Research Article

New Subclasses of Multivalent Analytic Functions Associated with a Linear Operator

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Making use of a linear operator, which is defined here by means of the Hadamard product (or convolution), we consider two subclasses $F_{p,n}(a, c, \lambda, A, B)$ and $G_{p,n}(a, c, \lambda, A, B)$ of multivalent analytic functions with negative coefficients in the open unit disk. Some modified Hadamard products, integral transforms, and the partial sums of functions belonging to these classes are studied.

1. Introduction

Let $A_{p,n}$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad (p, n \in N = \{1, 2, 3, \ldots\}), \quad (1)$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$.

For functions $f(z)$ and $g(z)$ analytic in U, we say that $f(z)$ is subordinate to $g(z)$ in U, written $f(z) \prec g(z)$ $(z \in U)$, if there exists an analytic function $w(z)$ in U such that

$$|w(z)| \le |z|, \quad f(z) = g(w(z)), \quad (z \in U). \quad (2)$$

Furthermore, if the function $g(z)$ is univalent in U, then

$$f(z) \prec g(z) \quad (z \in U) \iff f(0) = g(0),$$
$$f(U) \subseteq g(U). \quad (3)$$

In terms of the Pochhammer symbol $(b)_n$ given by $(b)_n = b(b+1)\cdots(b+n-1)$ $(n \in N)$, we define the function $\varphi_{p,n}(a, c; z)$ by

$$\varphi_{p,n}(a, c; z) = z^p + \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} z^k \quad (4)$$

$$(z \in U; c \notin \{0, -1, -2, \ldots\}).$$

Corresponding to $\varphi_{p,n}(a, c; z)$, we consider here a linear operator $L_{p,n}(a, c)$ on $A_{p,n}$ by the following usual Hadamard product (or convolution):

$$L_{p,n}(a, c) f(z) = \varphi_{p,n}(a, c; z) * f(z)$$

$$= z^p + \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} a_k z^k, \quad (5)$$

for $f(z)$ given by (1). For $p = n = 1$, $L_{1,1}(a, c)$ on $A_{1,1}$ was first defined by Carlson and Shaffer [1]. Its differential-integral representation can be found in [2]. We remark in passing that a much more general convolution operator than the operator $L_{p,1}(a, c)$ was introduced by Dziok and Srivastava [3].

Let $T_{p,n}$ denote the subclass of $A_{p,n}$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k, \quad (a_k \ge 0; p, n \in N). \quad (6)$$

We now consider the following two subclasses of the class $T_{p,n}$.

Definition 1. A function $f(z) \in T_{p,n}$ is said to be in the class $F_{p,n}(a, c, \lambda, A, B)$ if and only if

$$(1 - \lambda) \frac{L_{p,n}(a,c) f(z)}{z^p} + \lambda \frac{\left(L_{p,n}(a,c) f(z)\right)'}{pz^{p-1}}$$

$$\prec \frac{1 + Az}{1 + Bz} \quad (z \in U), \tag{7}$$

where

$$a > 0, \quad c > 0, \quad \lambda \geq 0, \quad -1 \leq B \leq 0, \quad B < A \leq 1. \tag{8}$$

Definition 2. A function $f(z) \in T_{p,n}$ is said to be in the class $G_{p,n}(a, c, \lambda, A, B)$ if and only if

$$\frac{zf'(z)}{p} \in F_{p,n}(a, c, \lambda, A, B). \tag{9}$$

For functions $f_j(z) \in T_{p,n}$ given by

$$f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k, \quad \left(a_{k,j} \geq 0; \, j = 1, 2\right), \tag{10}$$

we denote by $(f_1 \cdot f_2)(z)$ the modified Hadamard (or quasi-Hadamard) product of $f_1(z)$ and $f_2(z)$; that is,

$$(f_1 \cdot f_2)(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 \cdot f_1)(z). \tag{11}$$

The class

$$F_{p,n}\left(1, 1, \lambda, 1 - \frac{2\alpha}{p}, -1\right) = F_p(n, \lambda, \alpha),$$

$$(\lambda \geq 0; \, 0 \leq \alpha < p) \tag{12}$$

with $n = 1$ was introduced and studied earlier by Lee et al. [4] (and was further investigated by Aouf and Darwish [5], Aouf et al. [6], and Yaguchi et al. [7]). The class

$$G_{p,n}\left(1, 1, \lambda, 1 - \frac{2\alpha}{p}, -1\right) = G_p(n, \lambda, \alpha),$$

$$(\lambda \geq 0; \, 0 \leq \alpha < p) \tag{13}$$

with $n = 1$ was studied by Aouf [8] and Aouf et al. [6]. Recently, Aouf [9] investigated the modified Hadamard products of several functions in the classes $F_p(n, \lambda, \alpha)$ and $G_p(n, \lambda, \alpha)$ for $n \in N$.

In the present paper, we prove a number of theorems involving the modified Hadamard products, integral transforms, and the partial sums of functions in the classes $F_{p,n}(a, c, \lambda, A, B)$ and $G_{p,n}(a, c, \lambda, A, B)$. Some of our results are generalizations of the corresponding results in [4–9].

In proving our main results, we need the following lemmas.

