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Dynamic response of a beam subjected to moving load and moving mass supported by Pasternak foundation

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Abstract. This paper presents the dynamic response of an Euler- Bernoulli beam supported on two-parameter Pasternak foundation subjected to moving load as well as moving mass. Modal analysis along with Fourier transform technique is employed to find the analytical solution of the governing partial differential equation. Shape functions are assumed to convert the partial differential equation into a series of ordinary differential equations. The dynamic responses of the beam in terms of normalized deflection and bending moment have been investigated for different velocity ratios under moving load and moving mass conditions. The effect of moving load velocity on dynamic deflection and bending moment responses of the beam have been investigated. The effect of foundation parameters such as, stiffness and shear modulus on dynamic deflection and bending moment responses have also been investigated for both moving load and moving mass at constant speeds. Numerical results obtained from the study are presented and discussed.

Keywords: Beam vibration, moving load, moving mass, Pasternak foundation

1. Introduction

The dynamic behavior of beams on elastic foundations subjected to moving loads or masses has been investigated by many researchers in engineering, especially in Railway Engineering. The modern trend towards higher speeds in the railways has further intensified the research in order to accurately predict the vibration behavior of the railway track. These studies mostly considered the Winkler elastic foundation model that consists of infinite closely-spaced linear springs subjected to a moving load [1–5]. These models are also termed as one-parameter models [6]. These one-parameter models have been extensively employed in early studies to investigate the vibration of the beams due to moving loads. In the case of moving mass, studies are limited to single [7–19] or multiple span [20,21] beams with different boundary conditions and without any elastic supports. A very few studies considered one parameter foundation model for prediction of beam responses subjected to a moving mass [22–24]. However, these one parameter models do not accurately represent the continuous characteristics of practical foundations since it assumes no interaction between the lateral springs. Moreover, it also results in overlooking the influence of the soil on either side of the beam [25]. In order to overcome the limitations of one parameter model, several two-parameter models, also known as Pasternak models, have been proposed for the analysis of the dynamic behavior of beams under moving loads [26–28]. All of these models are mathematically equivalent and differ only in foundation parameters. However, dynamic response of the beam supported on a two parameter foundation model under a moving mass is

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not investigated so far. Moreover, the effects of shear modulus and foundation stiffness on deflection and bending moment responses of the beam supported by Pasternak foundation have also never been investigated in the presence of a moving mass.

In order to capture the distributed stresses accurately, a three-parameter model has been developed for cohesive and non-cohesive soil foundations [29–31]. This model offers the continuity in the vertical displacements at the boundaries between the loaded and the unloaded surfaces of the soil [32]. In the analysis of vibration of beams under the moving loads and masses, the beam has been modeled as either a Timoshenko beam [7,24,25,33–38], or an Euler-Bernoulli beam [1–6,9–15,20,27,28,39–41]. A Timoshenko beam model considers the shear deformation and rotational inertia of the beam. Chen and Huang [34] have graphically shown that an Euler-Bernoulli beam can accurately predict the response of the beam for foundation stiffness up to 10^8 N/m^2. Therefore, an Euler–Bernoulli beam has been considered in the present analysis, since the foundation stiffness considered is 4.078×10^6 N/m, which is much less than the suggested value. Moreover, an Euler–Bernoulli beam model can accurately predict the response since the depth and rotary inertia of the track can be considered small compared to the translational inertia [28].

The analytical solution of the vibration of infinite beams under the moving load has received considerable attention by researchers [1,5,28,42,43]. In case of two-parameter model, studies are scarce due to the model complexity and difficulties in estimating parameter values [28,42–44]. In recent years, there is a growing interest on the vibration of the beam under moving load in railway industry in view of the use of beam type structure as a simplified physical model for railway track and pavements [3,5,45].

Apart from the one, two or three parameter foundation models, viscoelastic and poroelastic half space models of the foundation are also common in the dynamic analysis of a beam due to a moving oscillating load [46,48], or moving point load [47,49–53]. These half space models can be single layer [48,49,51–53], or multiple layers [47, 50]. Responses of the beams in terms of displacements [46–53], bending moments [47], accelerations [48] and shear force [47] have been analyzed in these studies. Studies with multilayer half space show that the response calculated for the multi-layered case exhibits higher frequencies and larger amplitudes than the response obtained for a uniform half-space [47,50].

Fryba [1] presented a detailed solution of the moving load problem where the beam was modeled as infinitely long Euler-Bernoulli beam resting on Winkler foundation. A vast majority of the studies dealing with the moving load problems utilized the Fourier transformation method to solve the governing differential equations arising from either Euler-Bernoulli or Timoshenko theory [1,5,28,42,43]. The responses of the infinite beam under moving load supported on either Winkler or Pasternak foundation were studied by means of Fourier transforms and using Green's function in [2,43,54]. Mead and Mallik [55], and Cai et al. [56] presented an approximate "assumed mode" method to study the space-averaged response of infinitely long periodic beam subjected to convected loading and moving force, respectively. These methods are applicable to the forced vibration analysis of an infinite continuous beam subjected to arbitrary excitations. In order to consider the effect of non linearity in beam analysis, Finite Element Analysis (FEA) of an infinite beam has been carried out in [57–59]. In these studies, FEA has been adopted to perform the analysis of nonlinear dynamic structure under moving loads where the load varies with both time and space.

