Research Article

A Globally Convergent Inexact Newton-Like Cayley Transform Method for Inverse Eigenvalue Problems

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We propose an inexact Newton method for solving inverse eigenvalue problems (IEP). This method is globalized by employing the classical backtracking techniques. A global convergence analysis of this method is provided and the R-order convergence property is proved under some mild assumptions. Numerical examples demonstrate that the proposed method is very effective in solving the IEP with distinct eigenvalues.

1. Introduction

In the present paper, we consider inverse eigenvalue problems (IEP) which are defined as follows. Let $\mathbf{c} = (c_1, c_2, \ldots, c_n)^\top \in \mathbb{R}^n$ and let $\{A_i\}_{i=0}^n$ be a sequence of real symmetric $n \times n$ matrices. Define

$$A(\mathbf{c}) = A_0 + \sum_{i=1}^n c_i A_i \qquad (1)$$

and denote its eigenvalues by $\{\lambda_i(\mathbf{c})\}_{i=1}^n$ with the increasing order $\lambda_1(\mathbf{c}) \leq \lambda_2(\mathbf{c}) \leq \cdots \leq \lambda_n(\mathbf{c})$. Given n real numbers $\{\lambda_i^*\}_{i=1}^n$ which are arranged in increasing order $\lambda_1^* \leq \lambda_2^* \leq \cdots \leq \lambda_n^*$, the IEP is to find a vector $\mathbf{c}^* \in \mathbb{R}^n$ such that

$$\lambda_i(\mathbf{c}^*) = \lambda_i^*, \quad i = 1, 2, \ldots, n. \qquad (2)$$

Such vector $\mathbf{c}^*$ is called a solution of the IEP. This type of inverse problem arises in a variety of applications, for instance, the inverse Toeplitz eigenvalue problem [1, 2], inverse Sturm-Liouville's problem, inverse vibrating string problem, and the pole assignment problem; see [3–5] and the references therein for more details on these applications.

Define $f : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f(\mathbf{c}) = \left(\lambda_1(\mathbf{c}) - \lambda_1^*, \lambda_2(\mathbf{c}) - \lambda_2^*, \ldots, \lambda_n(\mathbf{c}) - \lambda_n^* \right)^\top. \qquad (3)$$

Then, solving IEP (2) is equivalent to solving the nonlinear equation $f(\mathbf{c}) = 0$ on $\mathbb{R}^n$. It is clear that $\mathbf{c}^*$ is a solution of the IEP if and only if $\mathbf{c}^*$ is a solution of the equation $f(\mathbf{c}) = 0$. Based on this equivalence, Newton's method can be applied to the IEP, and it converges quadratically [6]. As it is known, each iteration of Newton's method involves solving a complete eigenproblem for the matrix $A(\mathbf{c})$. To overcome this drawback, different Newton-like methods have been proposed and studied [7, 8]. To alleviate the over-solving problem, Bai et al. presented in [9] an inexact Cayley transform method for solving the nonlinear system $f(\mathbf{c}) = 0$. To avoid the computation of the approximate Jacobian equations, Shen and Li proposed in [10, 11] Ulm-like methods for solving the IEPs. However, all these numerical methods for solving the IEP converge only locally.

In this paper, we study the numerical methods with global convergence property for solving the IEP. Since the IEP is a nonlinear equation, we review some classical work on solving the general nonlinear equation $f(\mathbf{c}) = 0$. Among the inexact Newton-type methods where a line search procedure is used, an inexact Newton backtracking method was proposed in [12]. It performed backtracking along the inexact Newton step, and computational results on a large set of test problems have shown its robustness and efficiency [13, 14].

The purpose of the present paper is motivated by the inexact Newton backtracking method proposed in [12],

to propose an inexact Newton-type method which combines with the Cayley transform method for solving the IEP. In the backtracking procedure, we use the Rayleigh quotient instead of the classical merit function and therefore reduce the computational cost. Under the classical assumption, which is also used in [7, 9, 10], that the given eigenvalues are distinct and the Jacobian matrix $J(\mathbf{c}^*)$ is invertible, we show that this method is globally convergent. Some numerical examples are reported to illustrate the effectiveness of the proposed method with distinct eigenvalues.

The paper is organized as follows. In Section 2, a global inexact Newton-type algorithm is proposed. The global convergence analysis is given in Section 3. And finally in Section 4, some numerical examples are given to confirm the numerical effectiveness and the good performance of our algorithm.

2. A Globally Inexact Newton-Like Cayley Transform Method

In this section, we present our algorithm. Let $\mathbb{R}^{n \times n}$ denote the set of all real $n \times n$ matrices. Let $\| \cdot \|$ and $\| \cdot \|_F$ denote the 2-norm and the Frobenius norm in $\mathbb{R}^n$, respectively. The induced 2-norm in $\mathbb{R}^{n \times n}$ is also denoted by $\| \cdot \|$; that is,

$$\|A\| := \sup_{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}, \quad A \in \mathbb{R}^{n \times n}. \tag{4}$$

Then, we have $\|A\| \leq \|A\|_F$ for any $A \in \mathbb{R}^{n \times n}$. For $\mathbf{x} \in \mathbb{R}^n$ and a positive number r, we use $\mathbf{B}(\mathbf{x}, r)$ to stand for the open ball with radius r and center $\mathbf{x}$. Let $\lambda_1(\mathbf{c}) \leq \lambda_2(\mathbf{c}) \leq \cdots \leq \lambda_n(\mathbf{c})$ be the eigenvalues of matrix $A(\mathbf{c})$ and let $\{\mathbf{q}_i(\mathbf{c})\}_{i=1}^n$ be the normalized eigenvectors corresponding to $\{\lambda_i(\mathbf{c})\}_{i=1}^n$. Define $J(\mathbf{c}) = ([J(\mathbf{c})]_{ij})$ by

$$[J(\mathbf{c})]_{ij} := \mathbf{q}_i(\mathbf{c})^\top A_j \mathbf{q}_i(\mathbf{c}), \quad 1 \leq i, j \leq n. \tag{5}$$

Let $\{\lambda_i^*\}_{i=1}^n$ be given with $\lambda_1^* \leq \lambda_2^* \leq \cdots \leq \lambda_n^*$ and write $\boldsymbol{\lambda}^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)^\top$.

The Cayley transform method for computing approximately the eigenproblem of the matrix $A(\mathbf{c})$ was proposed in [6] and was applied in [9, 10]. We now recall this method and then apply it to our algorithm. Suppose that $\mathbf{c}^*$ is a solution of the IEP. Then, there exists an orthogonal matrix Q_* such that

$$Q_*^\top A(\mathbf{c}^*) Q_* = \mathrm{diag}\{\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*\} \triangleq \Lambda^*. \tag{6}$$

Assume that $\mathbf{c}$, $\boldsymbol{\rho}$, and P are the current approximations of $\mathbf{c}^*$, $\boldsymbol{\lambda}^*$, and Q_*, respectively. Define $e^Y := P^\top Q_*$, where Y is a skew-symmetric matrix. Then, (6) can be rewritten as

$$P^\top A(\mathbf{c}^*) P = e^Y \Lambda^* e^{-Y}$$

$$= \left(\mathbf{I} + Y + \frac{1}{2} Y^2 + \cdots \right) \Lambda^* \left(\mathbf{I} - Y + \frac{1}{2} Y^2 + \cdots \right)$$

$$= \Lambda^* + Y\Lambda^* - \Lambda^* Y + O\left(\|Y\|^2\right).$$

$$\tag{7}$$

Based on (7), we define the new approximation $\mathbf{c}^{\mathrm{new}}$ of $\mathbf{c}^*$ by neglecting the second-order terms in Y:

$$P^\top A\left(\mathbf{c}^{\mathrm{new}}\right) P = \Lambda^* + Y\Lambda^* - \Lambda^* Y. \tag{8}$$

By equating the diagonal elements in (8), we have

$$\lambda_i^* = \mathbf{p}_i(\mathbf{c})^\top A\left(\mathbf{c}^{\mathrm{new}}\right) \mathbf{p}_i(\mathbf{c}), \quad i = 1, 2, \ldots, n, \tag{9}$$

where $\{\mathbf{p}_i(\mathbf{c})\}_{i=1}^n$ are the column vectors of P. Thus, once we get $\mathbf{c}^{\mathrm{new}}$ by solving the Jacobian equation, we can obtain Y by equating the off-diagonal elements in (8); that is,

$$[Y]_{ij} := \frac{\mathbf{p}_i(\mathbf{c})^\top A\left(\mathbf{c}^{\mathrm{new}}\right) \mathbf{p}_j(\mathbf{c})}{\lambda_j^* - \lambda_i^*}, \quad 1 \leq i \neq j \leq n. \tag{10}$$

In order to update the new approximation P^{new} of Q_*, we construct an orthogonal matrix U using Cayley's transform

$$U := \left(\mathbf{I} + \frac{1}{2} Y \right) \left(\mathbf{I} - \frac{1}{2} Y \right)^{-1}, \tag{11}$$

and set $P^{\mathrm{new}} = PU$; that is, we can obtain P^{new} by solving

$$\left(\mathbf{I} + \frac{1}{2} Y \right) P^{\mathrm{new}} = \left(\mathbf{I} - \frac{1}{2} Y \right) P. \tag{12}$$

Finally, the new approximations of eigenvalues can be obtained by

$$\rho_i^{\mathrm{new}} := \mathbf{p}_i(\mathbf{c}^{\mathrm{new}})^\top A\left(\mathbf{c}^{\mathrm{new}}\right) \mathbf{p}_i(\mathbf{c}^{\mathrm{new}}), \quad i = 1, 2, \ldots, n, \tag{13}$$

where $\{\mathbf{p}_i(\mathbf{c}^{\mathrm{new}})\}_{i=1}^n$ are the column vectors of P^{new}.