Lemma 3 (see [10, 11]). *A function $f(z) \in T_{p,n}$ defined by (6) is in the class $F_{p,n}(a, c, \lambda, A, B)$ if and only if*

$$\sum_{k=p+n}^{\infty} (p + (k - p)\lambda) \frac{(a)_{k-p}}{(c)_{k-p}} a_k \leq p \frac{A - B}{1 - B}. \tag{14}$$

Lemma 4 (see [10, 11]). *A function $f(z) \in T_{p,n}$ defined by (6) is in the class $G_{p,n}(a, c, \lambda, A, B)$ if and only if*

$$\sum_{k=p+n}^{\infty} k(p + (k - p)\lambda) \frac{(a)_{k-p}}{(c)_{k-p}} a_k \leq p^2 \frac{A - B}{1 - B}. \tag{15}$$

Making use of Lemmas 3 and 4, we can show the following two results.

Corollary 5. *Let*

$$0 < a_1 \leq a_0, \quad 0 < c_0 \leq c_1, \quad 0 \leq \lambda_1 \leq \lambda_0,$$

$$-1 \leq B_1 \leq B_0 < A_0 \leq A_1 \leq 1, \quad B_0 \leq 0. \tag{16}$$

Then

$$F_{p,n}(a_0, c_0, \lambda_0, A_0, B_0) \subseteq F_{p,n}(a_1, c_1, \lambda_1, A_1, B_1),$$

$$G_{p,n}(a_0, c_0, \lambda_0, A_0, B_0) \subseteq G_{p,n}(a_1, c_1, \lambda_1, A_1, B_1). \tag{17}$$

Corollary 6. *Let $f(z) \in T_{p,n}$. Then $f(z) \in F_{p,n}(a_0, c_0, \lambda_0, A, B)$ $(G_{p,n}(a_0, c_0, \lambda_0, A, B))$ if and only if $(f * g)(z) \in F_{p,n}(a_1, c_1, \lambda_1, A, B)$ $(G_{p,n}(a_1, c_1, \lambda_1, A, B))$, where $\lambda_0 \geq 0, \lambda_1 \geq 0, a_j > 0$ and $c_j > 0$ $(j = 0, 1)$, and*

$$g(z) = z^p + \sum_{k=p+n}^{\infty} \frac{(p + (k - p)\lambda_0)(a_0)_{k-p}(c_1)_{k-p}}{(p + (k - p)\lambda_1)(a_1)_{k-p}(c_0)_{k-p}} z^k. \tag{18}$$

If we let

$$a = c = 1, \quad A = 1 - \frac{2\alpha}{p}, \quad 0 \leq \alpha < p, \quad B = -1, \tag{19}$$

then Lemmas 3 and 4 reduce to the following result.

Corollary 7. *Let $f(z) \in T_{p,n}$ be defined by (6). Then*

(i) *$f(z)$ is in the class $F_p(n, \lambda, \alpha)$ if and only if*

$$\sum_{k=p+n}^{\infty} (p + (k - p)\lambda) a_k \leq p - \alpha. \tag{20}$$

(ii) *$f(z)$ is in the class $G_p(n, \lambda, \alpha)$ if and only if*

$$\sum_{k=p+n}^{\infty} k(p + (k - p)\lambda) a_k \leq p(p - \alpha). \tag{21}$$

2. Modified Hadamard Products

Hereafter in this paper we assume that (8) is satisfied:

$$0 \leq \alpha < p, \quad 0 \leq \alpha_j < p, \quad -1 \leq B_j \leq 0,$$

$$B_j < A_j \leq 1, \quad (j = 1, 2, \ldots, m). \tag{22}$$

Theorem 8. *Let $f_j(z) \in F_{p,n}(a, c, \lambda, A_j, B_j)$ $(j = 1, 2, \ldots, m)$ and $a \geq c > 0$. Then*

$$(f_1 \cdot f_2 \cdot \ldots \cdot f_m)(z) \in F_{p,n}(a, c, \lambda, A(B), B), \tag{23}$$

where $m \geq 2$ and

$$A(B) = B + (1 - B) \left(\frac{p(c)_n}{(p + n\lambda)(a)_n} \right)^{m-1} \prod_{j=1}^{m} \left(\frac{A_j - B_j}{1 - B_j} \right).$$

(24)

The result is sharp; that is, $A(B)$ cannot be decreased for each $B \in [-1, 0]$.

Proof. By (24) we have $B < A(B) \leq 1$ for $a \geq c > 0$. Let

$$f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k \in F_{p,n}(a, c, \lambda, A_j, B_j)$$

(25)

$$(a_{k,j} \geq 0; \ j = 1, 2, \ldots, m).$$

Then Lemma 3 gives

$$\left(\frac{1 - B_j}{A_j - B_j} \right) \sum_{k=p+n}^{\infty} \frac{(p + (k - p)\lambda)(a)_{k-p}}{p(c)_{k-p}} a_{k,j}$$

(26)

$$\leq 1 \quad (j = 1, 2, \ldots, m),$$

and hence

$$\prod_{j=1}^{m} \left(\frac{1 - B_j}{A_j - B_j} \right)$$

$$\times \sum_{k=p+n}^{\infty} \left(\frac{(p + (k - p)\lambda)(a)_{k-p}}{p(c)_{k-p}} \right)^m a_{k,1} a_{k,2} \cdots a_{k,m}$$

$$\leq \prod_{j=1}^{m} \left\{ \left(\frac{1 - B_j}{A_j - B_j} \right) \sum_{k=p+n}^{\infty} \frac{(p + (k - p)\lambda)(a)_{k-p}}{p(c)_{k-p}} a_{k,j} \right\}$$

$$\leq 1.$$

(27)