In moving force problem, the magnitude of the moving force has been assumed to be constant by neglecting the inertia forces of a moving mass. However, in the case of moving mass, the interaction force consists of inertia of the mass, centrifugal force, etc. Hence, the velocity of the moving mass, structural flexibility, and the ratio of the moving mass to that of the structure play important roles on the overall interaction process. A closed-form solution to a moving mass problem is obtained by Michaltos et al. [9] by approximating the solution without the effect of the mass. By using the method of Green functions, the effects of the system parameters on the dynamic response of the beam subjected to a moving mass have been studied by Ting et al. [13], Foda and Abduljabbar [14], and Sadiku and Leipholz [16]. The method of the eigenfunction expansion in series or modal analysis has been employed by Akin and Mofld [18], Bowe and Mullarkey [19], Ichikawa et al. [20], Stanisic and Lafayette [23], and Lee [24]. Lou et al. [37], Yavari et al. [38], Vu-Quoc and Olsson [39], Bajer and Dyniewicz [40], and Cifuentes [41] investigated the dynamic response of single and multi span beams subjected to a moving mass by using finite element method.

In the present analysis, dynamic responses of an Euler-Bernoulli beam under constant moving load as well as moving mass are investigated. In case of moving force, the exact analysis has been validated by numerical method. However, only numerical method is employed in case of moving mass problem. The foundation representing the soil

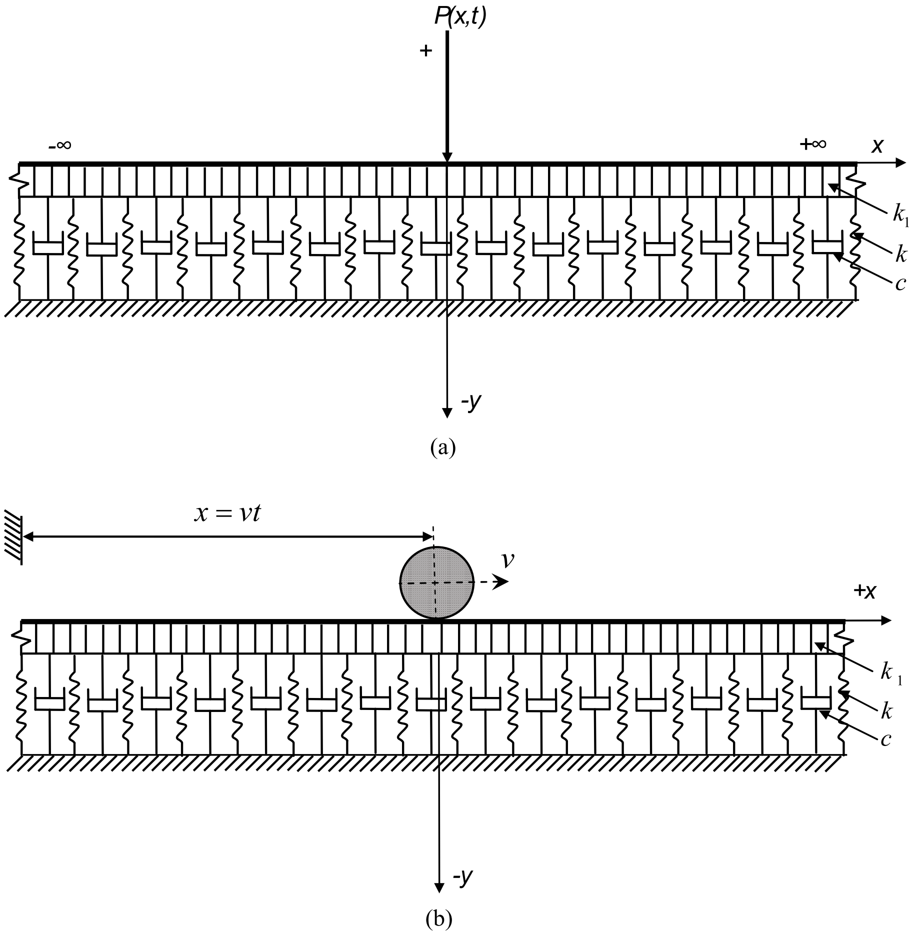


Fig. 1. Beam on Pasternak foundation subjected to a (a) moving load; and (b) moving mass.

is modeled as two-parameter Pasternak model. The beam and foundation both were assumed to be homogeneous and isotropic. Fourier transform technique is employed to find the analytical solution of the governing partial differential equation in case of moving load problem. The dynamic responses of the beam in terms of beam deflections, and bending moments have been obtained with different velocity ratios. The effects of shear modulus and foundation stiffness on deflection and bending moment responses have also been investigated.

