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Research Article

Initial-Boundary Value Problem for Fractional Partial Differential Equations of Higher Order

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The initial-boundary value problem for partial differential equations of higher-order involving the Caputo fractional derivative is studied. Theorems on existence and uniqueness of a solution and its continuous dependence on the initial data and on the right-hand side of the equation are established.

1. Introduction

Many problems in viscoelasticity [1–3], dynamical processes in self-similar structures [4], biosciences [5], signal processing [6], system control theory [7], electrochemistry [8], diffusion processes [9], and linear time-invariant systems of any order with internal point delays [10] lead to differential equations of fractional order. For more details of fractional calculus, see [11–15].

The study of existence and uniqueness, periodicity, asymptotic behavior, stability, and methods of analytic and numerical solutions of fractional differential equations have been studied extensively in a large cycle works (see, e.g., [16–42] and the references therein).

In the paper [43], Cauchy problem in a half-space $\{(x, y, t) : (x, y) \in \mathbb{R}^2, t > 0\}$ for partial pseudodifferential equations involving the Caputo fractional derivative was studied. The existence and uniqueness of a solution and its continuous dependence on the initial data and on the right-hand side of the equation were established.

In the paper [44], the initial-boundary value problem for heat conduction equation with the Caputo fractional derivative was studied. Moreover, in [45], the initial-boundary value problem for partial differential equations of higher order with the Caputo fractional derivative was studied in the case when the order of the fractional derivative belongs to the interval (0,1).

In the paper [46], the initial-boundary value problem in plane domain for partial differential equations of fourth order with the fractional derivative in the sense of Caputo was studied in the case when the order of fractional derivative belongs to the interval (1,2). The present paper generalizes results of [46] in the case of space domain for partial differential equations of higher order with a fractional derivative in the sense of Caputo.

The organization of this paper is as follows. In Section 2, we provide the necessary background and formulation of problem. In Section 3, the formal solution of problem is presented. In Sections 4 and 5, the solvability and the regular solvability of the problem are studied. Theorems on existence and uniqueness of a solution and its continuous dependence on the initial data and on the right-hand side of the equation are established. Finally, Section 6 is conclusion.

2. Preliminaries

In this section, we present some basic definitions and preliminary facts which are used throughout the paper.

Definition 2.1. If $g(t) \in C[a,b]$ and $\alpha > 0$, then the Riemann-Liouville fractional integral is defined by

$$I_{a+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function defined for any complex number z as

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt. \tag{2.2}$$

Definition 2.2. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $g : (a,b) \to R$ is defined by

$$^{c}D_{a+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \tag{2.3}$$

where $n = [\alpha] + 1$, (the notation $[\alpha]$ stands for the largest integer not greater than α).

Lemma 2.3 (see [13]). *Let $p, q \geq 0$, $f(t) \in L_1[0,T]$. Then,*

$$I_{0+}^{p} I_{0+}^{q} f(t) = I_{0+}^{p+q} f(t) = I_{0+}^{q} I_{0+}^{p} f(t) \tag{2.4}$$

is satisfied almost everywhere on $[0,T]$. Moreover, if $f(t) \in C[0,T]$, then (2.4) is true and $^{c}D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t)$ for all $t \in [0,T]$ and $\alpha > 0$.

Theorem 2.4 (see [47, page 123]). *Let $f(t) \in L_1(0,T)$. Then, the integral equation*

$$z(t) = f(t) + \lambda \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} z(\tau)d\tau \tag{2.5}$$

has a unique solution $z(t)$ defined by the following formula:

$$z(t) = f(t) + \lambda \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda(t-\tau)^{\alpha}\right) f(\tau)d\tau, \tag{2.6}$$

where $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}(z^k/\Gamma(k\alpha+\beta))$ is a Mittag-Leffler type function.

For the convenience of the reader, we give the proof of Theorem 2.4, applying the fixed-point iteration method. We denote

$$Bz(t) = \lambda \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} z(\tau)d\tau. \tag{2.7}$$

Then,

$$z(t) = \sum_{k=0}^{m-1} B^k f(t) + B^m z(t), \quad m = 1, \ldots, n. \tag{2.8}$$

The proof of this theorem is based on formula (2.8) and

$$B^m z(t) = \lambda^m \int_0^t \frac{(t-\tau)^{m\alpha-1}}{\Gamma(m\alpha)} z(\tau)d\tau, \tag{2.9}$$

for any $m \in N$. Let us prove (2.9) for any $m \in N$. For $m = 1$, it follows from (2.7) directly. Assume that (2.9) holds for some $m-1 \in N$. Then, applying (2.7) and (2.9) for $m-1 \in N$, we get

$$\begin{aligned}
B^m z(t) &= \lambda^{m-1} \int_0^t \frac{(t-s)^{(m-1)\alpha-1}}{\Gamma((m-1)\alpha)} Bz(s)ds \\
&= \lambda^{m-1} \int_0^t \frac{(t-s)^{(m-1)\alpha-1}}{\Gamma((m-1)\alpha)} \lambda \int_0^s (s-\tau)^{\alpha-1} z(\tau)d\tau\, ds \\
&= \frac{\lambda^m}{\Gamma(\alpha)\Gamma((m-1)\alpha)} \int_0^t \int_0^s (t-s)^{(m-1)\alpha-1}(s-\tau)^{\alpha-1} z(\tau)d\tau\, ds \\
&= \frac{\lambda^m}{\Gamma(\alpha)\Gamma((m-1)\alpha)} \int_0^t \int_\tau^t (t-s)^{(m-1)\alpha-1}(s-\tau)^{\alpha-1} ds z(\tau)d\tau.
\end{aligned} \tag{2.10}$$