Also, using Lemma 3,

$$h(z) = (f_1 \cdot f_2 \cdot \ldots \cdot f_m)(z)$$

$$= z^p - \sum_{k=p+n}^{\infty} a_{k,1} a_{k,2} \cdots a_{k,m} z^k \in F_{p,n}(a, c, \lambda, A, B)$$

(28)

if and only if

$$\left(\frac{1 - B}{A - B} \right) \sum_{k=p+n}^{\infty} \frac{(p + (k - p)\lambda)(a)_{k-p}}{p(c)_{k-p}} a_{k,1} a_{k,2} \cdots a_{k,m} \leq 1.$$

(29)

To prove the result of Theorem 8, it follows from (27) and (29) that we need to find the smallest A such that

$$\frac{(p + (k - p)\lambda)(a)_{k-p}(1 - B)}{p(c)_{k-p}(A - B)}$$

$$\leq \left(\frac{(p + (k - p)\lambda)(a)_{k-p}}{p(c)_{k-p}} \right)^m \prod_{j=1}^{m} \left(\frac{1 - B_j}{A_j - B_j} \right)$$

(30)

$$(k \geq p + n; \ m \geq 2);$$

that is, that

$$A \geq B + (1 - B) \left(\frac{p(c)_{k-p}}{(p + (k - p)\lambda)(a)_{k-p}} \right)^{m-1}$$

$$\times \prod_{j=1}^{m} \left(\frac{A_j - B_j}{1 - B_j} \right) \quad (k \geq p + n; \ m \geq 2).$$

(31)

Since

$$\frac{(c)_{k+1-p}}{(a)_{k+1-p}} \leq \frac{(c)_{k-p}}{(a)_{k-p}} \quad (k \geq p + n; \ a \geq c > 0),$$

(32)

we see that the right-hand side of (31) is a decreasing function of k. Consequently, taking $k = p + n$ in (31), we have $h(z) \in F_{p,n}(a, c, \lambda, A(B), B)$, where $A(B)$ is given by (24).

Furthermore, by considering the functions $f_j(z)$ defined by:

$$f_j(z) = z^p - \frac{p(c)_n (A_j - B_j)}{(p + n\lambda)(a)_n (1 - B_j)} z^{p+n}$$

(33)

$$(j = 1, 2, \ldots, m),$$

we have $f_j(z) \in F_{p,n}(a, c, \lambda, A_j, B_j)$ and

$$(f_1 \cdot f_2 \cdot \ldots \cdot f_m)(z)$$

$$= z^p - \left(\frac{p(c)_n}{(p + n\lambda)(a)_n} \right)^m \prod_{j=1}^{m} \left(\frac{A_j - B_j}{1 - B_j} \right) z^{p+n}.$$

(34)

Noting that

$$(p + n\lambda) \frac{(a)_n}{(c)_n} \left(\frac{p(c)_n}{(p + n\lambda)(a)_n} \right)^m \prod_{j=1}^{m} \left(\frac{A_j - B_j}{1 - B_j} \right)$$

$$= p \frac{A(B) - B}{1 - B},$$

(35)

we conclude that $A(B)$ cannot be decreased for each B. □

By using Lemma 4 instead of Lemma 3, the following theorem can be proved on the lines of the proof of Theorem 8. We omit the details involved.

Theorem 9. *Let $f_j(z) \in G_{p,n}(a, c, \lambda, A_j, B_j)$ $(j = 1, 2, \ldots, m)$ and $a \geq c > 0$. Then*

$$(f_1 \cdot f_2 \cdot \ldots \cdot f_m)(z) \in G_{p,n}(a, c, \lambda, A(B), B),$$

(36)

where $m \geq 2$ and

$$A(B) = B + (1 - B) \left(\frac{p^2(c)_n}{(p + n)(p + n\lambda)(a)_n} \right)^{m-1}$$

$$\times \prod_{j=1}^{m} \left(\frac{A_j - B_j}{1 - B_j} \right) \in (B, 1).$$

(37)

The result is sharp for the functions $f_j(z)$ defined by

$$f_j(z) = z^p - \frac{p^2(c)_n \left(A_j - B_j\right)}{(p+n)(p+n\lambda)(a)_n \left(1 - B_j\right)} z^{p+n} \tag{38}$$

$$\in G_{p,n}\left(a, c, \lambda, A_j, B_j\right) \quad (j = 1, 2, \ldots, m).$$

Theorem 10. Let $f_j(z) \in F_{p,n}(a, c, \lambda, A_j, B_j)$ $(j = 1, 2, \ldots, m)$ with $a \geq c > 0$ and $\lambda \geq 1$. Then

$$\left(f_1 \cdot f_2 \cdot \ldots \cdot f_m\right)(z) \in G_{p,n}\left(a, c, \lambda, A(B), B\right), \tag{39}$$

where $m \geq 2$ and

$$A(B) = B + (1 - B)\left(\frac{p+n}{p}\right)\left(\frac{p(c)_n}{(p+n\lambda)(a)_n}\right)^{m-1} \tag{40}$$

$$\times \prod_{j=1}^{m}\left(\frac{A_j - B_j}{1 - B_j}\right).$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2, \ldots, m)$ defined by (33).