2. Modeling of beam on Pasternak foundation

The governing equation of motion of an Euler-Bernoulli beam resting on two-parameter Pasternak foundation and subjected to a moving load or mass, as shown in Fig. 1, can be written as:

$$EI\frac{\partial^4 w}{\partial x^4} + \rho\frac{\partial^2 w}{\partial t^2} + c\frac{\partial w}{\partial t} - k_1\frac{\partial^2 w}{\partial x^2} + kw = F(x,t) \tag{1}$$

where $F\left(x,t\right) = P\delta(x - vt)$ in case of moving load, and

$$F\left(x,t\right) = \left[Mg - M\frac{\partial^2 w(vt,t)}{\partial t^2}\right]\delta(x - vt)$$

in case of moving mass, $w = w(x,t)$ is the transverse deflection of the beam, is the Young's modulus of elasticity of the beam material, I is the second moment of area of the beam cross section about its neutral axis, ρ is the mass per unit length of the beam, c is the coefficient of viscous damping per unit length of the beam, k_1 is the shear parameter of the beam, k is the spring constant of the foundation per unit length, $P\delta(x - vt)$ is the applied moving load per unit length, x is the space coordinate measured along the length of the beam, t is the time in second, M is the moving mass, g is the acceleration due to gravity, and δ is the Dirac delta function.

Defining the following,

$$a = \frac{\rho}{2EI}, b^2 = \frac{k}{EI}, c_1 = \frac{k_1}{2EI}, d = \frac{c}{EI},$$

Eq. (1) can be written as:

$$\frac{\partial^4 w}{\partial x^4} + 2a\frac{\partial^2 w}{\partial t^2} + d\frac{\partial w}{\partial t} - 2c_1\frac{\partial^2 w}{\partial x^2} + b^2 w = \frac{F}{EI}(x,t) \tag{2}$$

3. Method of analysis

3.1. Exact analysis

A Fourier transform pair is given by:

$$w * (\gamma, t) = \int_{-\infty}^{\infty} w(x,t)e^{-i\gamma x}dx \tag{3}$$

and,

$$w(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w * (\gamma, t)e^{+i\gamma x}d\gamma \tag{4}$$

where γ is a variable in complex plane. Multiplying both sides of Eq. (2) by $e^{-i\gamma x}$ and integrating by parts over x from $-\infty$ to $+\infty$, and assuming that w and its space derivatives vanish at $x = \pm\infty$, namely, for $x \to \pm\infty w(x) = w'(x) = w''(x) = w'''(x) = 0$, we get

$$\gamma^4 w * + b^2 w * + d\frac{dw*}{dt} - 2c_1\gamma^2 w * + 2a\frac{d^2 w*}{dt^2} = \frac{P}{EI}e^{-i\gamma vt} \tag{5}$$

The homogeneous part of the solution of Eq. (5) in the presence of light damping dies down and is neglected here. Thus, the steady state solution is given by only the particular integral of Eq. (5). Substituting $w* = W * e^{-i\gamma vt}$ in Eq. (5), we obtain

$$(\gamma^4 + b^2 - 2a\gamma^2 v^2 + 2c_1\gamma^2 - idv\gamma)W* = \frac{P}{EI} \tag{6}$$

or,

$$w* = \frac{P}{EI}\left[\frac{1}{\gamma^4 - 2a\gamma^2 v^2 + b^2 + 2c_1\gamma^2 - idv\gamma}\right]e^{-i\gamma vt} \tag{7}$$

Thus, from Eq. (4) one obtains

$$w(x,t) = \frac{P}{EI}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\gamma(x-vt)}}{\gamma^4 - 2a\gamma^2 v^2 + b^2 + 2c_1\gamma^2 - idv\gamma}d\gamma \tag{8}$$

The integral in Eq. (8) can be expressed as the limit (as shown in Fig. 2 (a)):

$$\int_{-\infty}^{\infty} F(\gamma)d\gamma = \lim_{\mathrm{R}\to\infty}\int_{-\mathrm{R}}^{+\mathrm{R}}\frac{e^{i\gamma(x-vt)}}{\gamma^4 - 2a\gamma^2 v^2 + b^2 + 2c_1\gamma^2 - idv\gamma}d\gamma \tag{9}$$

where R is the radius of semicircle C.