Note that (12) can be computed as follows. Compute $H := (I - (1/2)Y)P^\top$ and let $\mathbf{h}_i$ be the ith column of H at first. Then, solve $\mathbf{w}_i$, $i = 1, 2, \ldots, n$, iteratively from the n linear systems:

$$\left(I + \frac{1}{2} Y \right) \mathbf{w}_i = \mathbf{h}_i, \quad i = 1, 2, \ldots, n. \tag{14}$$

Finally, set $P^{\mathrm{new}} := [\mathbf{w}_1, \ldots, \mathbf{w}_n]^\top$. Since P is an orthogonal matrix and Y is skew-symmetric matrix, we see that P^{new} must be orthogonal. To maintain the orthogonality of P^{new}, (14) cannot be solved inexactly. One could expect that it requires only few iterations to solve each system of (14). This is due to the fact that as $\{\mathbf{c}^k\}$ converges to $\mathbf{c}^*$, $\|Y_k\|$ converges to zero; see [6, (3.64)]. Consequently, the coefficient matrix on the left-hand side of (14) approaches the identity matrix in the limit.

For solving the general nonlinear equation $f(\mathbf{x}) = 0$, linesearch techniques [15] are often used to enlarge the convergence basin of a locally convergent method. They are based on a globally convergent method for a problem of the form $\min_{\mathbf{x} \in \mathbb{R}^n} M(\mathbf{x})$, where M is an appropriately chosen merit function whose global minimum is a zero of f. In these cases, for a given direction $\mathbf{s} \in \mathbb{R}^n$, we have the iteration form $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha \mathbf{s}$, where $\alpha \in (0, 1]$ is such that $M(\mathbf{x}_k + \alpha \mathbf{s}) < M(\mathbf{x}_k)$. The existence of such an α is ensured if there exists an $\alpha_0 > 0$ such that $M(\mathbf{x}_k + \alpha \mathbf{s}) < M(\mathbf{x}_k)$ for all $\alpha < \alpha_0$.

In typical linesearch strategies, the step length α is chosen by using so-called backtracking approach. Among the backtracking method, inexact Newton backtracking methods (INB) [12] is a globally convergent process where the kth iteration of an inexact Newton method is embedded in a backtracking strategy. The merit function M of INB usually used $M := \|f\|$; see, for example, [12–14, 16]. Thanks to (3), this will involve computing $\lambda_i(\mathbf{c}^k)$ of $A(\mathbf{c}^k)$ which are costly to compute. Our intention here is to replace them by the Rayleigh quotient (see (13)). In Section 3 we will show that this replacement retain superlinear and global convergence.

The details of our algorithm for solving the IEP are specified as Algorithm 1. In Step 7, the following sufficient decrease in the merit function $\|\boldsymbol{\rho}(\mathbf{c}) - \boldsymbol{\lambda}^*\|$ based on the Rayleigh quotient is provided:

$$\left\|\boldsymbol{\rho}\left(\mathbf{c}^k + \Delta\mathbf{c}^k\right) - \boldsymbol{\lambda}^*\right\| \le \left(1 - \xi\left(1 - \eta_k\right)\right)\left\|\boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^*\right\|. \quad (15)$$

The while loop in Step 7 is also called backtracking loop below. Note that if we set

$$\bar{\eta}_k^1 := \frac{\left\|\boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^*\right\|}{\left\|\boldsymbol{\lambda}^*\right\|}, \qquad \bar{\eta}_k^2 := \frac{\left\|\boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^*\right\|}{\left\|\boldsymbol{\rho}\left(\mathbf{c}^{k-1}\right) - \boldsymbol{\lambda}^*\right\|}, \quad (16)$$

when $\mathbf{c}^k$ is away from an accumulation point $\mathbf{c}^*$, then $\bar{\eta}_k^1 > 1$ or $\bar{\eta}_k^2 > 1$ may be possible. Accordingly, the forcing term $\bar{\eta}_k$ given in (20) guarantees that $\bar{\eta}_k \in [0, 1)$ for each k.

3. Convergence Analysis

In this section, we analyze the global behavior of Algorithm 1. We will show that if the given eigenvalues are distinct and if there exists an accumulation point $\mathbf{c}^*$ of $\{\mathbf{c}^k\}$ such that the Jacobian matrix $J(\mathbf{c}^*)$ is invertible, then the iterations are guaranteed to remain near $\mathbf{c}^*$ and $\boldsymbol{\rho}(\mathbf{c}^*) - \boldsymbol{\lambda}^* = 0$, $\mathbf{c}^k \to \mathbf{c}^*$ as $k \to \infty$. Furthermore, for k sufficiently large, we have the equality $\mathbf{c}^{k+1} = \mathbf{c}^k + \Delta\bar{\mathbf{c}}^k$. Thus, we obtain that the ultimate rate of convergence is β which depends on the choices of the $\bar{\eta}_k$ given in (20).

Algorithm 1 (Inexact Newton-Like Backtracking Cayley Transform Method for IEP). For any $\mathbf{c}^0 \in \mathbb{R}^n$, $\eta_{\max} \in [0, 1)$, $\xi \in (0, 1)$, $0 < \theta_{\min} < \theta_{\max} < 1$. Compute the orthonormal eigenvectors $\{\mathbf{q}_i(\mathbf{c}^0)\}_{i=1}^n$ and the eigenvalues $\{\lambda_i(\mathbf{c}^0)\}_{i=1}^n$ of $A(\mathbf{c}^0)$. Let

$$\begin{aligned}
P_0 &:= \left[\mathbf{p}_1\left(\mathbf{c}^0\right), \mathbf{p}_2\left(\mathbf{c}^0\right), \ldots, \mathbf{p}_n\left(\mathbf{c}^0\right)\right] \\
&= \left[\mathbf{q}_1\left(\mathbf{c}^0\right), \mathbf{q}_2\left(\mathbf{c}^0\right), \ldots, \mathbf{q}_n\left(\mathbf{c}^0\right)\right], \\
\boldsymbol{\rho}\left(\mathbf{c}^0\right) &:= \left[\rho_1\left(\mathbf{c}^0\right), \rho_2\left(\mathbf{c}^0\right), \ldots, \rho_n\left(\mathbf{c}^0\right)\right]^\top \\
&= \left[\lambda_1\left(\mathbf{c}^0\right), \lambda_2\left(\mathbf{c}^0\right), \ldots, \lambda_n\left(\mathbf{c}^0\right)\right]^\top.
\end{aligned} \quad (17)$$

For $k = 0, 1, 2, \ldots$ until convergence do the following.

Step 1. Form $[J_k]_{ij} = \mathbf{p}_i(\mathbf{c}^k)^\top A_j \mathbf{p}_i(\mathbf{c}^k)$ for $1 \le i, j \le n$.

Step 2. Solve $\Delta\bar{\mathbf{c}}^k$ inexactly from the approximate Jacobian equation:

$$J_k\Delta\bar{\mathbf{c}}^k + \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* = 0 \quad (18)$$

such that

$$\left\|J_k\Delta\bar{\mathbf{c}}^k + \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^*\right\| \le \bar{\eta}_k\left\|\boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^*\right\|, \quad (19)$$

where $\bar{\eta}_0 \in (0, 1)$ and

$$\bar{\eta}_k := \min\left\{\frac{\left\|\boldsymbol{\rho}(\mathbf{c}^k) - \boldsymbol{\lambda}^*\right\|^\beta}{\left\|\boldsymbol{\lambda}^*\right\|^\beta}, \frac{\left\|\boldsymbol{\rho}(\mathbf{c}^k) - \boldsymbol{\lambda}^*\right\|^\beta}{\left\|\boldsymbol{\rho}(\mathbf{c}^{k-1}) - \boldsymbol{\lambda}^*\right\|^\beta}, \eta_{\max}\right\}, \quad (20)$$

$$\beta \in (1, 2], k = 1, 2, \ldots.$$

Step 3. Set $\Delta\mathbf{c}^k = \Delta\bar{\mathbf{c}}^k$ and $\eta_k = \bar{\eta}_k$.

Step 4. Form the skew-symmetric matrix $\overline{Y}_k$ by (10) with $\mathbf{c}^{\text{new}} = \mathbf{c}^k + \Delta\mathbf{c}^k$.

Step 5. Compute matrix $\overline{P}_k := [\mathbf{p}_1(\mathbf{c}^k + \Delta\mathbf{c}^k), \ldots, \mathbf{p}_n(\mathbf{c}^k + \Delta\mathbf{c}^k)]$ by solving (12).

Step 6. Compute $\boldsymbol{\rho}(\mathbf{c}^k + \Delta\mathbf{c}^k) = (\rho_1(\mathbf{c}^k + \Delta\mathbf{c}^k), \ldots, \rho_n(\mathbf{c}^k + \Delta\mathbf{c}^k))^\top$ by (13).