Proof. Obviously, $B < A(B) \leq 1$ for $m \geq 2$, $a \geq c > 0$, and $\lambda \geq 1$. By applying Lemma 4, we know that

$$\left(f_1 \cdot f_2 \cdot \ldots \cdot f_m\right)(z)$$

$$= z^p - \sum_{k=p+n}^{\infty} a_{k,1} a_{k,2} \cdots a_{k,m} z^k \in G_{p,n}(a, c, \lambda, A, B), \tag{41}$$

if and only if

$$\left(\frac{1-B}{A-B}\right) \sum_{k=p+n}^{\infty} \frac{k(p+(k-p)\lambda)(a)_{k-p}}{p^2(c)_{k-p}} a_{k,1} a_{k,2} \cdots a_{k,m} \leq 1. \tag{42}$$

Proceeding as in the proof of Theorem 8, we need to find the smallest A such that

$$A \geq B + (1 - B)\frac{k}{p}\left(\frac{p(c)_{k-p}}{(p+(k-p)\lambda)(a)_{k-p}}\right)^{m-1}$$

$$\times \prod_{j=1}^{m}\left(\frac{A_j - B_j}{1 - B_j}\right) \quad (k \geq p + n). \tag{43}$$

Defining the function $\varphi(x)$ by

$$\varphi(x) = \frac{x}{(p+(x-p)\lambda)^{m-1}} \quad (x \geq p + n), \tag{44}$$

we see that

$$\varphi'(x) = \frac{p(1-\lambda) - (m-2)\lambda x}{(p+(x-p)\lambda)^m} \leq 0 \quad (x \geq p + n), \tag{45}$$

for $m \geq 2$ and $\lambda \geq 1$. Hence, the right-hand side of (43) is a decreasing function of k. Thus, we arrive at $(f_1 \cdot f_2 \cdot \ldots \cdot f_m)(z) \in G_{p,n}(a, c, \lambda, A(B), B)$, where $A(B)$ is given by (40). $\square$

Sharpness can be verified easily.

By putting

$$a = c = 1, \quad A(B) = 1 - \frac{2\alpha}{p}, \quad B = -1,$$

$$A_j = 1 - \frac{2\alpha_j}{p}, \quad B_j = -1 \quad (j = 1, 2, \ldots, m). \tag{46}$$

Theorem 10 reduces to the following.

Corollary 11. Let $f_j(z) \in F_p(n, \lambda, \alpha_j)$ $(j = 1, 2, \ldots, m)$. Then $(f_1 \cdot f_2 \cdot \ldots \cdot f_m)(z) \in G_p(n, \lambda, \alpha)$, where $m \geq 2$, $\lambda \geq 1$, and

$$\alpha = p - \frac{p+n}{p(p+n\lambda)^{m-1}} \prod_{j=1}^{m} \left(p - \alpha_j\right). \tag{47}$$

The result is sharp for the functions

$$f_j(z) = z^p - \frac{p - \alpha_j}{p + n\lambda} z^{p+n} \in F_p\left(n, \lambda, \alpha_j\right) \tag{48}$$

$$(j = 1, 2, \ldots, m).$$

Theorem 12. Let $f_j(z) \in F_{p,n}(a, c, \lambda, A_j, B_j)$ $(j = 1, 2, \ldots, m)$,

$$g_j(z) \in F_{p,n}\left(a, c, \lambda, C_j, D_j\right),$$

$$\left(-1 \leq D_j \leq 0; \; D_j < C_j \leq 1; \; j = 1, 2, \ldots, m\right) \tag{49}$$

and $a \geq c > 0$. Then one has

(i) $\sum_{j=1}^{m}(f_j \cdot g_j)(z) - (m-1)z^p \in F_{p,n}(a, c, \lambda, A(B), B)$, where

$$A(B) = B + (1 - B)\frac{mp(c)_n \left(A_0 - B_0\right)\left(C_0 - D_0\right)}{(p+n\lambda)(a)_n \left(1 - B_0\right)\left(1 - D_0\right)},$$

$$A_0 = \max_{1 \leq j \leq m}\left\{A_j\right\}, \qquad B_0 = \min_{1 \leq j \leq m}\left\{B_j\right\}, \tag{50}$$

$$C_0 = \max_{1 \leq j \leq m}\left\{C_j\right\}, \qquad D_0 = \min_{1 \leq j \leq m}\left\{D_j\right\},$$

provided that $A(B) \leq 1$.

(ii) $(1/m)\sum_{j=1}^{m}(f_j \cdot g_j)(z) \in F_{p,n}(a, c, \lambda, \widetilde{A}(B), B)$, where

$$\widetilde{A}(B) = B + (1 - B)$$

$$\times \frac{p(c)_n \left(A_0 - B_0\right)\left(C_0 - D_0\right)}{(p+n\lambda)(a)_n \left(1 - B_0\right)\left(1 - D_0\right)} \in (B, 1]. \tag{51}$$

