |x|^2\\
&+ \left(\max_{1 \leq i,j \leq n} \{a_{ij}^+\} + \max_{1 \leq i,j \leq n} \{b_{ij}^+\} + \max_{1 \leq i,j \leq n} \{c_{ij}^+\} \right) \sqrt{n} |x|\\
&- \frac{\kappa}{2n} + \max_{t \in [0,\omega]} \left\{ \int_{\mathbb{Y}} \ln (1 - \gamma_d(t,u))\, \nu(du) \right\}.
\end{aligned} \tag{99}$$

Let

$$A = -\frac{1-p}{2n}\min_{1\le i\le n}\left\{\sigma_{ii}^2\right\},$$
$$B = \left(\max_{1\le i,j\le n}\left\{a_{ij}^+\right\} + \max_{1\le i,j\le n}\left\{b_{ij}^+\right\} + \max_{1\le i,j\le n}\left\{c_{ij}^+\right\}\right)\sqrt{n}, \quad (100)$$
$$C = \frac{\kappa}{2n} - \max_{t\in[0,\omega]}\left\{\int_{\mathbb{Y}} \ln\left(1-\gamma_d(t,u)\right)\nu(du)\right\}.$$

Then, $\Phi(x)$ is denoted by

$$\Phi(x) = A|x|^2 + B|x| - C. \quad (101)$$

Recall that $0 < p < 1$, so $A < 0$. Since $B \ge 0$ and $C > 0$, by (82), we may obtain that the discriminant $\Delta = B^2 + 4AC < 0$. Hence, there exists a positive constant $\eta > 0$ such that

$$\Phi(x) \le -\eta, \quad x \in \mathbb{R}_+^n. \quad (102)$$

Therefore, we obtain from (98) that

$$\begin{aligned}
&\ln V_1(x) \\
&\le \ln V_1(\xi(0)) - \eta t + \frac{\theta \ln(t+1)}{p} \\
&+ \max_{1\le i,j\le n}\left\{b_{ij}^+\right\}\sum_{j=1}^n \int_0^t \left[x_j\left(s-\tau_{ij}\right) - x_j(s)\right] ds \\
&+ \max_{1\le i,j\le n}\left\{c_{ij}^+\right\}\sum_{j=1}^n \int_0^t \left[\int_{-\infty}^0 x_j(s+\theta)\, d\mu_{ij}(\theta) - x_j(s)\right] ds \\
&+ M_1(t) + M_3(t).
\end{aligned} \quad (103)$$

By (81) and the definition of H, for $x \in \mathbb{R}_+^n$, we obtain that

$$H(x,t,u) \le \delta(t,u). \quad (104)$$

This implies that

$$|\ln(1-H(x,t,u))| \le -\ln(1-\delta(t,u)). \quad (105)$$

Then, by (H2), there exists a positive constant K_0 such that

$$\begin{aligned}
\langle M_3, M_3\rangle(t) &= \int_0^t \int_{\mathbb{Y}} \left[\ln(1-H(x(s),s,u))\right]^2 \nu(du)\, ds \\
&\le \int_0^t \int_{\mathbb{Y}} \left[\ln(1-\delta(t,u))\right]^2 \nu(du)\, ds \\
&\le K_0 t.
\end{aligned} \quad (106)$$

On the other hand,

$$\begin{aligned}
\langle M_1, M_1\rangle(t) &= \int_0^t Z_1^2(s)\, ds \\
&\le \int_0^t \frac{\left(\sum_{i=1}^n |\beta_i| x_i(s)\right)^2}{V_1^2(x)} ds \le \max_{1\le i\le n}\left\{\beta_i^2\right\} t.
\end{aligned} \quad (107)$$

Hence, by Lemma 10, we have

$$\lim_{t\to\infty}\frac{M_i(t)}{t} = 0 \quad \text{a.s.} \quad i = 1, 3. \quad (108)$$

In addition, we may estimate that from (49) and (51)

$$\int_0^t \left[x_j\left(s-\tau_{ij}\right) - x_j(s)\right] ds \le \tau_{ij} \sup_{u\le 0} \xi_j(u),$$
$$\int_0^t \left[\int_{-\infty}^0 x_j(s+\theta)\, d\mu_{ij}(\theta) - x_j(s)\right] ds \le \frac{\overline{\mu}_{ij}}{\lambda} \sup_{u\le 0} \xi_j(u), \quad (109)$$

which, together with (108), imply

$$\limsup_{t\to\infty} \frac{\ln V_1(x(t))}{t} \le -\eta \quad \text{a.s.} \quad (110)$$

Consequently, by inequality (10) we complete the proof of this Theorem. □

6. Discussion

Traditionally, the population dynamics are modeled by the deterministic models, which assume that parameters in the model are all deterministic irrespective of environmental fluctuations. However, population systems in the real word are often inevitably affected by environmental noises, which are important factors in an ecosystem. In particular, the population may suffer sudden abiotic catastrophes such as earthquakes, tsunami, hurricanes, floods, and fire. It is therefore useful to reveal how the environmental noises affect the population systems.

In this paper, we consider the effect of jump-diffusion random environmental perturbations on the asymptotic properties and extinction of delay Lotka-Volterra population dynamics. From condition (82) in Theorem 12, we can observe that if intensities β_i, σ_{ii} of white noises, the random downward jump magnitude $\gamma_i(t,u)$, or intensity ν (average rate of jump events arrival) is sufficiently large, the species will be extinct. Compared with the former results [17], this result gives an interesting and important condition under which the frequent nature disasters can force the population to become extinct.

Recalling protection of wildlife rare species, the southwest region Sichuan in China, is best-preserved panda habitat on earth, which belongs to Longmen Shan active fault zone. Both the 2008 M8.0 Wenchuan and the 2013 M7.0 Ya'an earthquakes occurred in this region. Earthquakes and secondary disasters caused by the earthquake such as mudslides, landslides, barrier lake, and other geological disasters may destroy a lot of woods and bamboo, which is natural habitat of endangered wildlife species. Because the population does not have enough time to adjust to these sudden and severe environmental perturbations, the death rate of these endangered wildlife species may increase suddenly and greatly. In this circumstance, $\gamma_i(t,u)$ represents random loss magnitude of the ith species caused by earthquake and ν stands for the intensity of occurrence of earthquake (average rate of earthquake arrival). By Theorem 12, if the earthquake destruction and the frequency of occurrence are large enough such that condition (82) is satisfied, then the wildlife species face extinction and we should do something to avoid its extinction.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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