Step 7. While $\|\boldsymbol{\rho}(\mathbf{c}^k + \Delta\mathbf{c}^k) - \boldsymbol{\lambda}^*\| > (1 - \xi(1 - \eta_k))\|\boldsymbol{\rho}(\mathbf{c}^k) - \boldsymbol{\lambda}^*\|$ do the following:

$$\begin{aligned}
&\text{choose } \theta \in \left[\theta_{\min}, \theta_{\max}\right], \\
&\text{then update } \Delta\mathbf{c}^k \longleftarrow \theta\Delta\mathbf{c}^k, \eta_k \longleftarrow 1 - \theta\left(1 - \eta_k\right).
\end{aligned} \quad (21)$$

Step 8. Set $\mathbf{c}^{k+1} = \mathbf{c}^k + \Delta\mathbf{c}^k$. As Steps 4–6, compute, respectively, the new approximations $P_{k+1} := [\mathbf{p}_1(\mathbf{c}^{k+1}), \ldots, \mathbf{p}_n(\mathbf{c}^{k+1})]$ and $\boldsymbol{\rho}(\mathbf{c}^{k+1}) := (\rho_1(\mathbf{c}^{k+1}), \ldots, \rho_n(\mathbf{c}^{k+1}))^\top$.

It is worth noting that if $\{\mathbf{c}^k\}$ has no accumulation point, $\{\mathbf{c}^k\}$ has one or more accumulation points and the Jacobian matrix is singular at each of them, or the vector $\Delta\bar{\mathbf{c}}^k$ computed by solving the Jacobian equation (18) is such that $\Delta\bar{\mathbf{c}}^k = 0$, then our algorithm fails.

It is clear that if $\boldsymbol{\rho}(\mathbf{c}^k) - \boldsymbol{\lambda}^* \to 0$ as $k \to \infty$ and $\mathbf{c}^*$ is an accumulation point of $\{\mathbf{c}^k\}$, then $\boldsymbol{\rho}(\mathbf{c}^*) - \boldsymbol{\lambda}^* = 0$. Let $\{P_k\}$ be generated by Algorithm 1 (see Step 7) and define $E_k := P_k - Q^*$ for each $k = 0, 1, \ldots$. The following lemma is taken from [17, Lemma 2].

Lemma 2 (see [17]). *For any $\mathbf{c}, \bar{\mathbf{c}} \in \mathbb{R}^n$, one has*

$$\|A(\mathbf{c}) - A(\bar{\mathbf{c}})\| \le L\|\mathbf{c} - \bar{\mathbf{c}}\|, \quad (22)$$

where $L := \left(\sum_{i=1}^n \|A_i\|^2\right)^{1/2}$.

Based on Lemma 2, the following lemma is a straightforward application of [9, Lemma 4].

Lemma 3. *Assume that* $\mathbf{c}^*$ *is an accumulation point of* $\{\mathbf{c}^k\}$. *Let the given eigenvalues* $\{\lambda_i^*\}_{i=1}^n$ *be distinct. Then*

$$\left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\| \leq \sqrt{n} L \left\| \mathbf{c}^k - \mathbf{c}^* \right\| + \mu_2 \left\| E_k \right\|, \qquad (23)$$

for any $k \in \mathbb{N}$, *where* $\mu_2 := 2\sqrt{n} \cdot \|A(\mathbf{c}^*)\|$.

As shown in [18, Theorem 2.3], in the case when the given eigenvalues $\{\lambda_i^*\}_{i=1}^n$ are distinct, the eigenvalues of $A(\mathbf{c})$ are distinct too for any point $\mathbf{c}$ in some neighborhood of $\mathbf{c}^*$. It follows that the function $f(\cdot)$ defined in (3) is analytic in the same neighborhood. However, if $\mathbf{c}$ is not near the solution $\mathbf{c}^*$, the analyticity of the function $f(\cdot)$ cannot be guaranteed.

For any symmetric matrix $X \in \mathbb{R}^{n \times n}$, set $\Lambda(X) := \mathrm{diag}(\lambda_1(X), \ldots, \lambda_n(X))$, where $\lambda_i(X)$, $i = 1, 2, \ldots, n$ are the eigenvalues of X. As proved by D. Sun and J. Sun [19, Theorem 4.7], $\Lambda(\cdot)$ is a strongly semismooth function. Based on this result, for any $\mathbf{c} \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ sufficiently small such that

$$\begin{aligned} &\left\| \boldsymbol{\rho}\left(\overline{\mathbf{c}}\right) - \boldsymbol{\rho}\left(\mathbf{c}\right) - J\left(\mathbf{c}\right)\left(\overline{\mathbf{c}} - \mathbf{c}\right) \right\| \\ &\qquad \leq \varepsilon \| \overline{\mathbf{c}} - \mathbf{c} \|^2, \quad \text{for any } \overline{\mathbf{c}} \in \mathbf{B}\left(\mathbf{c}, \delta\right). \end{aligned} \qquad (24)$$

The following lemma says that the backtracking loop in Step 7 of Algorithm 1 terminates after a finite number of steps.

Lemma 4. *There exists* $\eta_k^{\min} \in [0, 1)$ *such that, for any* $\eta_k \in [\eta_k^{\min}, 1)$, *there is* $\Delta\mathbf{c}^k$ *satisfying*

$$\left\| \boldsymbol{\rho}\left(\mathbf{c}^k + \Delta\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\| \leq \left[1 - \xi\left(1 - \eta_k\right)\right] \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\|. \quad (25)$$

Proof. By using the strong semismoothness of all eigenvalues of a real symmetric matrix [19], for any given $\varepsilon > 0$, there exists $\delta > 0$ sufficiently small such that

$$\left\| \boldsymbol{\rho}\left(\mathbf{c}^k + \Delta\mathbf{c}\right) - \boldsymbol{\rho}\left(\mathbf{c}^k\right) - J_k\Delta\mathbf{c} \right\| \leq \varepsilon \|\Delta\mathbf{c}\| \qquad (26)$$

whenever $\|\Delta\mathbf{c}\| \leq \delta$. Choose

$$\varepsilon = \frac{\left(1 - \xi\right)\left(1 - \overline{\eta}_k\right) \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\|}{\left\| \Delta\overline{\mathbf{c}}^k \right\|}, \qquad (27)$$

and set

$$\eta_k^{\min} := \max \left\{ \overline{\eta}_k, 1 - \frac{\left(1 - \overline{\eta}_k\right)\delta}{\left\| \Delta\overline{\mathbf{c}}^k \right\|} \right\}. \qquad (28)$$

For any $\eta_k \in [\eta_k^{\min}, 1)$, let $\Delta\mathbf{c}^k := ((1 - \eta_k)/(1 - \overline{\eta}_k))\Delta\overline{\mathbf{c}}^k$. Then, by the definition of $\eta_k^{\min}$ given in (28), we have

$$\begin{aligned} \left\| \Delta\mathbf{c}^k \right\| &= \frac{1 - \eta_k}{1 - \overline{\eta}_k} \left\| \Delta\overline{\mathbf{c}}^k \right\| \leq \frac{1 - \eta_k^{\min}}{1 - \overline{\eta}_k} \left\| \Delta\overline{\mathbf{c}}^k \right\| \\ &\leq \frac{\left(1 - \overline{\eta}_k\right)\delta}{\left\| \Delta\overline{\mathbf{c}}^k \right\|} \cdot \frac{1}{1 - \overline{\eta}_k} \left\| \Delta\overline{\mathbf{c}}^k \right\| = \delta. \end{aligned} \qquad (29)$$

On the other hand, by (19), one gets that

$$\begin{aligned} &\left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* + J_k\Delta\mathbf{c}^k \right\| \\ &\quad \leq \frac{\eta_k - \overline{\eta}_k}{1 - \overline{\eta}_k} \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\| + \frac{1 - \eta_k}{1 - \overline{\eta}_k} \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* + J_k\Delta\overline{\mathbf{c}}^k \right\| \\ &\quad \leq \frac{\eta_k - \overline{\eta}_k}{1 - \overline{\eta}_k} \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\| + \frac{1 - \eta_k}{1 - \overline{\eta}_k} \cdot \overline{\eta}_k \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\| \\ &\quad = \eta_k \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\|. \end{aligned} \qquad (30)$$

This together with (26) and (27) yields that

$$\begin{aligned} &\left\| \boldsymbol{\rho}\left(\mathbf{c}^k + \Delta\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\| \\ &\quad \leq \left\| \boldsymbol{\rho}\left(\mathbf{c}^k + \Delta\mathbf{c}^k\right) - \boldsymbol{\rho}\left(\mathbf{c}^k\right) - J_k\Delta\mathbf{c}^k \right\| + \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* + J_k\Delta\mathbf{c}^k \right\| \\ &\quad \leq \varepsilon \left\| \Delta\mathbf{c}^k \right\| + \eta_k \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\| \\ &\quad = \varepsilon \frac{1 - \eta_k}{1 - \overline{\eta}_k} \left\| \Delta\overline{\mathbf{c}}^k \right\| + \eta_k \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\| \\ &\quad = \left(1 - \xi\right)\left(1 - \eta_k\right) \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\| + \eta_k \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\| \\ &\quad = \left[1 - \xi\left(1 - \eta_k\right)\right] \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\|, \end{aligned} \qquad (31)$$

and the proof is completed. □

Next, we give sufficient conditions for Algorithm 1 not to break down in the backtracking loop in Step 7.