Proof. It is clear that $-1 \leq B_0 \leq B_j < A_j \leq A_0 \leq 1$, $B_0 \leq 0$, $-1 \leq D_0 \leq D_j < C_j \leq C_0 \leq 1$, $D_0 \leq 0$,

$$\frac{1-B_j}{A_j - B_j} \geq \frac{1-B_0}{A_0 - B_0} > 0,$$

$$\frac{1-D_j}{C_j - D_j} \geq \frac{1-D_0}{C_0 - D_0} > 0, \tag{52}$$

for $j = 1, 2, \ldots, m$. Let

$$f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k \in F_{p,n}\left(a, c, \lambda, A_j, B_j\right)$$

$$g_j(z) = z^p - \sum_{k=p+n}^{\infty} b_{k,j} z^k \in F_{p,n}\left(a, c, \lambda, C_j, D_j\right), \tag{53}$$

$$(j = 1, 2, \ldots, m).$$

Then Lemma 3 gives

$$\sum_{j=1}^{m} \left(f_j \cdot g_j\right)(z) - (m-1) z^p$$

$$= z^p - \sum_{k=p+n}^{\infty} \left(\sum_{j=1}^{m} a_{k,j} b_{k,j}\right) z^k \in F_{p,n}(a, c, \lambda, A, B), \tag{54}$$

if and only if

$$\left(\frac{1-B}{A-B}\right) \sum_{k=p+n}^{\infty} \left\{ \left(\frac{(p+(k-p)\lambda)(a)_{k-p}}{p(c)_{k-p}}\right) \left(\sum_{j=1}^{m} a_{k,j} b_{k,j}\right) \right\}$$

$$\leq 1. \tag{55}$$

Also, by Lemma 3 and (52), we deduce that

$$\left(\frac{1-B_0}{A_0-B_0}\right)\left(\frac{1-D_0}{C_0-D_0}\right)$$

$$\times \sum_{k=p+n}^{\infty} \left(\frac{(p+(k-p)\lambda)(a)_{k-p}}{p(c)_{k-p}}\right)^2 a_{k,j} b_{k,j}$$

$$\leq \left\{ \left(\frac{1-B_j}{A_j-B_j}\right) \sum_{k=p+n}^{\infty} \frac{(p+(k-p)\lambda)(a)_{k-p}}{p(c)_{k-p}} a_{k,j} \right\}$$

$$\times \left\{ \left(\frac{1-D_j}{C_j-D_j}\right) \sum_{k=p+n}^{\infty} \frac{(p+(k-p)\lambda)(a)_{k-p}}{p(c)_{k-p}} b_{k,j} \right\}$$

$$\leq 1 \quad (j = 1, 2, \ldots, m), \tag{56}$$

and hence

$$\frac{1}{m}\left(\frac{1-B_0}{A_0-B_0}\right)\left(\frac{1-D_0}{C_0-D_0}\right)$$

$$\times \sum_{k=p+n}^{\infty} \left\{ \left(\frac{(p+(k-p)\lambda)(a)_{k-p}}{p(c)_{k-p}}\right)^2 \left(\sum_{j=1}^{m} a_{k,j} b_{k,j}\right) \right\}$$

$$\leq 1. \tag{57}$$

To prove Theorem 12(i), it follows from (55) and (57) that we need to find the smallest A such that

$$A \geq B + (1-B)\left(\frac{mp(c)_{k-p}}{(p+(k-p)\lambda)(a)_{k-p}}\right)$$

$$\times \left(\frac{A_0-B_0}{1-B_0}\right)\left(\frac{C_0-D_0}{1-D_0}\right) \quad (k \geq p+n), \tag{58}$$

for $a \geq c > 0$. This leads to the assertion of Theorem 12(i). Analogously, we can prove Theorem 12(ii). □

In the special case when

$$a = c = 1, \quad A(B) = 1 - \frac{2\alpha}{p}, \quad \widetilde{A}(B) = 1 - \frac{2\widetilde{\alpha}}{p},$$

$$B = -1, \quad A_j = C_j = 1 - \frac{2\alpha_j}{p}, \tag{59}$$

$$B_j = D_j = -1 \quad (j = 1, 2, \ldots, m),$$

Theorem 12 reduces to the following.

Corollary 13. *Let $f_j(z) \in F_p(n, \lambda, \alpha_j)$ $(j = 1, 2, \ldots, m)$. Then*

(i) $\sum_{j=1}^{m}(f_j \cdot f_j)(z) - (m-1)z^p \in F_p(n, \lambda, \alpha)$, *where*

$$\alpha = p - \frac{m}{p+n\lambda}\left(p - \min_{1 \leq j \leq m}\{\alpha_j\}\right)^2, \tag{60}$$

   provided that $\alpha \geq 0$.

(ii) $(1/m)\sum_{j=1}^{m}(f_j \cdot f_j)(z) \in F_p(n, \lambda, \widetilde{\alpha})$, *where*

$$\widetilde{\alpha} = p - \frac{1}{p+n\lambda}\left(p - \min_{1 \leq j \leq m}\{\alpha_j\}\right)^2 \in [0, p). \tag{61}$$

Replacing Lemma 3 by Lemma 4 in the proof of Theorem 12, one can prove the following.

Theorem 14. *Let $f_j(z) \in G_{p,n}(a, c, \lambda, A_j, B_j)$ $(j = 1, 2, \ldots, m)$,*

$$g_j(z) \in G_{p,n}\left(a, c, \lambda, C_j, D_j\right),$$

$$\left(-1 \leq D_j \leq 0; D_j < C_j \leq 1; j = 1, 2, \ldots, m\right) \tag{62}$$

and $a \geq c > 0$. Then

(i) $\sum_{j=1}^{m}(f_j \cdot g_j)(z) - (m-1)z^p \in G_{p,n}(a, c, \lambda, A(B), B)$, *where*

$$A(B) = B + (1-B)\frac{mp^2(c)_n(A_0-B_0)(C_0-D_0)}{(p+n)(p+n\lambda)(a)_n(1-B_0)(1-D_0)}, \tag{63}$$

   and A_0, B_0, C_0, and D_0 are given as in Theorem 12, provided that $A(B) \leq 1$.