According to Cauchy's residue theorem, the integral in the counter-clock wise direction around the closed curve C consisting of segments $-R$, $+R$ and semicircle C at limit R $= \infty$ is

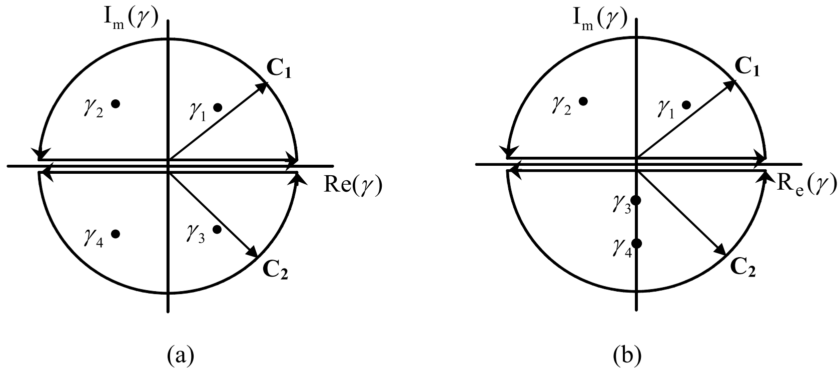


Fig. 2. Position of the poles for damping (a) less than (b) greater than the critical damping.

$$\frac{e^{i\gamma(x-vt)}}{\gamma^4-2a\gamma^2v^2+b^2+2c_1\gamma^2-idv\gamma}d\gamma=\lim_{R\to\infty}\int_{-R}^{+R}\frac{e^{i\gamma(x-vt)}}{\gamma^4-2a\gamma^2v^2+b^2+2c_1\gamma^2-idv\gamma}d\gamma$$
$$+\int_C\frac{e^{i\gamma(x-vt)}}{\gamma^4-2a\gamma^2v^2+b^2+2c_1\gamma^2-idv\gamma}d\gamma=2\pi i\sum_{j=1}^{n}residueF(\gamma)|_{\gamma=\gamma_j} \tag{10}$$

where $n=4$, is the number of poles in the circle.

The integral in Eq. (8) is evaluated by contour integration as discussed above [60], and we obtain

$$w(x,t)=\frac{P}{EI}\frac{1}{2\pi}(2\pi i)\sum \text{residues at four simple poles.} \tag{11}$$

Considering ($k_1=0$ and $d=0$), Let $s(\gamma)=\gamma^4-2a\gamma^2v^2+b^2$

$$\text{Poles at } \gamma:s(\gamma)=0\Rightarrow\gamma^4-2a\gamma^2v^2+b^2=0. \tag{12}$$

$$\text{Therefore, } \gamma^2_{1,2,3,4}=av^2\pm\sqrt{(av^2)^2-b^2} \tag{13}$$

Assume $v^2<\frac{b}{a}$ where the critical velocity is given by $v^2_{cr}=\frac{b}{a}$

$$\text{For } v<v_{cr},\gamma^2_{1,3}=av^2+i\sqrt{b^2-(av^2)^2}. \tag{14}$$

and

$$\gamma^2_{2,4}=av^2-i\sqrt{b^2-(av^2)^2}. \tag{15}$$

From Eq. (14), let the two roots of γ be

$$\gamma_1=p+ir, and \gamma_3=q-ir \tag{16}$$

Similarly, from Eq. (15) the other two roots of γ are

$$\gamma_2=-p+ir,\text{and}\gamma_4=-q-ir. \tag{17}$$

Let, the poles of the function of the complex variable in the integrand of Eq. (8) be in the form as shown in Fig. 2(a).

The poles are given by

$$\gamma_1=p+ir,\gamma_2=-p+ir,\gamma_3=q-ir,\gamma_4=-q-ir. \tag{18}$$

The values of p,q, and r are determined by equating the roots of the denominator inside the integral of Eq. (8) as:

$$\gamma^4-2a\gamma^2v^2+2c_1\gamma^2-idv\gamma+b^2=(\gamma-\gamma_1)(\gamma-\gamma_2)(\gamma-\gamma_3)(\gamma-\gamma_4) \tag{19}$$

Substituting the values of γ_i' s in terms of p, q and r from Eq. (18) into Eq. (19), and equating the coefficients of similar terms, we get

$$2r^2 - p^2 - q^2 = -2av^2 + 2c_1, \tag{20}$$

$$2ri(q^2 - p^2) = -idv, \tag{21}$$

$$(p^2 + r^2)(q^2 + r^2) = b^2. \tag{22}$$

From Eqs (20) and (21), we get

$$p^2 = av^2 - c_1 + r^2 + \frac{1}{4r}dv, \text{and, } q^2 = av^2 - c_1 + r^2 - \frac{1}{4r}dv, \tag{23}$$

By substituting the p and q values in Eq. (22), we get

$$r^6 + (av^2 - c_1)r^4 + \left(\frac{1}{4}a^2v^2 - \frac{1}{2}av^2 - \frac{1}{4}b^2 + \frac{1}{4}c_1^2\right)r^2 - \frac{1}{16}d^2v^2 = 0 \tag{24}$$

3.1.1. *Calculating the residues*

For $\xi(= x - vt) > 0$, as shown in Fig. 2(a), integrating in the upper half plane:

$$w(x,t) = \frac{iP}{EI}\,[\text{Residues at } \gamma_1 + \text{ Residues at } \gamma_2].$$

or,

$$w(x,t) = \frac{iP}{EI}\left[\frac{e^{i\gamma_1\xi}}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_4)} + \frac{e^{i\gamma_2\xi}}{(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3)(\gamma_2 - \gamma_4)}\right]. \tag{25}$$