Lemma 5. *If* $\boldsymbol{\rho}(\mathbf{c}^k) - \boldsymbol{\lambda}^* \neq 0$ *and there exists* $\Gamma > 0$ *for which*

$$\left\| \Delta\mathbf{c}^k \right\| \leq \Gamma\left(1 - \eta_k\right) \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\|, \qquad (32)$$

then the backtracking loop terminates.

Proof. For constant Γ in (32) and the given $\xi \in (0, 1)$, choosing $\varepsilon = (1 - \xi)/\Gamma$, there exists $\delta > 0$ sufficiently small such that (26) holds whenever $\|\Delta\mathbf{c}\| \leq \delta$. We choose $\eta_k \in [\overline{\eta}_k, 1]$ satisfying

$$1 - \eta_k < \frac{\delta}{\Gamma \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\|}. \qquad (33)$$

It follows that

$$\left\| \Delta\mathbf{c}^k \right\| \leq \Gamma\left(1 - \eta_k\right) \left\| \boldsymbol{\rho}\left(\mathbf{c}^k\right) - \boldsymbol{\lambda}^* \right\| < \delta, \qquad (34)$$

which gives $\mathbf{c}^k + \Delta\mathbf{c}^k \in \mathbf{B}(\mathbf{c}^k, \delta)$. Thus, we have

$$\left\| \boldsymbol{\rho}\left(\mathbf{c} + \Delta\mathbf{c}^k\right) - \boldsymbol{\rho}\left(\mathbf{c}^k\right) - J_k\Delta\mathbf{c}^k \right\| \leq \frac{1 - \xi}{\Gamma} \left\| \Delta\mathbf{c}^k \right\|. \qquad (35)$$

This together with (19) gives that

$$
\begin{aligned}
&\left\| \boldsymbol{\rho} \left(\mathbf{c}^k + \Delta \mathbf{c}^k \right) - \boldsymbol{\lambda}^* \right\| \\
&\leq \left\| \boldsymbol{\rho} \left(\mathbf{c}^k \right) - \boldsymbol{\lambda}^* + J_k \Delta \mathbf{c}^k \right\| + \left\| \boldsymbol{\rho} \left(\mathbf{c}^k + \Delta \mathbf{c}^k \right) - \boldsymbol{\rho} \left(\mathbf{c}^k \right) - J_k \Delta \mathbf{c}^k \right\| \\
&\leq \eta_k \left\| \boldsymbol{\rho} \left(\mathbf{c}^k \right) - \boldsymbol{\lambda}^* \right\| + (1 - \xi)(1 - \eta_k) \left\| \boldsymbol{\rho} \left(\mathbf{c}^k \right) - \boldsymbol{\lambda}^* \right\| \\
&= (1 - \xi(1 - \eta_k)) \left\| \boldsymbol{\rho} \left(\mathbf{c}^k \right) - \boldsymbol{\lambda}^* \right\|.
\end{aligned}
\tag{36}
$$

It follows that the backtracking loop terminates. This completes the proof. $\qquad \square$

The next lemma gives condition under which (32) is satisfied.

Lemma 6. *Assume that the Jacobian matrix J_k is invertible and set $M_k := \|J_k^{-1}\|$. Then there exists Γ such that (32) holds.*

Proof. By using condition (19), one has that

$$
\begin{aligned}
&\left\| \Delta \mathbf{c}^k \right\| \\
&\leq \left\| J_k^{-1} \right\| \left\| J_k \Delta \mathbf{c}^k \right\| \\
&\leq M_k \left(\left\| \boldsymbol{\rho} \left(\mathbf{c}^k \right) - \boldsymbol{\lambda}^* \right\| + \left\| \boldsymbol{\rho} \left(\mathbf{c}^k \right) - \boldsymbol{\lambda}^* + J_k \Delta \mathbf{c}^k \right\| \right) \\
&\leq M_k (1 + \eta_k) \left\| \boldsymbol{\rho} \left(\mathbf{c}^k \right) - \boldsymbol{\lambda}^* \right\| \\
&= M_k \frac{1 + \eta_k}{1 - \eta_k} (1 - \eta_k) \left\| \boldsymbol{\rho} \left(\mathbf{c}^k \right) - \boldsymbol{\lambda}^* \right\| \\
&\leq M_k \frac{1 + \eta_{\max}}{1 - \eta_{\max}} (1 - \eta_k) \left\| \boldsymbol{\rho} \left(\mathbf{c}^k \right) - \boldsymbol{\lambda}^* \right\|.
\end{aligned}
\tag{37}
$$

We finish the proof by taking $\Gamma := M_k(1 + \eta_{\max})/(1 - \eta_{\max})$. $\qquad \square$

Lemmas 5 and 6 yield the result below.

Corollary 7. *Assume that $\boldsymbol{\rho}(\mathbf{c}^k) - \boldsymbol{\lambda}^* \neq 0$ and J_k are invertible. Set $M_k := \|J_k^{-1}\|$ and $\Gamma := M_k(1 + \eta_{\max})/(1 - \eta_{\max})$. Then, the backtracking loop in Step 7 of Algorithm 1 terminates with*

$$
1 - \eta_k \geq \min \left\{ 1 - \bar{\eta}_k, \frac{\delta \theta_{\min}}{\Gamma \left\| \boldsymbol{\rho} \left(\mathbf{c}^k \right) - \boldsymbol{\lambda}^* \right\|} \right\},
\tag{38}
$$

for any $\delta > 0$ small enough such that (26) holds whenever $\|\Delta \mathbf{c}\| \leq \delta$.

Proof. Suppose that η_k is the final value determined by the while-loop. If $\eta_k = \bar{\eta}_k$, then (38) is trivial. Assume that $\eta_k \neq \bar{\eta}_k$; that is, the body of the while-loop has been executed at least once. Denoting the penultimate value by η_k^-, then it follows from (33) that $1 - \eta_k^- \geq \delta/\Gamma\|\boldsymbol{\rho}(\mathbf{c}^k) - \boldsymbol{\lambda}^*\|$. Thus, we have

$$
1 - \eta_k = \theta (1 - \eta_k^-) \geq \theta_{\min} (1 - \eta_k^-) \geq \frac{\delta \theta_{\min}}{\Gamma \left\| \boldsymbol{\rho} \left(\mathbf{c}^k \right) - \boldsymbol{\lambda}^* \right\|}.
\tag{39}
$$

This completes the proof. $\qquad \square$

Lemma 8. *Assume that $\mathbf{c}^*$ is an accumulation point of $\{\mathbf{c}^k\}$ such that there exists a constant Γ independent of k for which*

$$
\left\| \Delta \mathbf{c}^k \right\| \leq \Gamma (1 - \eta_k) \left\| \boldsymbol{\rho} \left(\mathbf{c}^k \right) - \boldsymbol{\lambda}^* \right\|,
\tag{40}
$$

whenever $\mathbf{c}^k$ is sufficiently near $\mathbf{c}^*$ and k is sufficient large. Then $\mathbf{c}^k \to \mathbf{c}^*$ as $k \to \infty$.

Proof. Suppose that $\mathbf{c}^k \nrightarrow \mathbf{c}^*$. Then, there exists $\delta > 0$ sufficiently small such that there are infinitely many k for which $\mathbf{c}^k \notin \mathbf{B}(\mathbf{c}^*, \delta)$ and (40) hold whenever $\mathbf{c}^k \in \mathbf{B}(\mathbf{c}^*, \delta)$ for k sufficiently large.

Since $\mathbf{c}^*$ is an accumulation point of $\{\mathbf{c}^k\}$, there exists subsequence $\{\mathbf{c}^{k_i}\} \subset \{\mathbf{c}^k\}$ such that $\mathbf{c}^{k_i} \in \mathbf{B}(\mathbf{c}^*, \delta/i)$ for i sufficiently large. Choose $\ell_i > 0$ satisfying $k_i + \ell_i < k_{i+1}$ and $\mathbf{c}^{k_i + \ell_i} \notin \mathbf{B}(\mathbf{c}^*, \delta)$. It follows that

$$
\begin{aligned}
&\left\| \boldsymbol{\rho} \left(\mathbf{c}^{k_i + j - 1} \right) - \boldsymbol{\lambda}^* \right\| \\
&\leq \frac{1}{\Gamma \left(1 - \eta_{k_i + j - 1} \right)} \left(\left\| \boldsymbol{\rho} \left(\mathbf{c}^{k_i + j - 1} \right) - \boldsymbol{\lambda}^* \right\| - \left\| \boldsymbol{\rho} \left(\mathbf{c}^{k_i + j} \right) - \boldsymbol{\lambda}^* \right\| \right),
\end{aligned}
$$
$$
j = 1, \ldots, \ell_i.
\tag{41}
$$