Performing the change of variables $s-\tau=(t-\tau)p$, we get

$$\begin{aligned}\int_{\tau}^{t}(t-s)^{(m-1)\alpha-1}(s-\tau)^{\alpha-1}ds &= (t-\tau)^{m\alpha-1}\int_{0}^{1}(1-p)^{(m-1)\alpha-1}p^{\alpha-1}dp\\ &= (t-\tau)^{m\alpha-1}B((m-1)\alpha,\alpha)\\ &= \frac{(t-\tau)^{m\alpha-1}}{\Gamma(m\alpha)}\Gamma((m-1)\alpha)\Gamma(\alpha).\end{aligned} \tag{2.11}$$

Then,

$$B^{m}z(t)=\lambda^{m}\int_{0}^{t}\frac{(t-\tau)^{m\alpha-1}}{\Gamma(m\alpha)}z(\tau)d\tau. \tag{2.12}$$

So, identity (2.9) holds for $m\in N$. Therefore, by induction identity (2.9) holds for any $m\in N$.

In the space domain, $\Omega=\{(x,y,t):0<x<p,0<y<q,0<t<T\}$, we consider the initial-boundary value problem:

$$\begin{aligned}&(-1)^{k}{}^{c}D_{0+}^{\alpha}u+\frac{\partial^{2k}u}{\partial x^{2k}}+\frac{\partial^{2k}u}{\partial y^{2k}}=f(x,y,t),\quad 0<x<p,\,0<y<q,\,0<t<T,\\ &\frac{\partial^{2m}u(0,y,t)}{\partial x^{2m}}=\frac{\partial^{2m}u(p,y,t)}{\partial x^{2m}}=0,\quad m=0,1,\ldots,k-1,\,0\le y\le q,\,0\le t\le T,\\ &\frac{\partial^{2m}u(x,0,t)}{\partial y^{2m}}=\frac{\partial^{2m}u(x,q,t)}{\partial y^{2m}}=0,\quad m=0,1,\ldots,k-1,\,0\le x\le p,\,0\le t\le T,\\ &u(x,y,0)=\varphi(x,y),\qquad u_t(x,y,0)=\psi(x,y),\quad 0\le x\le p,\,0\le y\le q\end{aligned} \tag{2.13}$$

for partial differential equations of higher order with the fractional derivative order $\alpha\in(1,2)$ in the sense of Caputo. Here, $k(k\ge1)$ is a fixed positive integer number.

3. The Construction of the Formal Solution of (2.13)

We seek a solution of problem (2.13) in the form of Fourier series:

$$u(x,y,t)=\sum_{n,m=1}^{\infty}u_{nm}(t)v_{nm}(x,y), \tag{3.1}$$

expanded along a complete orthonormal system:

$$v_{nm}(x,y)=\frac{2}{\sqrt{pq}}\sin\frac{n\pi}{p}x\sin\frac{m\pi}{q}y,\quad 1\le n,\ m<\infty. \tag{3.2}$$

We denote

$$\Omega_0 = \overline{\Omega} \cap (t = 0) = \{(x, y, 0) : 0 \le x \le p,\, 0 \le y \le q\},$$
$$\frac{n\pi}{p} = \nu_n, \quad \frac{m\pi}{q} = \mu_m, \quad \nu_n^{2k} + \mu_m^{2k} = \lambda_{nm}^{2k}, \quad 1 \le n, m < \infty. \tag{3.3}$$

We expand the given function $f(x, y, t)$ in the form of a Fourier series along the functions $v_{nm}(x, y), 1 \le n, m < \infty$:

$$f(x, y, t) = \sum_{n,m=1}^{\infty} f_{nm}(t) v_{nm}(x, y), \tag{3.4}$$

where

$$f_{nm}(\mathrm{t}) = \int_0^p \int_0^q f(x, y, t) v_{nm}(x, y)\, dy\, dx, \quad 1 \le n, m < \infty. \tag{3.5}$$

Substituting (3.1) and (3.4) into (2.13), we obtain

$$(-1)^k\, {}^cD_{0+}^{\alpha} u_{nm}(t) + (-1)^k \lambda_{nm}^{2k} u_{nm}(t) = f_{nm}(t). \tag{3.6}$$