Substituting γ_i',s in terms of p, q and r(as chosen in Eq. (18)), we finally get

$$\begin{aligned} w(x,t) &= \frac{P}{EIp(A_1^2 + A_2^2)}\left[(A_1 - A_2 i)e^{i\gamma_1\xi} + (A_1 + A_2 i)e^{i\gamma_2\xi}\right] \\ &= \frac{2P}{EIp(A_1^2 + A_2^2)}e^{-r\xi}(A_1\cos p\xi + A_2 \sin p\xi) \end{aligned} \tag{26}$$

Similarly, for

$$\begin{aligned} \xi < 0, w(x,t) &= \frac{P}{EIq(A_3^2 + A_4^2)}\left[(A_3 + A_4 i)e^{i\gamma_3\xi} + (A_3 - A_4 i)e^{i\gamma_4\xi}\right] \\ &= \frac{2P}{EIq(A_3^2 + A_4^2)}e^{r\xi}(A_3\cos q\xi - A_4 \sin q\xi). \end{aligned} \tag{27}$$

where,

$$A_1 = pr, A_3 = qr, A_2 = r^2 - \frac{1}{4}(p^2 - q^2), A_4 = r^2 + \frac{1}{4}(p^2 - q^2). \tag{28}$$

When the damping is above the critical damping, both p^2 and q^2 in Eq. (23) become negative. Therefore, the poles can be designated in this case as shown in Fig. 2(b).

$$\gamma_1 = p + ir, \quad \gamma_2 = -p + ir, \quad \gamma_3 = -(r - q)i, \gamma_4 = -(r + q)i. \tag{29}$$

3.2. *Modal analysis*

The deflection mode of continuous beam with simply supported boundary conditions can be derived from the Euler-Bernoulli equation of the beam. The deflection modes and the natural frequencies of an Euler beam can be expressed as:

$$Y_k(x) = \sin\left(\frac{k\pi x}{l}\right);\ \text{and}\ \omega_k = \left(\frac{k\pi}{l}\right)^2\sqrt{\frac{EI}{m_r}}, k = 1, 2, 3 \ldots\ldots\ldots K \tag{30}$$

where $Y_k(x)$ is the deflection mode, ω_k is the corresponding natural frequency and l is the beam length.

The above equations are obtained for the following boundary conditions:

$$Y(0) = Y(l) = 0; \text{ and } Y''(0) = Y''(l) = 0 \tag{31}$$

Assume a solution of Eq. (1) in the form of a series:

$$w_r\,(x,t) = \sum_{k=1}^{K} Y_k(x) q_k(t) \quad k = 1,2,3\ldots K \tag{32}$$

where $Y_k(x)$ are the eigenfunctions of the beam, $q_k(t)$ are the functions of time which have to be found and k is the number of contributed modes. Recalling Eq. (1) for moving mass problem as:

$$EI\frac{\partial^4 w}{\partial x^4} + \rho\frac{\partial^2 w}{\partial t^2} + c\frac{\partial w}{\partial t} - k_1\frac{\partial^2 w}{\partial x^2} + kw = \left[Mg - M\frac{\partial^2 w(vt,t)}{\partial t^2}\right]\delta(x - vt) \tag{33}$$

The derivative $\frac{\partial^2 w(vt,t)}{\partial t^2}$ is:

$$\frac{\partial^2 w(vt,t)}{\partial t^2} = \left[v^2\frac{\partial^2 w(x,t)}{\partial x^2} + 2v\frac{\partial^2 w(x,t)}{\partial x \partial t} + \frac{\partial^2 w(x,t)}{\partial t^2}\right]_{x=vt} \tag{34}$$

It has been stated that the first two terms of R.H.S of Eq. (34) can be neglected in case of low speed range [13,40]. The analysis of moving mass in the present study is carried out in low speed range, which is practical speed range for typical North American railway. That's why, first two terms of Eq. (34) is also ignored in the present study.

The substitution of Eq. (30) together with Eq. (32) into the Eq. (33) and using a simplified subscript for differentiation yields:

$$\begin{aligned} &EI\left(\frac{k\pi}{l}\right)^4\sum_{k=1}^{K} Y_k(x)q_k(t) + \rho\sum_{k=1}^{K} Y_k(x)\ddot{q}_k(t) + c\sum_{k=1}^{K} Y_k(x)\dot{q}_k(t) - k_1\left(\frac{k\pi}{l}\right)^2\sum_{k=1}^{K} Y_k(x)q_k(t) \\ &\quad + k\sum_{k=1}^{K} Y_k(x)q_k(t) = \left[Mg - M\sum_{k=1}^{K} Y_k(vt)\ddot{q}_k(t)\right] \end{aligned} \tag{35}$$