(ii) $(1/m) \sum_{j=1}^{m} (f_j \cdot g_j)(z) \in G_{p,n}(a, c, \lambda, \widetilde{A}(B), B)$, where

$$\widetilde{A}(B) = B + (1 - B)$$

$$\times \frac{p^2(c)_n (A_0 - B_0)(C_0 - D_0)}{(n + p)(p + n\lambda)(a)_n (1 - B_0)(1 - D_0)} \in (B, 1). \tag{64}$$

As a special case of Theorem 14, one has the following.

Corollary 15. *Let $f_j(z) \in G_p(n, \lambda, \alpha_j)$ $(j = 1, 2, \ldots, m)$. Then*

(i) $\sum_{j=1}^{m} (f_j \cdot f_j)(z) - (m-1)z^p \in G_p(n, \lambda, \alpha)$, where

$$\alpha = p - \frac{mp}{(p+n)(p+n\lambda)} \left(p - \min_{1 \le j \le m} \{\alpha_j\} \right)^2, \tag{65}$$

provided that $\alpha \ge 0$.

(ii) $(1/m) \sum_{j=1}^{m} (f_j \cdot f_j)(z) \in G_p(n, \lambda, \widetilde{\alpha})$, where

$$\widetilde{\alpha} = p - \frac{p}{(p+n)(p+n\lambda)} \left(p - \min_{1 \le j \le m} \{\alpha_j\} \right)^2 \in (0, p). \tag{66}$$

3. Integral Operator

Theorem 16. *Let*

$$0 \le \lambda_1 \le \lambda_0, \qquad 0 < c_0 \le a_0, \qquad 0 < a_1 \le c_1 \tag{67}$$
$$\left(or \ 0 < a_1 \le a_0 \ and \ 0 < c_0 \le c_1 \right).$$

If $f(z) \in F_{p,n}(a_0, c_0, \lambda_0, A, B)$, then the function $I(z)$ defined by

$$I(z) = \frac{\mu + p}{z^\mu} \int_0^z t^{\mu - 1} f(t) \, dt \quad (\mu > -p) \tag{68}$$

belongs to $F_{p,n}(a_1, c_1, \lambda_1, C(D), D)$, where $-1 \le D \le 0$ and

$$C(D) = D + (1 - D)$$

$$\times \frac{(p + n\lambda_1)(a_1)_n (c_0)_n (\mu + p)(A - B)}{(p + n\lambda_0)(c_1)_n (a_0)_n (\mu + p + n)(1 - B)}. \tag{69}$$

The result is sharp; that is, the number $C(D)$ cannot be decreased for each D.

Proof. Note that $D < C(D) < 1$. For

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \in F_{p,n}(a_0, c_0, \lambda_0, A, B), \tag{70}$$

it follows from (68) that

$$I(z) = z^p - \sum_{k=p+n}^{\infty} \frac{\mu + p}{\mu + k} a_k z^k \quad (z \in U; \mu > -p). \tag{71}$$

To prove the result of Theorem 16, we need to find the smallest C such that

$$C \ge D + (1 - D)$$

$$\times \frac{(p + (k - p)\lambda_1)(a_1)_{k-p}(c_0)_{k-p}(\mu + p)(A - B)}{(p + (k - p)\lambda_0)(c_1)_{k-p}(a_0)_{k-p}(\mu + k)(1 - B)}$$

$$(k \ge p + n), \tag{72}$$

where we have used Lemma 3. In view of (67), it is easy to know that the right-hand side of (72) is a decreasing function of k. Therefore, we conclude that

$$I(z) \in F_{p,n}(a_1, c_1, \lambda_1, C(D), D), \tag{73}$$

where $C(D)$ is given by (69).

Furthermore, it can easily be verified that the result is sharp, with the extremal function

$$f(z) = z^p - \frac{p(c_0)_n (A - B)}{(p + n\lambda_0)(a_0)_n (1 - B)}$$

$$\times z^{p+n} \in F_{p,n}(a_0, c_0, \lambda_0, A, B). \tag{74}$$

$\square$

With the aid of Lemma 4 (instead of Lemma 3) and using the same steps as in the proof of Theorem 16, we can prove the following.

Theorem 17. *Let (67) in Theorem 16 be satisfied. If $f(z) \in G_{p,n}(a_0, c_0, \lambda_0, A, B)$, then the function $I(z)$ defined by (68) belongs to $G_{p,n}(a_1, c_1, \lambda_1, C(D), D)$, where $C(D)$ $(-1 \le D \le 0)$ is the same as in Theorem 16. The result is sharp for the function*

$$f(z) = z^p - \frac{p^2(c_0)_n (A - B)}{(p + n)(p + n\lambda_0)(a_0)_n (1 - B)}$$

$$\times z^{p+n} \in G_{p,n}(a_0, c_0, \lambda_0, A, B). \tag{75}$$

If we let

$$a_0 = c_0 = a_1 = c_1 = 1, \qquad \lambda_0 = \lambda_1 = \lambda \ge 0,$$

$$A = 1 - \frac{2\alpha}{p}, \qquad B = D = -1, \qquad C(D) = 1 - \frac{2\beta(\alpha)}{p}, \tag{76}$$

then Theorem 17 yields the following.