Then, by (40), we have, for i sufficiently large,

$$
\begin{aligned}
\frac{\delta}{2} &\leq \left\| \mathbf{c}^{k_i + \ell_i} - \mathbf{c}^{k_i} \right\| \\
&\leq \sum_{k=k_i}^{k_i + \ell_i - 1} \left\| \Delta \mathbf{c}^k \right\| \leq \sum_{k=k_i}^{k_i + \ell_i - 1} \Gamma (1 - \eta_k) \left\| \boldsymbol{\rho} \left(\mathbf{c}^k \right) - \boldsymbol{\lambda}^* \right\| \\
&\leq \frac{\Gamma}{t} \left(\left\| \boldsymbol{\rho} \left(\mathbf{c}^{k_i} \right) - \boldsymbol{\lambda}^* \right\| - \left\| \boldsymbol{\rho} \left(\mathbf{c}^{k_{i+1}} \right) - \boldsymbol{\lambda}^* \right\| \right).
\end{aligned}
\tag{42}
$$

Note that $\mathbf{c}^{k_i} \to \mathbf{c}^*$ as $i \to \infty$. It follows that $\|\boldsymbol{\rho}(\mathbf{c}^{k_i}) - \boldsymbol{\lambda}^*\| - \|\boldsymbol{\rho}(\mathbf{c}^{k_{i+1}}) - \boldsymbol{\lambda}^*\| \to 0$, which is a contradiction. Therefore, $\mathbf{c}^k \to \mathbf{c}^*$ as $k \to \infty$. $\qquad \square$

Lemma 9. *Assume that $\mathbf{c}^*$ is an accumulation point of $\{\mathbf{c}^k\}$ such that $J(\mathbf{c}^*)$ is invertible. Set $M = \|J(\mathbf{c}^*)^{-1}\|$ and let δ_1 be such that $0 < \delta_1 \leq 1/2\mu_1 M$, where $\mu_1 := 2n \cdot \max_{1 \leq j \leq n} \|A_j\|$. Suppose that $\|E_k\| \leq \delta_1$ for k sufficiently large. Then, for all k sufficiently large,*

$$
\left\| J_k^{-1} \right\| \leq \frac{\left\| J(\mathbf{c}^*)^{-1} \right\|}{1 - \mu_1 \delta_1 \left\| J(\mathbf{c}^*)^{-1} \right\|} \leq 2M.
\tag{43}
$$

Proof. By the definitions of $[J_k]_{ij}$ and $[J(\mathbf{c}^*)]_{ij}$, for all k sufficiently large,

$$
\begin{aligned}
&\left| [J_k]_{ij} - [J(\mathbf{c}^*)]_{ij} \right| \\
&\leq \left| \mathbf{p}_i(\mathbf{c}^k)^\top A_j \mathbf{p}_i(\mathbf{c}^k) - \mathbf{p}_i(\mathbf{c}^k)^\top A_j \mathbf{q}_i(\mathbf{c}^*) \right| \\
&\quad + \left| \mathbf{p}_i(\mathbf{c}^k)^\top A_j \mathbf{q}_i(\mathbf{c}^*) - \mathbf{q}_i(\mathbf{c}^*)^\top A_j \mathbf{q}_i(\mathbf{c}^*) \right|
\end{aligned}
$$

$$\le \left\| \mathbf{p}_i(\mathbf{c}^k)^\top \right\| \left\| A_j \right\| \left\| \mathbf{p}_i(\mathbf{c}^k) - \mathbf{q}_i(\mathbf{c}^*) \right\|$$

$$+ \left\| \mathbf{p}_i(\mathbf{c}^k)^\top - \mathbf{q}_i(\mathbf{c}^*)^\top \right\| \left\| A_j \right\| \left\| \mathbf{q}_i(\mathbf{c}^*) \right\|$$

$$= 2 \left\| A_j \right\| \left\| \mathbf{p}_i(\mathbf{c}^k) - \mathbf{q}_i(\mathbf{c}^*) \right\|, \quad 1 \le i, j \le n.$$

$$(44)$$

Then, we have, for all k sufficiently large,

$$\left\| J_k - J(\mathbf{c}^*) \right\| \le \left\| J_k - J(\mathbf{c}^*) \right\|_F$$

$$\le 2n \cdot \max_{1 \le j \le n} \left\| A_j \right\| \cdot \max_{1 \le i \le n} \left\| \mathbf{p}_i(\mathbf{c}^k) - \mathbf{q}_i(\mathbf{c}^*) \right\|.$$

$$(45)$$

Noting that $\mathbf{p}_i(\mathbf{c}^k) - \mathbf{q}_i(\mathbf{c}^*)$ is the ith column of E_k, then $\|\mathbf{p}_i(\mathbf{c}^k) - \mathbf{q}_i(\mathbf{c}^*)\| \le \|E_k\|$ for $i = 1, \dots, n$. So, for all k sufficiently large,

$$\left\| J(\mathbf{c}^*)^{-1} \right\| \left\| J_k - J(\mathbf{c}^*) \right\| \le \mu_1 \delta_1 M \le \frac{1}{2} < 1. \quad (46)$$

It follows from Banach lemma that J_k is invertible and $\|J_k^{-1}\| \le 2M$ for all k sufficiently large. This completes the proof. □

Lemmas 6, 8, and 9 yield the result below.

Corollary 10. *Assume that $\mathbf{c}^*$ is an accumulation point of $\{\mathbf{c}^k\}$ such that $J(\mathbf{c}^*)$ is invertible. Set $M := \|J(\mathbf{c}^*)^{-1}\|$ and let δ_1 be determined by Lemma 9. Assume that $\|E_k\| \le \delta_1$ for k sufficiently large. Then $\mathbf{c}^k \to \mathbf{c}^*$ as $k \to \infty$.*

Proof. By Lemma 9, we know J_k is invertible and $\|J_k^{-1}\| \le 2M$ for all k sufficiently large. From the proof of Lemma 6, (40) holds for the constant $\Gamma = 2M(1 + \eta_{\max})/(1 - \eta_{\max})$ independent of k. Therefore, $\mathbf{c}^k \to \mathbf{c}^*$ as $k \to \infty$ follows from Lemma 8. □

Corollary 11. *Assume that $\mathbf{c}^*$ is an accumulation point of $\{\mathbf{c}^k\}$ such that $J(\mathbf{c}^*)$ is invertible. Then, $\rho(\mathbf{c}^k) - \lambda^* \to 0$ as $k \to \infty$. Moreover, for all k sufficiently large, one has $\eta_k = \overline{\eta}_k$.*

Proof. If $\Delta\mathbf{c}^k$ is computed in the backtracking loop, the backtracking terminates with η_k such that (38) holds. Since $\mathbf{c}^k \to \mathbf{c}^*$ by Corollary 10, we have $\mathbf{c}^k \in \mathbf{B}(\mathbf{c}^*, \delta)$ for all k sufficiently large. Thus, the series $\sum_{k=0}^\infty (1 - \eta_k)$ is divergent. Then, we have

$$\prod_{1 \le i < k} \left[1 - \xi(1 - \eta_i) \right] \le \prod_{1 \le i < k} e^{-\xi(1-\eta_i)}$$

$$= e^{-\xi \cdot \sum_{i=1}^{k-1}(1-\eta_i)} \longrightarrow 0, \quad k \longrightarrow \infty,$$

$$(47)$$

and so

$$\left\| \rho(\mathbf{c}^k) - \lambda^* \right\|$$

$$\le \left[1 - \xi(1 - \eta_{k-1}) \right] \left\| \rho(\mathbf{c}^{k-1}) - \lambda^* \right\|$$

$$\le \left\| \rho(\mathbf{c}^0) - \lambda^* \right\| \prod_{0 \le i < k} \left[1 - \xi(1 - \eta_i) \right] \longrightarrow 0, \quad k \longrightarrow \infty.$$

$$(48)$$

This together with (38) yields that $\eta_k = \overline{\eta}_k$ for all k sufficiently large. □

Lemma 12. *Assume that $\mathbf{c}^*$ is an accumulation point of $\{\mathbf{c}^k\}$ such that $J(\mathbf{c}^*)$ is invertible. Then, there exists $\delta_2 > 0$ such that*

$$\left\| \rho(\mathbf{c}^k) - \lambda^* - J_k(\mathbf{c}^k - \mathbf{c}^*) \right\| \le \mu_3 \|E_k\|_F^2 \quad (49)$$

whenever $\mathbf{c}^k \in \mathbf{B}(\mathbf{c}^*, \delta_2)$ and k sufficiently large, where $\mu_3 := 2\max_{1 \le i \le n} |\lambda_i^*|$.