Multiplying both sides of Eq. (35) by $Y_p(x)$ and integrating along the beam length and employing the orthogonality property among the normal modes, it becomes after some rearrangement

$$\begin{aligned} &\frac{EI}{\rho}\left(\frac{k\pi}{l}\right)^4 q_k(t) + \ddot{q}_k(t) + \frac{c}{\rho}\dot{q}_k(t) - \frac{k_1}{\rho}\left(\frac{k\pi}{l}\right)^2 q_k(t) + \frac{k}{\rho}q_k(t) = \\ &\frac{2}{\rho l}\left[Mg - M\sum_{k=1}^{K} Y_k(vt)\ddot{q}_k(t)\right] Y_p(vt) \end{aligned} \tag{36}$$

for $k = 1,2,3\ldots K$.

In case of moving load problem, Eq. (36) can be written as:

$$\frac{EI}{\rho}\left(\frac{k\pi}{l}\right)^4 q_k(t) + \ddot{q}_k(t) + \frac{c}{\rho}\dot{q}_k(t) - \frac{k_1}{\rho}\left(\frac{k\pi}{l}\right)^2 q_k(t) + \frac{k}{\rho}q_k(t) = \frac{2}{\rho l}P\sum_{k=1}^{K} Y_k(vt)Y_p(vt) \tag{37}$$

for $k = 1,2,3\ldots K$.

Equations (36) and (37) are sets of coupled ordinary differential equations and a numerical procedure is employed to solve them. A MATLAB library function "ode 45" is applied to solve the differential equations in time domain. The equations are solved simultaneously to obtain the deflection and bending moment responses for both moving mass and moving load conditions.

The validity of the present model was examined by comparing the responses in terms of deflection and bending moment of the beam with the analytical results presented in [28]. In this reported study, an infinite Euler–Bernoulli beam of constant cross-section resting on an elastic foundation with both one and two parameters was considered. The beam was subjected to a constant point load moving with a constant speed along the beam. Dynamic responses