By Lemma 2.3, we have that

$${}^cD_{0+}^{\alpha} u_{nm}(t) = I_{0+}^{2-\alpha} u_{nm}''(t), \tag{3.7}$$

where

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau) d\tau \tag{3.8}$$

is Riemann-Liouville integral of fractional order α. Using (3.6) and (3.7), we get the following equation:

$$I_{0+}^{2-\alpha} u_{nm}''(t) + \lambda_{nm}^{2k} u_{nm}(t) = (-1)^k f_{nm}(t). \tag{3.9}$$

Applying the operator I_{0+}^{α} to this equation, we get the following Volterra integral equation of the second kind:

$$u_{nm}(t) = \frac{-\lambda_{nm}^{2k}}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} u_{nm}(\tau) d\tau + u_{nm}(0) + t u_{nm}'(0) + (-1)^k I_{0+}^{\alpha} f_{nm}(t). \tag{3.10}$$

According to the Theorem 2.4, (3.10) has a unique solution $u_{nm}(t)$ defined by the following formula:

$$\begin{aligned} u_{nm}(t) = {} & \frac{(-1)^k}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f_{nm}(\tau)d\tau \\ & + u_{nm}(0)\left[1-\lambda_{nm}^{2k}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{nm}^{2k}(t-\tau)^{\alpha}\right)d\tau\right] \\ & + u'_{nm}(0)\left[t-\lambda_{nm}^{2k}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{nm}^{2k}(t-\tau)^{\alpha}\right)\tau d\tau\right] \\ & - \frac{\lambda_{nm}^{2k}}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{nm}^{2k}(t-\eta)^{\alpha}\right)d\eta\int_0^{\eta} (\eta-\tau)^{\alpha-1} f_{nm}(\tau)d\tau. \end{aligned} \tag{3.11}$$

Using the formula (see, e.g., [27, page 118] and [47, page 120])

$$\begin{gathered} \frac{1}{\Gamma(\beta)}\int_0^z t^{\mu-1}E_{\alpha,\mu}(\lambda t^{\alpha})(z-t)^{\beta-1}dt = z^{\mu+\beta-1}E_{\alpha,\mu+\beta}(\lambda z^{\alpha}), \\ \frac{1}{\Gamma(\mu)} + zE_{\alpha,\alpha+\mu}(z) = E_{\alpha,\mu}(z), \end{gathered} \tag{3.12}$$

we get

$$\begin{aligned} & -\frac{\lambda_{nm}^{2k}}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{nm}^{2k}(t-\eta)^{\alpha}\right)d\eta\int_0^{\eta} (\eta-\tau)^{\alpha-1} f_{nm}(\tau)\ d\tau \\ & = \int_0^t f_{nm}(\tau)\left\{-\frac{\lambda_{nm}^{2k}}{\Gamma(\alpha)}\int_{\tau}^t (t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{nm}^{2k}(t-\eta)^{\alpha}\right)(\eta-\tau)^{\alpha-1}d\eta\right\}d\tau \\ & = \int_0^t f_{nm}(\tau)\left\{-\frac{\lambda_{nm}^{2k}}{\Gamma(\alpha)}\int_0^{t-\tau} z^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{nm}^{2k}z^{\alpha}\right)(t-\tau-z)^{\alpha-1}dz\right\}d\tau \\ & = -\int_0^t f_{nm}(\tau)\lambda_{nm}^{2k}(t-\tau)^{2\alpha-1}E_{\alpha,2\alpha}\left(-\lambda_{nm}^{2k}(t-\tau)^{\alpha}\right)d\tau \\ & = \int_0^t (t-\tau)^{\alpha-1} f_{nm}(\tau)\left\{-\frac{1}{\Gamma(\alpha)}+E_{\alpha,\alpha}\left(-\lambda_{nm}^{2k}(t-\tau)^{\alpha}\right)\right\}d\tau, \\ & \quad -\lambda_{nm}^{2k}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{nm}^{2k}(t-\tau)^{\alpha}\right)d\tau \\ & = -\lambda_{nm}^{2k}\int_0^t z^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{nm}^{2k}z^{\alpha}\right)(t-z)^{1-1}dz \end{aligned}$$

$$
\begin{aligned}
&= \Gamma(1)\lambda_{nm}^{2k} t^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda_{nm}^{2k} t^{\alpha}\right) = E_{\alpha,1}\left(-\lambda_{nm}^{2k} t^{\alpha}\right) - 1, \\
&\quad -\lambda_{nm}^{2k} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{nm}^{2k} (t-\tau)^{\alpha}\right) \tau \, d\tau \\
&= -\lambda_{nm}^{2k} \int_0^t z^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{nm}^{2k} z^{\alpha}\right) (t-z)^{2-1} dz \\
&= \Gamma(2)\lambda_{nm}^{2k} t^{\alpha+1} E_{\alpha,\alpha+2}\left(-\lambda_{nm}^{2k} t^{\alpha}\right) = t E_{\alpha,2}\left(-\lambda_{nm}^{2k} t^{\alpha}\right) - t.
\end{aligned}
\tag{3.13}
$$