Proof. Set $H_k := Q(\mathbf{c}^*)^\top P_k - \mathbf{I}$. Then, we have, for all k sufficiently large,

$$P_k^\top A(\mathbf{c}^*) P_k = (\mathbf{I} + H_k)^\top \Lambda^* (\mathbf{I} + H_k)$$

$$= \Lambda^* + \Lambda^* H_k + H_k^\top \Lambda^* + H_k^\top \Lambda^* H_k. \quad (50)$$

It follows that, for all k sufficiently large,

$$\mathbf{p}_i(\mathbf{c}^k)^\top A(\mathbf{c}^*) \mathbf{p}_i(\mathbf{c}^k)$$

$$= \lambda_i^* + 2\lambda_i^* [H_k]_{ii} + \sum_{j=1}^n \lambda_j^* [H_k]_{ij}^2, \quad i = 1, 2, \dots, n. \quad (51)$$

Set

$$v_i := 2\lambda_i^* [H_k]_{ii} + \sum_{j=1}^n \lambda_j^* [H_k]_{ij}^2, \quad i = 1, 2, \dots, n, \quad (52)$$

and write $\mathbf{v} = (v_1, v_2, \dots, v_n)^\top$. It follows that $J_k \mathbf{c}^* = \lambda^* - \mathbf{a}^k + \mathbf{v}$, where

$$\left[\mathbf{a}^k \right]_i = \mathbf{p}_i(\mathbf{c}^k)^\top A_0 \mathbf{p}_i(\mathbf{c}^k), \quad i = 1, 2, \dots, n. \quad (53)$$

In view of that $J_k \mathbf{c}^k = \rho(\mathbf{c}^k) - \mathbf{a}^k$ for any $k \in \mathbb{N}$, one has that, for all k sufficiently large,

$$\mathbf{v} = \rho(\mathbf{c}^k) - \lambda^* - J_k(\mathbf{c}^k - \mathbf{c}^*). \quad (54)$$

Since

$$\mathbf{I} + H_k H_k^\top + H_k H_k^\top = (\mathbf{I} + H_k)(\mathbf{I} + H_k)^\top$$

$$= Q(\mathbf{c}^*)^\top P_k P_k^\top Q(\mathbf{c}^*) = \mathbf{I}, \quad (55)$$

we have

$$[H_k]_{ii} = -\frac{1}{2} \sum_{j=1}^n [H_k]_{ij}^2, \quad i = 1, 2, \dots, n. \quad (56)$$

It follows from (52) that, for all k sufficiently large,

$$
\begin{aligned}
v_i^2 &= \left(2\lambda_i^*[H_k]_{ii} + \sum_{j=1}^n \lambda_j^*[H_k]_{ij}^2 \right)^2 \\
&\leq 2\left[4(\lambda_i^*)^2[H_k]_{ii}^2 + \left(\sum_{j=1}^n \lambda_j^*[H_k]_{ij}^2 \right)^2 \right].
\end{aligned}
\tag{57}
$$

Combining (56) with (57), one has that, for all k sufficiently large,

$$
\begin{aligned}
\sum_{i=1}^n v_i^2 &\leq 2\left[\sum_{i=1}^n 4(\lambda_i^*)^2[H_k]_{ii}^2 + \sum_{i=1}^n \left(\sum_{j=1}^n \lambda_j^*[H_k]_{ij}^2 \right)^2 \right] \\
&= 2\left[\sum_{i=1}^n (\lambda_i^*)^2 \left(\sum_{j=1}^n [H_k]_{ij}^2 \right)^2 + \sum_{i=1}^n \left(\sum_{j=1}^n \lambda_j^*[H_k]_{ij}^2 \right)^2 \right] \\
&\leq 2\left[\max_{1\leq i\leq n}|\lambda_i^*|^2 \cdot \sum_{i=1}^n \left(\sum_{j=1}^n [H_k]_{ij}^2 \right)^2 \right. \\
&\qquad \left. + \max_{1\leq i\leq n}|\lambda_i^*|^2 \cdot \sum_{i=1}^n \left(\sum_{j=1}^n [H_k]_{ij}^2 \right)^2 \right] \\
&\leq 4\max_{1\leq i\leq n}|\lambda_i^*|^2 \cdot \left(\sum_{i=1}^n \sum_{j=1}^n [H_k]_{ij}^2 \right)^2 \\
&= 4\max_{1\leq i\leq n}|\lambda_i^*|^2 \cdot \|H_k\|_F^4,
\end{aligned}
\tag{58}
$$

which gives

$$
\|\mathbf{v}\| \leq 2\max_{1\leq i\leq n}|\lambda_i^*| \cdot \|H_k\|_F^2.
\tag{59}
$$

Note that,

$$
\|H_k\|_F = \left\| Q(\mathbf{c}^*)^\top P_k - \mathbf{I} \right\|_F = \|P_k - Q(\mathbf{c}^*)\|_F = \|E_k\|_F.
\tag{60}
$$

Therefore, we obtain from (54) that (49) holds for all k sufficiently large. This completes the proof. □

Lemma 13. *Assume that $\mathbf{c}^*$ is an accumulation point of $\{\mathbf{c}^k\}$ such that $J(\mathbf{c}^*)$ is invertible. Then, there exist $\delta_3 > 0$, $\delta_4 \in (0, \delta_1]$ sufficiently small such that $\|E_k\| \leq \delta_4$ and*

$$
\left\| \mathbf{c}^{k+1} - \mathbf{c}^* \right\| \leq 2M\left(\mu_3\|E_k\|_F^2 + \left(\frac{2}{\mu_3} \right)^\beta \|\rho(\mathbf{c}^k) - \lambda^*\|^\beta \right)
\tag{61}
$$

whenever $\mathbf{c}^k \in \mathbf{B}(\mathbf{c}^*, \delta_3)$ and k sufficiently large, where δ_1 is determined by Lemma 9.

Proof. Thanks to Corollary 11, we have $\eta_k = \overline{\eta}_k$ and $\Delta\mathbf{c}^k = \Delta\overline{\mathbf{c}}^k$ for all k sufficiently large. Thus, the Jacobian equation (18) is equivalent to, for all k sufficiently large,

$$
J_k\mathbf{c}^{k+1} - \lambda^* + \mathbf{a}^k = 0.
\tag{62}
$$

Assume that the residual of this approximate Jacobian equation is defined by $\mathbf{r}^k$; that is, for all k sufficiently large,

$$
\mathbf{r}^k = J_k\mathbf{c}^{k+1} - \lambda^* + \mathbf{a}^k.
\tag{63}
$$

This together with $J_k\mathbf{c}^* - \lambda^* + \mathbf{a}^k = \mathbf{v}$ gives, for all k sufficiently large,

$$
J_k\left(\mathbf{c}^* - \mathbf{c}^{k+1} \right) = \mathbf{v} - \mathbf{r}^k.
\tag{64}
$$

By (19), Lemmas 9 and 12, we obtain, for all k sufficiently large,

$$
\begin{aligned}
\left\| \mathbf{c}^{k+1} - \mathbf{c}^* \right\| &\leq \left\| J_k^{-1} \right\| \left(\|\mathbf{v}\| + \|\mathbf{r}^k\| \right) \\
&\leq 2M\left(\mu_3\|E_k\|_F^2 + \frac{\|\rho(\mathbf{c}^k) - \lambda^*\|^{\beta+1}}{\|\lambda^*\|^\beta} \right).
\end{aligned}
\tag{65}
$$

It follows from Lemma 3 that

$$
\left\| \rho(\mathbf{c}^k) - \lambda^* \right\| \leq \sqrt{n}L\left\| \mathbf{c}^k - \mathbf{c}^* \right\| + \mu_2\|E_k\|, \quad \forall k \in \mathbb{N}.
\tag{66}
$$

Thus, we can choose $\delta_3 > 0$ and $0 < \delta_4 \leq \delta_1$ sufficiently small such that

$$
\sqrt{n}L\delta_3 + \mu_2\delta_4 \leq 1,
\tag{67}
$$

whenever $\|\mathbf{c}^k - \mathbf{c}^*\| \leq \delta_3$ and $\|E_k\|_F \leq \delta_4$. Therefore, combining this with the definition of μ_3 given in Lemma 12, (61) follows. □

Lemma 14 (see [6]). *There exist two positive numbers δ_5 and ω_1 such that, for any orthogonal matrix P with $\|P - Q(\mathbf{c}^*)\| < \delta_5$, the skew-symmetric X defined by $e^X := P^\top Q(\mathbf{c}^*)$ satisfies $\|X\| \leq \omega_1\|P - Q(\mathbf{c}^*)\|$.*

Based on Lemma 14, by using the similar arguments in the proof of [10, Lemma 5], we can obtain the following lemma. If we write $e^{X_k} := P_k^\top Q^*$, then there exists $C > 0$ such that $\|Y_k\| \leq C(\|\mathbf{c}^{k+1} - \mathbf{c}^*\| + \|X_k\|)$.

Lemma 15. *Suppose that the given eigenvalues $\{\lambda_i^*\}_{i=1}^n$ are distinct and the Jacobian matrix $J(\mathbf{c}^*)$ is invertible. Then, there exist $\omega_2 > 0$ and $0 < \delta_6 < \min\{\delta_5, 1/(1 + \omega_1)C\}$ such that, for k sufficiently large, if $\|\mathbf{c}^{k+1} - \mathbf{c}^*\| \leq \delta_6$ and $\|E_k\| \leq \delta_6$, then*

$$
\|E_{k+1}\| \leq \omega_2\left(\left\| \mathbf{c}^{k+1} - \mathbf{c}^* \right\| + \|E_k\|^2 \right),
\tag{68}
$$

where ω_1 is determined by Lemma 14.

In order to prove our global convergence result for Algorithm 1, we introduce some notations. Let $\zeta_1 := 2(\sqrt{n}L + \mu_2)/\mu_3$ and $\zeta_2 := 2M(\mu_3 + \zeta_1^\beta)$. Set

$$
\begin{aligned}
\tau := \min\Big\{ &1, \left(2M\left(\mu_3 + \zeta_1^\beta \right) \right)^{-(1/(\beta-1))} \\
&\times \left(\omega_2\left(\zeta_2 + 1 \right) \right)^{-(1/(\beta-1))} \Big\},
\end{aligned}
\tag{69}
$$

$$
\delta := \min\{\tau, \delta_2, \delta_3, \delta_4, \delta_6\}.
\tag{70}
$$

Our main global convergence result is as follows.