Corollary 18. *Let $f(z) \in G_p(n, \lambda, \alpha)$. Then the function $I(z)$ defined by (68) belongs to $G_p(n, \lambda, \beta(\alpha))$, where*

$$\beta(\alpha) = \frac{\alpha(\mu + p) + np}{\mu + p + n}. \tag{77}$$

The number $\beta(\alpha)$ cannot be increased for each $\alpha \in [0, p)$.

4. Partial Sums

In this section, we let $f(z) \in T_{p,n}$ be given by (6) and define the partial sums $s_1(z)$ and $s_m(z)$ by

$$s_1(z) = z^p, \qquad s_m(z) = z^p - \sum_{k=p+n}^{p+n+m-2} a_k z^k, \tag{78}$$

$$(m \in N \setminus \{1\}).$$

Also we make the notation simple by writing

$$\beta_k = \frac{(p + (k-p)\lambda)(a)_{k-p}(1-B)}{p(c)_{k-p}(A-B)} \tag{79}$$

$$(k = p+n, p+n+1, \ldots).$$

Theorem 19. *Let $f(z) \in F_{p,n}(a, c, \lambda, A, B)$ and $a \geq c > 0$. Then for $z \in U$, one has the following.*

$$\text{Re} \frac{f(z)}{s_m(z)} > 1 - \frac{1}{\beta_{p+n+m-1}} \quad (m \in N), \tag{80}$$

$$\text{Re} \frac{s_m(z)}{f(z)} > \frac{\beta_{p+n+m-1}}{1 + \beta_{p+n+m-1}} \quad (m \in N). \tag{81}$$

The results are sharp for each $m \in N$.

Proof. Let $a \geq c > 0$ and β_k be given by (79). Then $\beta_{k+1} \geq \beta_k \geq 1$ for $k \geq p+n$, and so it follows from Lemma 3 that

$$\sum_{k=p+n}^{p+n+m-2} a_k + \beta_{p+n+m-1} \cdot \sum_{k=p+n+m-1}^{\infty} a_k$$

$$\leq \sum_{k=p+n}^{\infty} \beta_k a_k \leq 1 \quad (m \geq 2), \tag{82}$$

for $f(z) \in F_{p,n}(a, c, \lambda, A, B)$.

If we put

$$p_1(z) = 1 + \beta_{p+n+m-1} \left(\frac{f(z)}{s_m(z)} - 1 \right)$$

$$= 1 - \frac{\beta_{p+n+m-1} \cdot \sum_{k=p+n+m-1}^{\infty} a_k z^{k-p}}{1 - \sum_{k=p+n}^{p+n+m-2} a_k z^{k-p}} \tag{83}$$

$$(z \in U; m \geq 2),$$

then $p(0) = 1$ and

$$\left| \frac{p_1(z) - 1}{p_1(z) + 1} \right|$$

$$\leq \frac{\beta_{p+n+m-1} \cdot \sum_{k=p+n+m-1}^{\infty} a_k}{2 - 2\sum_{k=p+n}^{p+n+m-2} a_k - \beta_{p+n+m-1} \cdot \sum_{k=p+n+m-1}^{\infty} a_k} \tag{84}$$

$$\leq 1 \quad (z \in U; m \geq 2)$$

because of (82). Hence, we have $\text{Re} \, p_1(z) > 0$ for $z \in U$, which implies that (80) holds true for $m \geq 2$.

Similarly, by setting

$$p_2(z) = \left(1 + \beta_{p+n+m-1}\right) \frac{s_m(z)}{f(z)} - \beta_{p+n+m-1} \quad (z \in U), \tag{85}$$

it follows from (82) that

$$\left| \frac{p_2(z) - 1}{p_2(z) + 1} \right|$$

$$\leq \frac{\left(1 + \beta_{p+n+m-1}\right) \cdot \sum_{k=p+n+m-1}^{\infty} a_k}{2 - 2\sum_{k=p+n}^{p+n+m-2} a_k - \left(\beta_{p+n+m-1} - 1\right) \cdot \sum_{k=p+n+m-1}^{\infty} a_k}$$

$$\leq 1 \quad (z \in U; m \geq 2). \tag{86}$$

Therefore, we see that $\text{Re} \, p_2(z) > 0$ for $z \in U$, that is, that (81) holds for $m \geq 2$.

For $m = 1$, replacing (82) by

$$\beta_{p+n} \sum_{k=p+n}^{\infty} a_k \leq \sum_{k=p+n}^{\infty} \beta_k a_k \leq 1 \tag{87}$$

and proceeding as the above, we know that (80) and (81) are also true.

Furthermore, by taking the function

$$f(z) = z^p - \frac{z^{p+n+m-1}}{\beta_{p+n+m-1}}, \tag{88}$$

we find that $s_m(z) = z^p$,

$$\text{Re} \frac{f(z)}{s_m(z)} \longrightarrow 1 - \frac{1}{\beta_{p+n+m-1}} \quad \text{as } z \longrightarrow 1,$$

$$\text{Re} \frac{s_m(z)}{f(z)} \longrightarrow \frac{\beta_{p+n+m-1}}{1 + \beta_{p+n+m-1}} \quad \text{as } z \longrightarrow \exp\left(\frac{\pi i}{n+m-1}\right). \tag{89}$$

The proof of Theorem 19 is thus completed. □

By virtue of Theorem 19 and Definition 2, we easily have the following.