From these three formulas and (3.11), it follows that

$$
\begin{aligned}
u_{nm}(t) &= u_{nm}(0) E_{\alpha,1}\left(-\lambda_{nm}^{2k} t^{\alpha}\right) + t u_{nm}'(0) E_{\alpha,2}\left(-\lambda_{nm}^{2k} t^{\alpha}\right) \\
&\quad + (-1)^k \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{nm}^{2k} (t-\tau)^{\alpha}\right) f_{nm}(\tau) d\tau.
\end{aligned}
\tag{3.14}
$$

For $u_{nm}(0)$ and $u_{nm}'(0)$, we expand the given functions $\varphi(x,y)$ and $\psi(x,y)$ in the form of a Fourier series along the functions $v_{nm}(x,y), 1 \le n, m < \infty$:

$$
\begin{aligned}
\varphi(x,y) &= \sum_{n,m=1}^{\infty} \varphi_{nm} v_{nm}(x,y) \ , \\
\psi(x,y) &= \sum_{n,m=1}^{\infty} \psi_{nm} v_{nm}(x,y),
\end{aligned}
\tag{3.15}
$$

where

$$
\begin{aligned}
\varphi_{nm} &= \int_0^p \int_0^q \varphi(x,y) v_{nm}(x,y) dy\, dx, \\
\psi_{nm} &= \int_0^p \int_0^q \psi(x,y) v_{nm}(x,y) dy\, dx.
\end{aligned}
\tag{3.16}
$$

Using (2.13), (3.14), (3.16), we obtain

$$
\begin{aligned}
u_{nm}(t) &= E_{\alpha,1}\left(-\lambda_{nm}^{2k} t^{\alpha}\right) \varphi_{nm} + t E_{\alpha,2}\left(-\lambda_{nm}^{2k} t^{\alpha}\right) \psi_{nm} \\
&\quad + (-1)^k \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{nm}^{2k} (t-\tau)^{\alpha}\right) f_{nm}(\tau) d\tau.
\end{aligned}
\tag{3.17}
$$

So, the unique solution of (3.10) is defined by (3.17). Consequently, the unique solution of problem (2.13) is defined by (3.1).

Applying the formula (3.17), the Cauchy-Schwarz inequality, and the estimate (see [13, page 136])

$$\left|E_{\alpha,\beta}(z)\right| \leq \frac{M}{1+|z|}, \quad M = \text{const} > 0,\ \operatorname{Re} z < 0, \tag{3.18}$$

we get the following inequality:

$$|u_{nm}(t)| \leq C_0\left(\left|\varphi_{nm}\right| + \left|\psi_{nm}\right| + \left(\int_0^t \left|f_{nm}(t)\right|^2 dt\right)^{1/2}\right) \tag{3.19}$$

for the solution of (3.10) for any $t, t \in [0,T]$. Here, $C_0 = \max\{M, TM, M(T^{\alpha-1/2}/\sqrt{2\alpha-1})\}$.

4. Solvability of (2.13) in $L_2(\Omega)$ Space

Now, we will prove that the solution $u(x,y,t)$ of problem (2.13) continuously depends on $\varphi(x,y)$, $\psi(x,y)$, and $f(x,y,t)$.

Theorem 4.1. *Suppose $\varphi(x,y) \in L_2(\Omega_0)$, $\psi(x,y) \in L_2(\Omega_0)$, and $f(x,y,t) \in L_2(\Omega)$, then the series* (3.1) *converges in $L_2(\Omega)$ to $u \in L_2(\Omega)$ and for the solution of problem* (2.13), *the following stability inequality*

$$\|u\|_{L_2(\Omega)} \leq C_1\left(\|\varphi\|_{L_2(\Omega_0)} + \|\psi\|_{L_2(\Omega_0)} + \|f\|_{L_2(\Omega)}\right) \tag{4.1}$$

holds, where C_1 does not depend on $\varphi(x,y)$, $\psi(x,y)$, and $f(x,y,t)$.

Proof. We consider the sum:

$$u_N(x,y,t) = \sum_{n,m=1}^{N} u_{nm}(t) v_{nm}(x,y), \tag{4.2}$$

where N is a natural number. For the positive integer number L, we have that

$$\begin{aligned} \|u_{N+L} - u_N\|^2_{L_2(\Omega)} &= \left\|\sum_{n,m=N+1}^{N+L} u_{nm}(\cdot) v_{nm}(\cdot,\cdot)\right\|^2_{L_2(\Omega)} \\ &= \sum_{n,m=N+1}^{N+L} \int_0^T |u_{nm}(t)|^2 dt. \end{aligned} \tag{4.3}$$