Theorem 16. *Assume that $\{\mathbf{c}^k\}$ is generated by Algorithm 1. Suppose that the given eigenvalues $\{\lambda_i^*\}_{i=1}^n$ are distinct and $\mathbf{c}^*$ is an accumulation point of $\{\mathbf{c}^k\}$ such that $J(\mathbf{c}^*)$ is invertible. Then, $\boldsymbol{\rho}(\mathbf{c}^k) - \boldsymbol{\lambda}^* \to 0$ and $\mathbf{c}^k \to \mathbf{c}^*$ as $k \to \infty$. Moreover, the convergence is of R-order β.*

Proof. It follows immediately from Corollaries 10 and 11 that $\boldsymbol{\rho}(\mathbf{c}^k) - \boldsymbol{\lambda}^* \to 0$ and $\mathbf{c}^k \to \mathbf{c}^*$ as $k \to \infty$. For the τ given in (69), there exists k_0 sufficiently large such that $\|\mathbf{c}^{k_0} - \mathbf{c}^*\| \le \tau$ and $\|E_{k_0}\| \le \tau$. Set $\gamma := \delta/\tau$, where δ and τ are given in (70) and (69), respectively. Then, $\gamma \le 1$. We will show that, for all $k \ge k_0$ sufficiently large,

$$\left\| \mathbf{c}^k - \mathbf{c}^* \right\| \le \tau \cdot \gamma^{\beta^{k-k_0}}, \tag{71}$$

$$\left\| E_k \right\| \le \tau \cdot \gamma^{\beta^{k-k_0}}. \tag{72}$$

Suppose that (71) and (72) hold for some $k = \ell \ge k_0$. Consider the case $k = \ell + 1$. Thanks to Lemma 3, we have, for all $\ell \ge k_0$,

$$
\begin{aligned}
\left\| \boldsymbol{\rho}\left(\mathbf{c}^\ell\right) - \boldsymbol{\lambda}^* \right\| &\le \sqrt{n}L \left\| \mathbf{c}^\ell - \mathbf{c}^* \right\| + \mu_2 \left\| E_\ell \right\| \\
&\le \sqrt{n}L\tau \cdot \gamma^{\beta^{\ell-k_0}} + \mu_2\tau \cdot \gamma^{\beta^{\ell-k_0}} \\
&= \left(\sqrt{n}L + \mu_2 \right) \tau \cdot \gamma^{\beta^{\ell-k_0}}.
\end{aligned}
\tag{73}
$$

Then, by using Lemma 13, one has that, for all $\ell \ge k_0$,

$$
\begin{aligned}
&\left\| \mathbf{c}^{\ell+1} - \mathbf{c}^* \right\| \\
&\le 2M \left(\mu_3 \left\| E_\ell \right\|_F^2 + \left(\frac{2}{\mu_3} \right)^\beta \left\| \boldsymbol{\rho}\left(\mathbf{c}^\ell\right) - \boldsymbol{\lambda}^* \right\| \right) \\
&\le 2M\mu_3\tau^2 \left(\gamma^{\beta^{\ell-k_0}} \right)^2 + 2M \left(\frac{2}{\mu_3} \right)^\beta \left(\sqrt{n}L + \mu_3 \right)^\beta \tau^\beta \gamma^{\beta^{\ell-k_0+1}} \\
&\le 2M \left[\mu_3 + \left(\frac{2\left(\sqrt{n}L + \mu_2\right)}{\mu_3} \right)^\beta \right] \tau^\beta \gamma^{\beta^{\ell-k_0+1}} \\
&\le \tau \gamma^{\beta^{\ell-k_0+1}},
\end{aligned}
\tag{74}
$$

where the last inequality follows from the definition of τ in (69). By Lemma 15, we have, for all $\ell \ge k_0$,

$$
\begin{aligned}
\left\| E_{\ell+1} \right\| &\le \omega_2 \left(\left\| \mathbf{c}^{\ell+1} - \mathbf{c}^* \right\| + \left\| E_\ell \right\|^2 \right) \\
&\le 2M\omega_2 \left(\mu_3 + \zeta_1^\beta \right) \tau^\beta \gamma^{\beta^{\ell-k_0+1}} + \omega_2\tau^2 \left(\gamma^{\beta^{\ell-k_0}} \right)^2 \\
&\le \omega \left(2M \left(\mu_3 + \zeta_1^\beta \right) + 1 \right) \tau^\beta \cdot \gamma^{\beta^{\ell-k_0+1}} \\
&= \omega_2 \left(\zeta_2 + 1 \right) \tau^\beta \cdot \gamma^{\beta^{\ell-k_0+1}} \le \tau \cdot \gamma^{\beta^{\ell-k_0+1}}.
\end{aligned}
\tag{75}
$$

Therefore, we conclude that (71) and (72) hold for all $k \ge k_0$. Moreover, we see from (71) that $\mathbf{c}^k$ converges to $\mathbf{c}^*$ with R-order β. This completes the proof. □

If $\{\mathbf{c}^k\}$ generated by Algorithm 1 converges to a solution at which the Jacobian matrix is invertible, then the ultimate rate of convergence is governed by the choices of the η_k ($k = 0, 1, \ldots$) as in the local theory of [9].

4. Numerical Examples

In this section, we illustrate the effectiveness of Algorithm 1 in solving IEP on three examples. The tests were carried out in MATLAB 7.0 running on a PC Intel Pentium P6200 of 2.13 GHz CPU.

The given parameters used in our algorithm were $\eta_0 = 0.5$, $\eta_{\max} = 0.9$, $\xi = 10^{-4}$, $\theta_{\min} = 0.1$, and $\theta_{\max} = 0.9$. In the while loop, we choose $\theta \in [\theta_{\min}, \theta_{\max}]$ to minimize $\|\boldsymbol{\rho}(\mathbf{c}^k + \theta\Delta\mathbf{c}^k) - \boldsymbol{\lambda}^*\|$ if 80 iterations of the backtracking loop fail to produce the sufficient decrease in $\|\boldsymbol{\rho}(\mathbf{c}) - \boldsymbol{\lambda}^*\|$.

Linear systems (14) and (19) are all solved iteratively by the QMR method [20] using the MATLAB qmr function. In order to guarantee the orthogonality of P^{new} in (14), this system is solved up to machine precision eps ($\approx 2.2 \times 10^{-16}$). The inner loop stopping tolerance for (18) is given by (20). The stopping criterion of the outer iteration in our algorithm is

$$\left\| P_k^\top A(\mathbf{c}^k)P_k - \Lambda_* \right\|_F \le 10^{-10}. \tag{76}$$

Example 1. This is an inverse Toeplitz eigenvalue problem (see [2] for more details on this inverse problem) with distinct eigenvalues. The basis matrices $\{A_i\}_{i=1}^5$ are given as follows:

$$
A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},
$$

$$
A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{77}
$$

$$
A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.
$$

The given real eigenvalues and a solution, respectively, are

$$
\begin{aligned}
\boldsymbol{\lambda}^* &= (-5.2361, -1.5876, -0.7639, -0.5555, 18.1431)^\top, \\
\mathbf{c}^* &= (2, 3, 4, 5, 6)^\top.
\end{aligned}
\tag{78}
$$

TABLE 1: Numerical results for Example 1.

$\mathbf{c}^0$		$\beta = 1.5$	$\beta = 1.8$	$\beta = 2.0$
(a)	errs	$1.0659e - 02$	$1.0659e - 02$	$1.0659e - 02$
		$1.1323e - 05$	$1.1323e - 05$	$1.1323e - 05$
		$8.0567e - 12$	$8.0567e - 12$	$8.0567e - 12$
	ite.	6	6	6
	$\mathbf{c}^*$	$(2.0000, 3.2926, 3.4471, 4.9014, 6.5529)^\top$		
(b)	errs	$3.8846e - 03$	$3.8846e - 03$	$3.8846e - 03$
		$1.2491e - 06$	$1.2491e - 06$	$1.2491e - 06$
		$7.9990e - 14$	$7.9990e - 14$	$7.9990e - 14$
	ite.	9	9	9
	$\mathbf{c}^*$	$(2, 3, 4, 5, 6)^\top$		
(c)	errs	$5.0389e - 05$	$5.0389e - 05$	$5.0389e - 05$
		$3.5838e - 10$	$3.5838e - 10$	$3.5838e - 10$
		$5.0337e - 15$	$5.0337e - 15$	$5.0337e - 15$
	ite.	13	13	13
	$\mathbf{c}^*$	$(2.0000, 3.2926, 3.4471, 4.9014, 6.5529)^\top$		
(d)	errs	$1.3806e - 04$	$1.3806e - 04$	$1.3806e - 04$
		$6.4118e - 09$	$6.4118e - 09$	$6.4118e - 09$
		$9.8493e - 15$	$9.8493e - 15$	$9.8493e - 15$
	ite.	75	75	75
	$\mathbf{c}^*$	$(2.0000, 3.2926, 3.4471, 4.9014, 6.5529)^\top$		
(e)	errs	$1.2028e - 04$	$7.7624e - 03$	$7.7624e - 03$
		$3.8608e - 10$	$6.1243e - 06$	$6.1243e - 06$
		$5.3680e - 15$	$2.1634e - 12$	$2.1634e - 12$
	ite.	13	5	5
	$\mathbf{c}^*$	$(2, 3, 4, 5, 6)^\top$		

In Table 1, we report our numerical results for various starting points:

$$\text{(a)} \quad \mathbf{c}^0 = (1, 2, 3, 4, 5)^\top,$$

$$\text{(b)} \quad \mathbf{c}^0 = (1, 5, 10, 15, 20)^\top,$$

$$\text{(c)} \quad \mathbf{c}^0 = (11, 12, 13, 14, 15)^\top, \tag{79}$$

$$\text{(d)} \quad \mathbf{c}^0 = (21, 38, 46, 63, 81)^\top,$$

$$\text{(e)} \quad \mathbf{c}^0 = (101, 112, 123, 134, 145)^\top,$$

where $\mathbf{c}^0$, errs, ite. and $\mathbf{c}^*$ stand for the starting point, the error value of the left-hand side of (76) for the last three iterates of the algorithm, the number of outer iteration, and the accumulation point corresponding to the starting point.