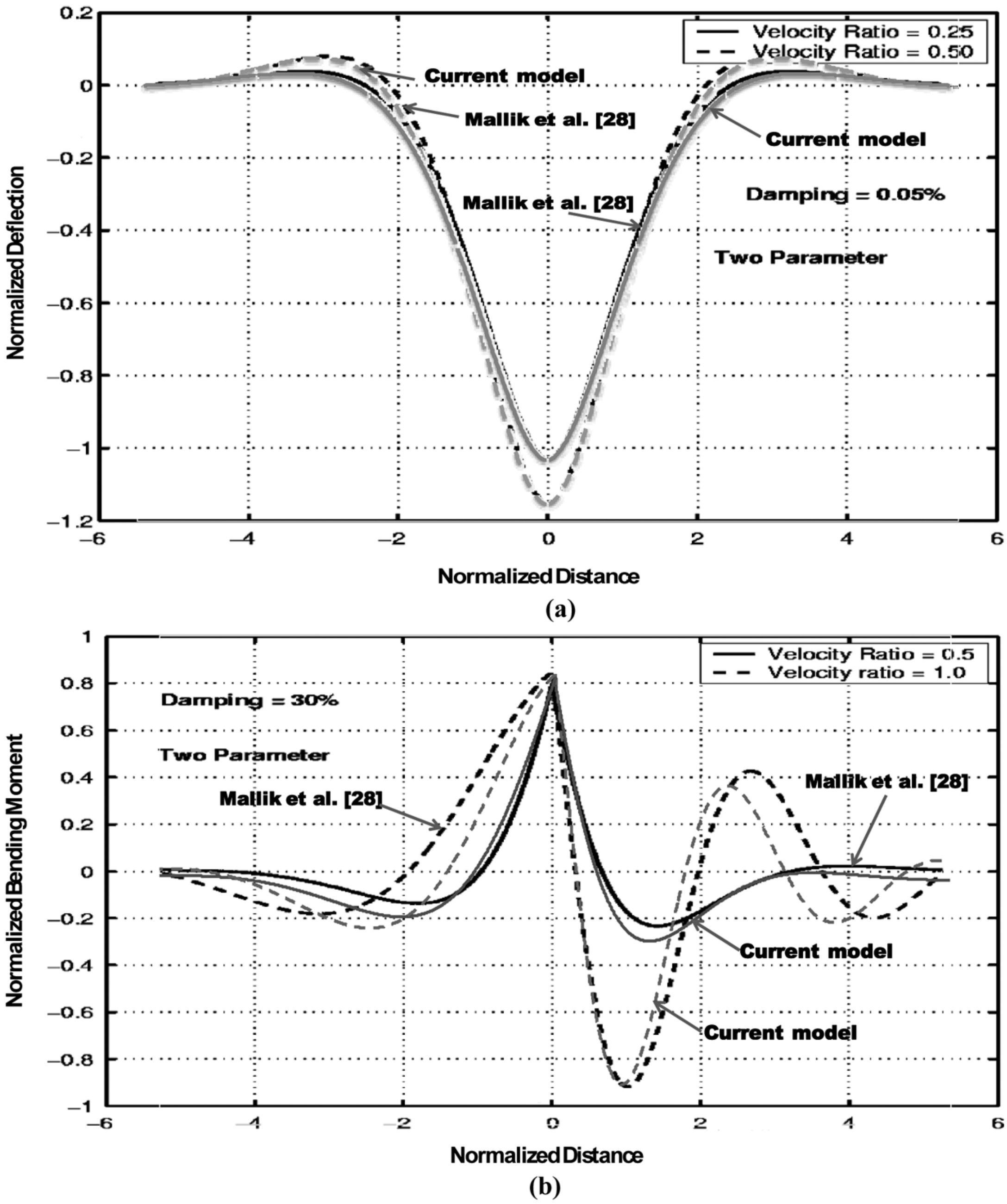
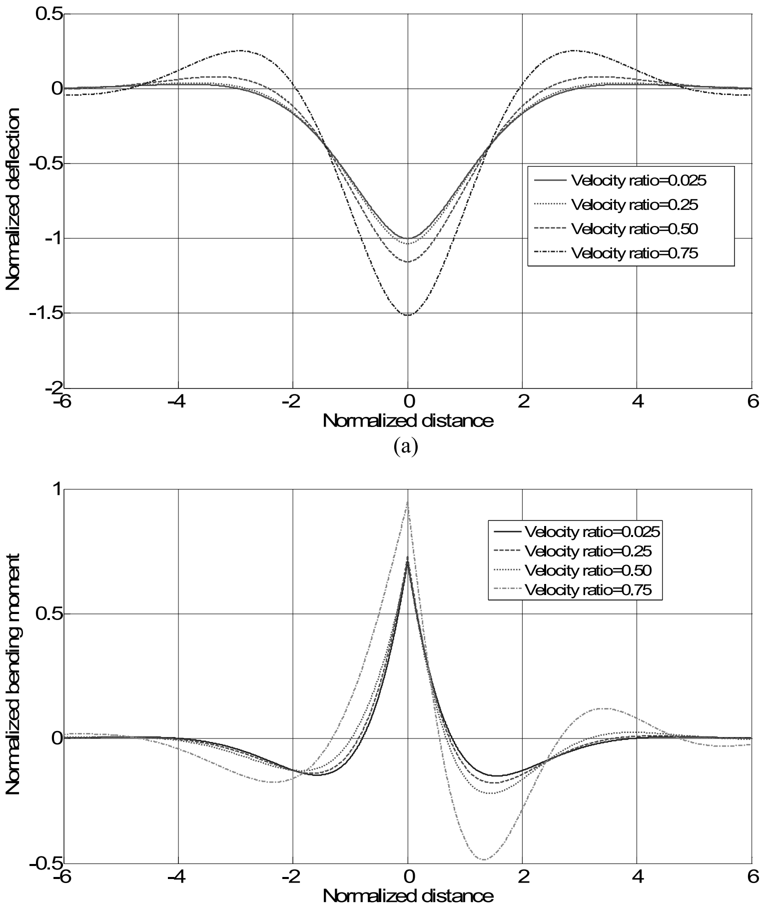


Fig. 3. Comparison of beam (a) deflection and (b) bending moment responses of the present model with that reported by Mallik et al. [28].

such as beam deflection, bending moment have been obtained for different velocity ratios. All the parameters considered in the validation are taken from Mallik et al. [28] and presented in Table 1. The variations in beam deflection and bending moment responses obtained from the present study are evaluated and compared with those reported in [28], as shown in Fig. 3. It can be seen that the deflection and bending moment response predicted by the current model agrees reasonably well with the response reported in [28]. The amplitude of deflection and bending moment and the period of vibrations predicted by both models are also in very good agreement, although some differences in the bending moment responses become evident.

Table 1
Simulation parameters [28]

Parameters	Values
ρ (kg/m)	25
EI (N-m^2)	1.75×10^6
K (N/m^2)	40.78×10^5
k_1 (N)	66.6875×10^4
P (N/m)	99.36×10^3



(a) (b)

Fig. 4. Normalized (a) deflection and (b) bending moment responses with different velocity ratios (c/c_{cr} = 0.0005).

4. Results and discussions

Numerical computations have been carried out for both exact and modal analysis methods described in Section 3. The normalized dynamic responses in terms of deflection, bending moment are obtained with respect to the normalized position of the beam. The deflection is normalized by dividing the response obtained by its maximum value for the static case. The normalized distance is obtained by multiplying the distance along the beam (ahead and behind the load) by the coefficient of characteristic wavelength, which can be expressed as: $\lambda = \sqrt{b/2}$. The velocity at which the deflection response shoots up to infinity is known as critical velocity, which can be expressed as [28]: $v_{cr} = \sqrt{(b + c_1)/a}$. The normalized bending moment is obtained by dividing the bending moment response by its maximum value for the static case. the parameters used in the present study are given in Table 1 [28].