Applying (3.19), we get

$$\sum_{n,m=1}^{\infty}\int_0^T |u_{nm}(t)|^2 dt \leq 3C_0^2\left(\sum_{n,m=1}^{\infty}\left|\varphi_{nm}\right|^2+\sum_{n,m=1}^{\infty}\left|\psi_{nm}\right|^2+\sum_{n,m=1}^{\infty}\int_0^T\left|f_{nm}(t)\right|^2 dt\right) \\ = C^2\left(\|\varphi\|_{L_2(\Omega_0)}^2+\|\psi\|_{L_2(\Omega_0)}^2+\|f\|_{L_2(\Omega)}^2\right), \tag{4.4}$$

where $C^2 = 3TC_0^2$. Therefore, $\sum_{n,m=N+1}^{N+L}\int_0^T |u_{nm}(t)|^2 dt \to 0$ as $N \to \infty$. Consequently, the series (3.1) converges in $L_2(\Omega)$ to $u(x,y,t) \in L_2(\Omega)$. Inequality (4.1) for the solution of problem (2.13) follows from the estimate (4.4). Theorem 4.1 is proved. □

5. The Regular Solvability of (2.13)

In this section, we will study theregular solvability of problem (2.13).

Lemma 5.1. *Suppose $\varphi(x,y) \in C^1(\overline{\Omega}_0), \varphi_{xy}(x,y) \in L_2(\Omega_0), \psi(x,y) \in C^1(\overline{\Omega}_0), \psi_{xy}(x,y) \in L_2(\Omega_0), \varphi(x,y) = 0$ on $\partial\Omega_0, \psi(x,y) = 0$ on $\partial\Omega_0, f(x,y,t) \in C^2(\overline{\Omega}), f_{xxy}(x,y,t) \in C(\overline{\Omega}) f_{xyy}(x,y,t) \in C(\overline{\Omega}_0), f_{xxyy}(x,y,t) \in C(\overline{\Omega}_0)$, and $f(x,y,t) = 0$ on $\partial\Omega \times [0,T]$. Then, for any $\varepsilon \in (0,1)$, the following estimates*

$$|u_{nm}(t)| \leq C_1\left(\frac{\left|\varphi_{nm}\right|}{\nu_n^k\mu_m^k}+\frac{\left|\psi_{nm}\right|}{\nu_n^k\mu_m^k}+\frac{1}{\nu_n^{k+1}\mu_m^{k+1}}+\frac{1}{\nu_n^{k+1-\varepsilon}\mu_m^{k+1}}+\frac{1}{\nu_n^{k+1}\mu_m^{k+1-\varepsilon}}\right), \tag{5.1}$$

$$\lambda_{nm}^{2k}|u_{nm}(t)| \leq C_2\left(\frac{\left|\varphi_{nm}^{(1,1)}\right|}{\nu_n\mu_m}+\frac{\left|\psi_{nm}^{(1,1)}\right|}{\nu_n\mu_m}+\frac{1}{\nu_n^2\mu_m^2}+\frac{1}{\nu_n^{2-\varepsilon}\mu_m^2}+\frac{1}{\nu_n^2\mu_m^{2-\varepsilon}}\right) \tag{5.2}$$

hold, where C_1 and C_2 do not depend on $\varphi(x,y)$ and $\psi(x,y)$.

Proof. Integrating by parts with respect to x and y in (3.5), (3.16), we get

$$\varphi_{nm} = \frac{1}{\nu_n\mu_m}\varphi_{nm}^{(1,1)}, \tag{5.3}$$

$$\psi_{nm} = \frac{1}{\nu_n\mu_m}\psi_{nm}^{(1,1)}, \tag{5.4}$$

$$f_{nm}(t) = \frac{1}{\nu_n\mu_m}f_{nm}^{(1,1,0)}(t), \tag{5.5}$$

$$f_{nm}(t) = \frac{1}{\nu_n^2\mu_m^2}f_{nm}^{(2,2,0)}(t), \tag{5.6}$$

where

$$\varphi_{nm}^{(1,1)} = \int_0^p \int_0^q \frac{\partial^2 \varphi(x,y)}{\partial x \partial y} v_{nm}(x,y)\,dy\,dx,$$
$$\psi_{nm}^{(1,1)} = \int_0^p \int_0^q \frac{\partial^2 \psi(x,y)}{\partial x \partial y} v_{nm}(x,y)\,dy\,dx,$$
$$f_{nm}^{(1,1,0)}(t) = \int_0^p \int_0^q \frac{\partial^2 f(x,y,t)}{\partial x \partial y} v_{nm}(x,y)\,dy\,dx,$$
$$f_{nm}^{(2,2,0)}(t) = \int_0^p \int_0^q \frac{\partial^4 f(x,y,t)}{\partial x^2 \partial y^2} v_{nm}(x,y)\,dy\,dx. \tag{5.7}$$