Corollary 20. *Let $f(z) \in G_{p,n}(a, c, \lambda, A, B)$ and $a \geq c > 0$. Then we have*

$$\text{Re} \frac{f'(z)}{s'_m(z)} > 1 - \frac{1}{\beta_{p+n+m-1}} \quad (z \in U; m \in N),$$

$$\text{Re} \frac{s'_m(z)}{f'(z)} > \frac{\beta_{p+n+m-1}}{1 + \beta_{p+n+m-1}} \quad (z \in U; m \in N). \tag{90}$$

The results are sharp for each $m \in N$.

Theorem 21. *Let $f(z) \in F_{p,n}(a, c, \lambda, A, B)$ with $a \geq c > 0$ and $\lambda \geq 1$. Then one has*

$$\operatorname{Re} \frac{f'(z)}{s'_m(z)} > 1 - \frac{p+n+m-1}{p\beta_{p+n+m-1}} \quad (z \in U; \; m \in N), \quad (91)$$

$$\operatorname{Re} \frac{s'_m(z)}{f'(z)} > \frac{p\beta_{p+n+m-1}}{p+n+m-1+p\beta_{p+n+m-1}} \quad (z \in U; m \in N). \tag{92}$$

The results are sharp for each $m \in N$.

Proof. Let $a \geq c > 0$, $\lambda \geq 1$ and β_k be given by (79). Then it is easy to verify that

$$\frac{\beta_{k+1}}{k+1} \geq \frac{\beta_k}{k} \geq \frac{1}{p} \quad (k \geq p+n), \tag{93}$$

and hence we deduce from Lemma 3 that

$$\sum_{k=p+n}^{p+n+m-2} \frac{k}{p} a_k + \frac{\beta_{p+n+m-1}}{p+n+m-1} \cdot \sum_{k=p+n+m-1}^{\infty} ka_k$$
$$\leq \sum_{k=p+n}^{\infty} \beta_k a_k \leq 1 \quad (m \geq 2), \tag{94}$$

for $f(z) \in F_{p,n}(a, c, \lambda, A, B)$.

Defining the function $q_1(z)$ by

$$q_1(z) = 1 + \frac{p\beta_{p+n+m-1}}{p+n+m-1} \left(\frac{f'(z)}{s'_m(z)} - 1 \right) \quad (z \in U), \tag{95}$$

it follows from (94) that

$$\left| \frac{q_1(z) - 1}{q_1(z) + 1} \right|$$
$$\leq \left(\frac{p\beta_{p+n+m-1}}{p+n+m-1} \cdot \sum_{k=p+n+m-1}^{\infty} ka_k \right)$$
$$\times \left(2p - 2 \sum_{k=p+n}^{p+n+m-2} ka_k - \frac{p\beta_{p+n+m-1}}{p+n+m-1} \right. \tag{96}$$
$$\left. \cdot \sum_{k=p+n+m-1}^{\infty} ka_k \right)^{-1}$$
$$\leq 1 \quad (z \in U; \; m \geq 2).$$

This leads to the inequality (91) for $m \geq 2$.

Similarly, for the function $q_2(z)$ defined by

$$q_2(z) = \left(1 + \frac{p\beta_{p+n+m-1}}{p+n+m-1} \right) \frac{s'_m(z)}{f'(z)} - \frac{p\beta_{p+n+m-1}}{p+n+m-1}, \tag{97}$$

we deduce from (94) that

$$\left| \frac{q_2(z) - 1}{q_2(z) + 1} \right|$$
$$\leq \left(\left(1 + \frac{p\beta_{p+n+m-1}}{p+n+m-1} \right) \cdot \sum_{k=p+n+m-1}^{\infty} ka_k \right)$$
$$\times \left(2p - 2 \sum_{k=p+n}^{p+n+m-2} ka_k - \left(\frac{p\beta_{p+n+m-1}}{p+n+m-1} - 1 \right) \right.$$
$$\left. \cdot \sum_{k=p+n+m-1}^{\infty} ka_k \right)^{-1}$$
$$\leq 1 \quad (z \in U; \; m \geq 2). \tag{98}$$

This yields the inequality (92) for $m \geq 2$.

For $m = 1$, replacing (94) by

$$\frac{\beta_{p+n}}{p+n} \sum_{k=p+n}^{\infty} ka_k \leq \sum_{k=p+n}^{\infty} \beta_k a_k \leq 1, \tag{99}$$

we know that (91) and (92) are also true.

Furthermore, the bounds in (91) and (92) are the best possible for the function $f(z)$ defined by (88). □

Finally, Theorem 21 yields the following.

Corollary 22. *Let $f(z) \in G_{p,n}(a, c, \lambda, A, B)$ with $a \geq c > 0$ and $\lambda \geq 1$. Then*

$$\operatorname{Re} \frac{f'(z) + zf''(z)}{s'_m(z) + zs''_m(z)} > 1 - \frac{p+n+m-1}{p\beta_{p+n+m-1}}$$
$$(z \in U; \; m \in N),$$
$$\operatorname{Re} \frac{s'_m(z) + zs''_m(z)}{f'(z) + zf''(z)} > \frac{p\beta_{p+n+m-1}}{p+n+m-1+p\beta_{p+n+m-1}} \tag{100}$$
$$(z \in U; \; m \in N).$$

The results are sharp for each $m \in N$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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