Example 2. This is a Toeplitz-plus-Hankel inverse eigenvalue problem (see [1] for more details on this inverse problem)

with distinct eigenvalues. The basis matrices $\{A_i\}_{i=1}^7$ are given as follows:

$$A_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\tag{80}$$

The given real eigenvalues are

$$\boldsymbol{\lambda}^* = (-35.4513, -13.6805, -9.5675,$$
$$-8.5489, 8.7666, 11.8220, 26596)^\top,$$

(81)

and $\mathbf{c}^* = (2, 3, 4, 5, 6, 7, 8)^\top$ is a solution. In Table 2, we report our numerical results for various starting points:

$$(a) \quad \mathbf{c}^0 = (1, 2, 3, 4, 5, 6, 7)^\top,$$

$$(b) \quad \mathbf{c}^0 = (1, 3, 5, 7, 9, 11, 13)^\top,$$

$$(c) \quad \mathbf{c}^0 = (11, 13, 15, 17, 19, 21, 23)^\top,$$ (82)

$$(d) \quad \mathbf{c}^0 = (50, 52, 56, 58, 62, 65, 68)^\top,$$

$$(e) \quad \mathbf{c}^0 = (101, 102, 103, 104, 106, 108, 110)^\top.$$

Example 3 (see [10]). Given $B = [b_{ij}]_{8 \times 8} := \mathbf{I} + VV^\top$, where

$$V = \begin{bmatrix} 1 & -1 & -3 & -5 & -6 \\ 1 & 1 & -2 & -5 & -17 \\ 1 & -1 & -1 & 5 & 18 \\ 1 & 1 & 1 & 2 & 0 \\ 1 & -1 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 & -1 \\ 2.5 & 0.2 & 0.3 & 0.5 & 0.6 \\ 2 & -0.2 & 0.3 & 0.5 & 0.8 \end{bmatrix}_{8 \times 5},$$

(83)

the basis matrices $\{A_k\}_{k=1}^8$ are defined from B as follows:

$$A_k = b_{kk} e_k e_k^\top + \sum_{j=1}^{k-1} b_{kj} \left(e_k e_k^\top + e_j e_k^\top \right), \quad k = 1, 2, \ldots, 8,$$

(84)

where e_k is the kth column of the identity matrix $\mathbf{I}$. The given real eigenvalues and a solution, respectively, are

$$\boldsymbol{\lambda}^* = (1.0e + 04) \times (-1.1672, -0.3139, -0.0140, -0.0008,$$
$$0.0003, 0.0075, 0.0327, 1.7372)^\top,$$

(85)

and $\mathbf{c}^* = (2, 3, 4, 5, 6, 7, 8, 9)^\top$. Table 3 shows the numerical results for the following various starting points:

$$(a) \quad \mathbf{c}^0 = (1, 1, 1, 1, 1, 1, 1, 1)^\top,$$

$$(b) \quad \mathbf{c}^0 = (2.5, 2.5, 2.5, 2.5, 2.5, 2.5, 2.5, 2.5)^\top,$$

(86)

$$(c) \quad \mathbf{c}^0 = (1, 2, 3, 4, 5, 6, 7, 8)^\top,$$

$$(d) \quad \mathbf{c}^0 = (2, 4, 6, 8, 10, 12, 14, 16)^\top,$$

TABLE 2: Numerical results for Example 2.

$\mathbf{c}^0$		$\beta = 1.5$	$\beta = 1.8$	$\beta = 2.0$
(a)		8.2730e − 03	8.2730e − 03	8.2730e − 03
	errs	2.2866e − 05	2.2866e − 05	2.2866e − 05
		5.5845e − 11	5.5845e − 11	5.5845e − 11
	ite.	6	6	6
	$\mathbf{c}^*$		$(2, 3, 4, 5, 6, 7, 8)^\top$	
(b)		8.2129e − 03	4.0190e − 03	1.7235e − 04
	errs	2.0027e − 05	1.5988e − 06	3.7839e − 09
		5.0828e − 11	2.8276e − 13	1.9386e − 14
	ite.	7	7	7
	$\mathbf{c}^*$	$(2.8703, 2.0639, 4.9434, 4.1053, 6.7530, 6.3287, 8.4794)^\top$		
(c)		3.4941e − 03	3.4941e − 03	3.4941e − 03
	errs	2.5608e − 06	2.5608e − 06	2.5608e − 06
		6.7550e − 13	6.7550e − 13	6.7550e − 13
	ite.	11	11	11
	$\mathbf{c}^*$	$(2.8703, 2.0639, 4.9434, 4.1053, 6.7530, 6.3287, 8.4794)^\top$		
(d)		2.9914e − 04	2.9914e − 04	2.9914e − 04
	errs	8.2616e − 09	8.2616e − 09	8.2616e − 09
		1.8946e − 14	1.8946e − 14	1.8946e − 14
	ite.	13	13	13
	$\mathbf{c}^*$	$(1.1787, -0.0035, 2.0401, 1.6976, 5.3794, 6.2068, 8.5273)^\top$		
(e)		2.1008e − 04	2.1008e − 04	2.1008e − 04
	errs	1.5703e − 08	1.5703e − 08	1.5703e − 08
		2.2331e − 14	2.2331e − 14	2.2331e − 14
	ite.	13	13	13
	$\mathbf{c}^*$	$(2.8703, 2.0639, 4.9434, 4.1053, 6.7530, 6.3287, 8.4794)^\top$		

where

$$\mathbf{c}_1^* = (2, 3, 4, 5, 6, 7, 8, 9)^\top,$$

$$\mathbf{c}_2^* = (-2.7065, -7.0690, 8.4130, 1.1038, -3.6637,$$
$$8.8346, 5.5078, 9.1273)^\top,$$

(87)

$$\mathbf{c}_3^* = (1.2877, 2.4677, 4.5379, -6.1037, 1.7930,$$
$$1.1093, 1.1024, 5.3320)^\top$$

are the three accumulation points of this problem.

We observe from Tables 1–3 that our algorithm is convergent for different starting points. We also see that our algorithm converges to a solution of the IEP, which is not necessarily equal to the original one. An interesting question is to consider the performance of the algorithm when the starting point $\mathbf{c}^0$ is chosen as random vector, which needs future study.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Table 3: Numerical results for Example 3.

$\mathbf{c}^0$		$\beta = 1.5$	$\beta = 1.8$	$\beta = 2.0$
(a)		$5.2594e-03$	$2.2949e-04$	$6.3777e-02$
	errs	$1.5831e-08$	$2.1853e-10$	$5.2251e-05$
		$9.8210e-12$	$5.3338e-12$	$1.1737e-11$
	ite.	17	17	17
	$\mathbf{c}^*$	$\mathbf{c}_2^*$	$\mathbf{c}_3^*$	$\mathbf{c}_2^*$
(b)		$2.3747e-02$	$2.4283e-02$	$5.3326e-02$
	errs	$6.6052e-07$	$6.8845e-07$	$3.5462e-05$
		$9.2871e-12$	$7.8077e-12$	$1.2096e-11$
	ite.	13	14	14
	$\mathbf{c}^*$	$\mathbf{c}_2^*$	$\mathbf{c}_2^*$	$\mathbf{c}_2^*$
(c)		$5.0900e-04$	$5.0902e-04$	$5.0902e-04$
	errs	$5.5760e-09$	$5.5819e-09$	$5.5827e-09$
		$1.4600e-11$	$1.3617e-11$	$1.7533e-11$
	ite.	10	10	10
	$\mathbf{c}^*$	$\mathbf{c}_1^*$	$\mathbf{c}_1^*$	$\mathbf{c}_1^*$
(d)		$1.7014e-02$	$1.7014e-02$	$1.7014e-02$
	errs	$6.9575e-07$	$6.9575e-07$	$6.9575e-07$
		$1.5224e-11$	$1.7256e-11$	$6.2077e-11$
	ite.	9	9	9
	$\mathbf{c}^*$	$\mathbf{c}_1^*$	$\mathbf{c}_1^*$	$\mathbf{c}_1^*$

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