Under the assumptions of Lemma 5.1, it follows that the functions $f_{nm}^{(1,1,0)}(t)$ and $f_{nm}^{(2,2,0)}(t)$ are bounded, that is,

$$\left|f_{nm}^{(1,1,0)}(t)\right| \leq N_1, \qquad \left|f_{nm}^{(2,2,0)}(t)\right| \leq N_2, \tag{5.8}$$

where $N_1 = \text{const} > 0$, $N_2 = \text{const} > 0$. Let $0 < t_0 \leq t \leq T$, where t_0 is a sufficiently small number. For sufficiently large n and m, the following inequalities are true:

$$\ln \lambda_{nm}^{\varepsilon} < \lambda_{nm}^{\varepsilon} < \nu_n^{\varepsilon} + \mu_m^{\varepsilon}, \quad 0 < \varepsilon < 1,$$
$$1 + \lambda_{nm}^{2k} T^{\alpha} < 2\lambda_{nm}^{2k} T^{\alpha}. \tag{5.9}$$

Using (3.16), (5.8), (5.9), and (3.17), we get

$$\begin{aligned} |u_{nm}(t)| &\leq M\left(\frac{1}{2t_0^{\alpha}}\frac{|\varphi_{nm}|}{\nu_n^k \mu_m^k} + \frac{1}{2t_0^{\alpha-1}}\frac{|\psi_{nm}|}{\nu_n^k \mu_m^k} - \frac{N_1}{\alpha \nu_n^{k+1}\mu_m^{k+1}}\int_0^t \frac{d\left(1+\lambda_{nm}^{2k}(t-\tau)^{\alpha}\right)}{1+\lambda_{nm}^{2k}(t-\tau)^{\alpha}}\right) \\ &\leq M\left(\frac{1}{2t_0^{\alpha}}\frac{|\varphi_{nm}|}{\nu_n^k \mu_m^k} + \frac{1}{2t_0^{\alpha-1}}\frac{|\psi_{nm}|}{\nu_n^k \mu_m^k} + \frac{N_1(\ln 2T^{\alpha} + (2k/\varepsilon)\ln \lambda_{nm}^{\varepsilon})}{\alpha \nu_n^{k+1}\mu_m^{k+1}}\right) \\ &\leq C_1\left(\frac{|\varphi_{nm}|}{\nu_n^k \mu_m^k} + \frac{|\psi_{nm}|}{\nu_n^k \mu_m^k} + \frac{1}{\nu_n^{k+1}\mu_m^{k+1}} + \frac{1}{\nu_n^{k+1-\varepsilon}\mu_m^{k+1}} + \frac{1}{\nu_n^{k+1}\mu_m^{k+1-\varepsilon}}\right), \end{aligned} \tag{5.10}$$

where $C_1 = \max\{M/2t_0^{\alpha}, M/2t_0^{\alpha-1}, MN_1 \ln 2T^{\alpha}/\alpha, 2kMN_1/\alpha\varepsilon\}$. Thus, inequality (5.1) is obtained. Now, we will prove inequality (5.2). Using (5.3), (5.4), (5.6), (5.8), (5.9), and (3.17), we get

$$\begin{aligned}
\lambda_{nm}^{2k}|u_{nm}(t)| &\le M\left(\frac{|\varphi_{nm}|}{t^{\alpha}} + \frac{|\varphi_{nm}|}{t^{\alpha-1}} + \lambda_{nm}^{2k}\int_0^t \frac{(t-\tau)^{\alpha-1} f_{nm}(\tau)}{1+\lambda_{nm}^{2k}(t-\tau)^{\alpha}} d\tau\right) \\
&\le M\left(\frac{1}{t_0^{\alpha}}\frac{\left|\varphi_{nm}^{(1,1)}\right|}{\nu_n\mu_m} + \frac{\left|\varphi_{nm}^{(1,1)}\right|}{t_0^{\alpha-1}\nu_n\mu_m} - \frac{N_2}{\alpha}\int_0^t \frac{d\left(1+\lambda_{nm}^{2k}(t-\tau)^{\alpha}\right)}{\nu_n^2\mu_m^2\left(1+\lambda_{nm}^{2k}(t-\tau)^{\alpha}\right)}\right) \\
&\le \frac{M}{t_0^{\alpha}}\frac{\left|\varphi_{nm}^{(1,1)}\right|}{\nu_n\mu_m} + \frac{M}{t_0^{\alpha-1}}\frac{\left|\varphi_{nm}^{(1,1)}\right|}{\nu_n\mu_m} + \frac{2MN_2\ln T^{\alpha}}{\alpha\nu_n^2\mu_m^2} + \frac{2kMN_2}{\alpha\varepsilon\nu_n^{2-\varepsilon}\mu_m^2} + \frac{2kMN_2}{\alpha\varepsilon\nu_n^2\mu_m^{2-\varepsilon}} \\
&\le C_2\left(\frac{\left|\varphi_{nm}^{(1,1)}\right|}{\nu_n\mu_m} + \frac{\left|\varphi_{nm}^{(1,1)}\right|}{\nu_n\mu_m} + \frac{1}{\nu_n^2\mu_m^2} + \frac{1}{\nu_n^{2-\varepsilon}\mu_m^2} + \frac{1}{\nu_n^2\mu_m^{2-\varepsilon}}\right),
\end{aligned} \tag{5.11}$$

where $C_2 = \max\{M/t_0^{\alpha}, M/t_0^{\alpha-1}, MN_2 \ln 2T^{\alpha}/\alpha, 2kMN_2/\alpha\varepsilon\}$. Lemma 5.1 is proved. □

Theorem 5.2. *Suppose that the assumptions of Lemma 5.1 hold. Then, there exists a regular solution of problem* (2.13).

Proof. We will prove uniform and absolute convergence of series (3.1) and

$$\frac{\partial^{2k}u(x,y,t)}{\partial x^{2k}} = \sum_{n,m=1}^{\infty}(-1)^k\nu_n^{2k}u_{nm}(t)v_{nm}(x,y), \tag{5.12}$$

$$\frac{\partial^{2k}u(x,y,t)}{\partial x^{2k}} = \sum_{n,m=1}^{\infty}(-1)^k\mu_n^{2k}u_{nm}(t)v_{nm}(x,y), \tag{5.13}$$

$${}^{c}D_{0+}^{\alpha}u(x,y,t) = -\sum_{n,m=1}^{\infty}(-1)^k\lambda_{nm}^{2k}u_{nm}(t)v_{nm}(x,y) + \sum_{n,m=1}^{\infty}f_{nm}(t)v_{nm}(x,y). \tag{5.14}$$

The series

$$\sum_{n,m=1}^{\infty}|u_{nm}(t)| \tag{5.15}$$

is majorant for the series (3.1). From (5.1), it follows that the series (5.15) uniformly converges. Actually,

$$\sum_{n,m=1}^{\infty}|u_{nm}(t)| \le C\sum_{n,m=1}^{\infty}\left(\frac{|\varphi_{nm}|}{\nu_n^k\mu_m^k} + \frac{|\psi_{nm}|}{\nu_n^k\mu_m^k} + \frac{1}{\nu_n^k\mu_n^k} + \frac{1}{\nu_n^{k+1-\varepsilon}\mu_n^{k+1}} + \frac{1}{\nu_n^{k+1}\mu_n^{k+1-\varepsilon}}\right). \tag{5.16}$$

Applying the Cauchy-Schwarz inequality and the Parseval equality, we obtain

$$\begin{aligned}\sum_{n,m=1}^{\infty}\frac{|\varphi_{nm}|}{\nu_n^k\mu_m^k} &\le \left(\sum_{n,m=1}^{\infty}\frac{1}{\nu_n^{2k}\mu_m^{2k}}\right)^{1/2}\left(\sum_{n,m=1}^{\infty}|\varphi_{nm}|^2\right)^{1/2}\\ &= \frac{p^k q^k}{\pi^{2k}}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\sum_{m=1}^{\infty}\frac{1}{m^{2k}}\right)^{1/2}\|\varphi\|_{L_2(\Omega_0)}.\end{aligned} \tag{5.17}$$

Analogously, we get

$$\sum_{n,m=1}^{\infty}\frac{|\psi_{nm}|}{\nu_n^k\mu_m^k} \le \frac{p^k q^k}{\pi^{2k}}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\sum_{m=1}^{\infty}\frac{1}{m^{2k}}\right)^{1/2}\|\psi\|_{L_2(\Omega_0)}. \tag{5.18}$$

Since $2k \ge 2$, then the series $\sum_{n=1}^{\infty}(1/n^{2k}),\sum_{m=1}^{\infty}(1/m^{2k})$ converges by the integral test. Further, $k+1-\varepsilon > 1$, then the series

$$\sum_{n,m=1}^{\infty}\frac{1}{\nu_n^{k+1}\mu_m^{k+1}}, \qquad \sum_{n,m=1}^{\infty}\frac{1}{\nu_n^{k+1-\varepsilon}\mu_m^{k+1}}, \qquad \sum_{n,m=1}^{\infty}\frac{1}{\nu_n^{k+1}\mu_m^{k+1-\varepsilon}} \tag{5.19}$$

converges also by the integral test for any $k \ge 1$ and $\varepsilon \in (0,1)$.

Consequently, the series (3.1) absolutely and uniformly converges in the domain $\Omega_{t_0} = \Omega \times [t_0, T]$ for any $t_0 \in (0,T)$. At $t = 0$, the series (3.1) converges and has a sum equal to $\varphi(x,y)$. Since $\nu_n^{2k} < \lambda_{nm}^{2k}, \mu_m^{2k} < \lambda_{nm}^{2k}$, then the series

$$\sum_{n,m=1}^{\infty}\lambda_{nm}^{2k}|u_{nm}| \tag{5.20}$$

is majorant for the series (5.12), (5.13) and for the first series from (5.14). From (5.2), it follows that the series (5.20) uniformly converges. Indeed, using the Parseval equality and Cauchy-Schwarz inequality, we get

$$\sum_{n,m=1}^{\infty}\frac{\left|\varphi_{nm}^{(1,1)}\right|}{\nu_n\mu_m} \le \left(\sum_{n=1}^{\infty}\frac{1}{\nu_n^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\right)^{1/2}\left(\sum_{n,m=1}^{\infty}\left|\varphi_{nm}^{(1,1)}\right|^2\right)^{1/2} = \frac{pq}{6}\left\|\frac{\partial^2\varphi}{\partial x\partial y}\right\|_{L_2(\Omega_0)}. \tag{5.21}$$

Analogously, we conclude that

$$\sum_{n,m=1}^{\infty}\frac{\left|\psi_{nm}^{(1,1)}\right|}{\nu_n\mu_m} \le \frac{pq}{6}\left\|\frac{\partial^2\psi}{\partial x\partial y}\right\|_{L_2(\Omega_0)}. \tag{5.22}$$

The series

$$\sum_{n,m=1}^{\infty}\left(\frac{1}{\nu_n^2\mu_m^2}+\frac{1}{\nu_n^{2-\varepsilon}\mu_m^2}+\frac{1}{\nu_n^2\mu_m^{2-\varepsilon}}\right) \tag{5.23}$$

converges for any $\varepsilon \in (0,1)$ according to the integral test. The series

$$\sum_{n,m=1}^{\infty}\left|f_{nm}(t)\right| \tag{5.24}$$

is majorant for the second series from (5.14). From (5.6) and (5.8), it follows that the series (5.14) uniformly converges. Indeed,

$$\sum_{n,m=1}^{\infty}\left|f_{nm}(t)\right| = \sum_{n,m=1}^{\infty}\frac{1}{\nu_n^2\mu_m^2}\left|f_{nm}^{2,2,0}(t)\right| \le N_2\sum_{n,m=1}^{\infty}\frac{1}{\nu_n^2\mu_m^2} = \frac{N_2p^2q^2}{36}. \tag{5.25}$$

Adding equality (5.12), (5.13), and (5.14), we note that the solution (3.1) satisfies equation (2.13). The solution (3.1) satisfies boundary conditions owing to properties of the functions $v_{nm}(x,y)$. Simple computations show that

$$\begin{gathered}\lim_{t\to 0}E_{\alpha,1}\left(-\lambda_{nm}^{2k}t^{\alpha}\right)=1,\\ \lim_{t\to 0}\frac{d}{dt}E_{\alpha,1}\left(-\lambda_{nm}^{2k}t^{\alpha}\right)=0,\\ \lim_{t\to 0}E_{\alpha,2}\left(-\lambda_{nm}^{2k}t^{\alpha}\right)=1,\\ \lim_{t\to 0}t\frac{d}{dt}E_{\alpha,2}\left(-\lambda_{nm}^{2k}t^{\alpha}\right)=0.\end{gathered} \tag{5.26}$$

Consequently, $\lim_{t\to 0}u_{nm}(t)=\varphi_{nm}$, $\lim_{t\to 0}u'_{nm}(t)=\psi_{nm}$. Hence, we conclude that the solution (3.1) satisfies initial conditions. Theorem 5.2 is proved. □

6. Conclusion

In this paper, the initial-boundary value problem (2.13) for partial differential equations of higher order involving the Caputo fractional derivative is studied. Theorems on existence and uniqueness of a solution and its continuous dependence on the initial data and on the right-hand side of the equation are established. Of course, such type of results have been established for the initial-boundary value problem:

$$
\begin{aligned}
&(-1)^k\, {}^cD_{0+}^{\alpha} u + \frac{\partial^{2k}u}{\partial x^{2k}} + \frac{\partial^{2k}u}{\partial y^{2k}} + u = f(x,y,t), \quad 0 < x < p,\ 0 < y < q,\ 0 < t < T, \\
&\frac{\partial^{2m+1}u(0,y,t)}{\partial x^{2m}} = \frac{\partial^{2m+1}u(p,y,t)}{\partial x^{2m}} = 0, \quad m = 0,1,\ldots,k-1,\ 0 \le y \le q,\ 0 \le t \le T, \\
&\frac{\partial^{2m+1}u(x,0,t)}{\partial y^{2m}} = \frac{\partial^{2m+1}u(x,q,t)}{\partial y^{2m}} = 0, \quad m = 0,\ 1,\ldots,k-1,\ 0 \le x \le p,\ 0 \le t \le T, \\
&u(x,y,0) = \varphi(x,y), \quad u_t(x,y,0) = \psi(x,y), \quad 0 \le x \le p,\ 0 \le y \le q
\end{aligned}
\tag{6.1}
$$

for partial differential equations of higher order with a fractional derivative of order $\alpha \in (1,2)$ in the sense of Caputo. Here, $k(k \ge 1)$ is a fixed positive integer number.

Moreover, applying the result of the papers [15, 23], the first order of accuracy difference schemes for the numerical solution of nonlocal boundary value problems (2.13) and (6.1) can be presented. Of course, the stability inequalities for the solution of these difference schemes have been established without any assumptions about the grid steps τ in t and h in the space variables.

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