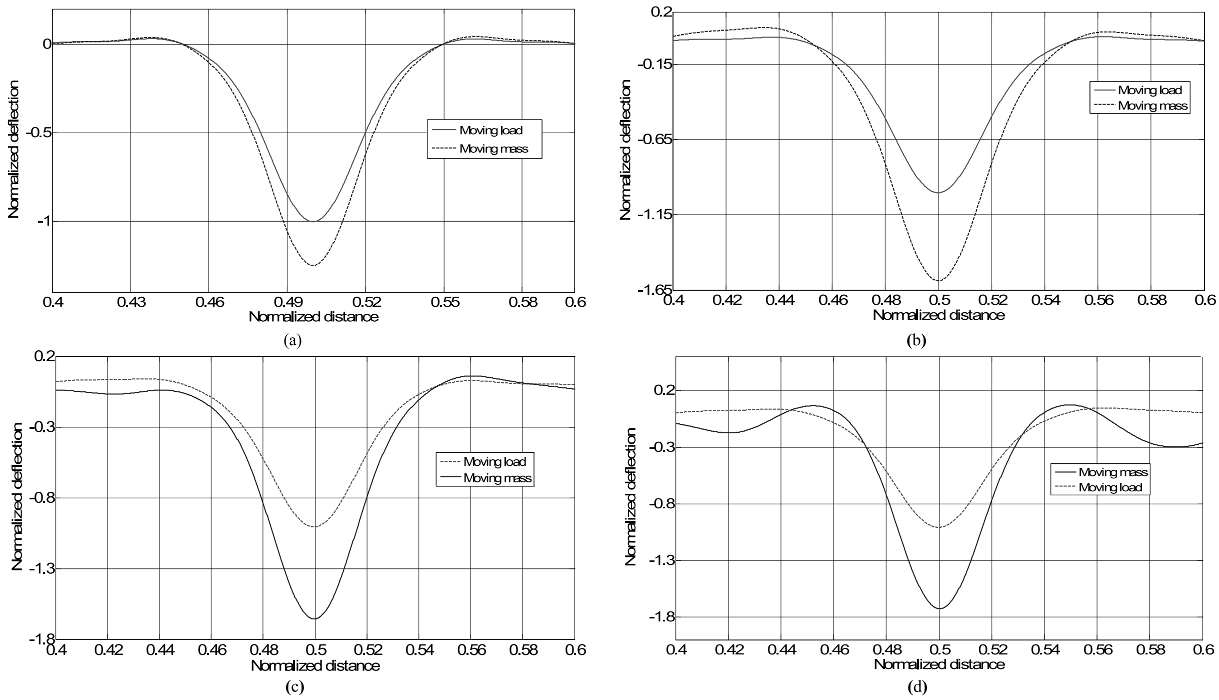


Fig. 5. Normalized deflection response with different velocity ratios; (a) velocity ratio = 0.025; (b) velocity ratio = 0.05; (c) velocity ratio = 0.075; (d) velocity ratio = 0.1.

The normalized dynamic deflection of an infinite long beam resting on two-parameter Pasternak foundation subjected to a moving load with various velocity ratios (v/v_{cr}) is shown in Fig. 4. In Fig. 4(a), the results show that the deflections are symmetric about the contact point with moving load for all the selected velocity ratios and the maximum deflections occur beneath the load and die down far away from the load. The negative value of the deflection signifies the downward deflection with the moving load acting downward. A small amount of uplift (positive deflection) occurs at distances away from the load. The figure further shows that the maximum dynamic deflection for both upward and downward deflection increases with increase in the velocity ratios of the moving load.

The variations in normalized bending moment for velocities less than the critical velocity along the beam are shown in Fig. 4(b). The figure shows small variations in the maximum negative bending moment in front of and behind the load below the critical velocity. The figure further shows that maximum negative bending moment occurs at a point ahead of the load and the magnitude of this peak negative bending moment increases with increase in speed. However, the maximum positive bending moment occurs at the point of the load. With decreasing speed, the maximum negative bending moments both in front and rear of the load shift away from the load.

4.1. Modal analysis

Dynamic response of an Euler-Bernoulli beam resting on Pasternak foundation is further investigated using the modal analysis approach under both moving load and moving mass conditions. In case of "moving mass" condition, the mass of the wheel is considered to exert force on the beam equivalent to the moving load. The parameters listed in Table 1 are employed in this study. It has been suggested that a total of 10 to 15 modes would be sufficient for evaluating deflection and bending moment of the beam [21]. In this study, 60 modes are considered in order to have accurate dynamic analysis of the beam in terms of deflection and bending moment. The normalized dynamic response in terms of deflection, bending moment for both "moving load" and "moving mass" conditions is obtained with respect to the normalized position of the beam.

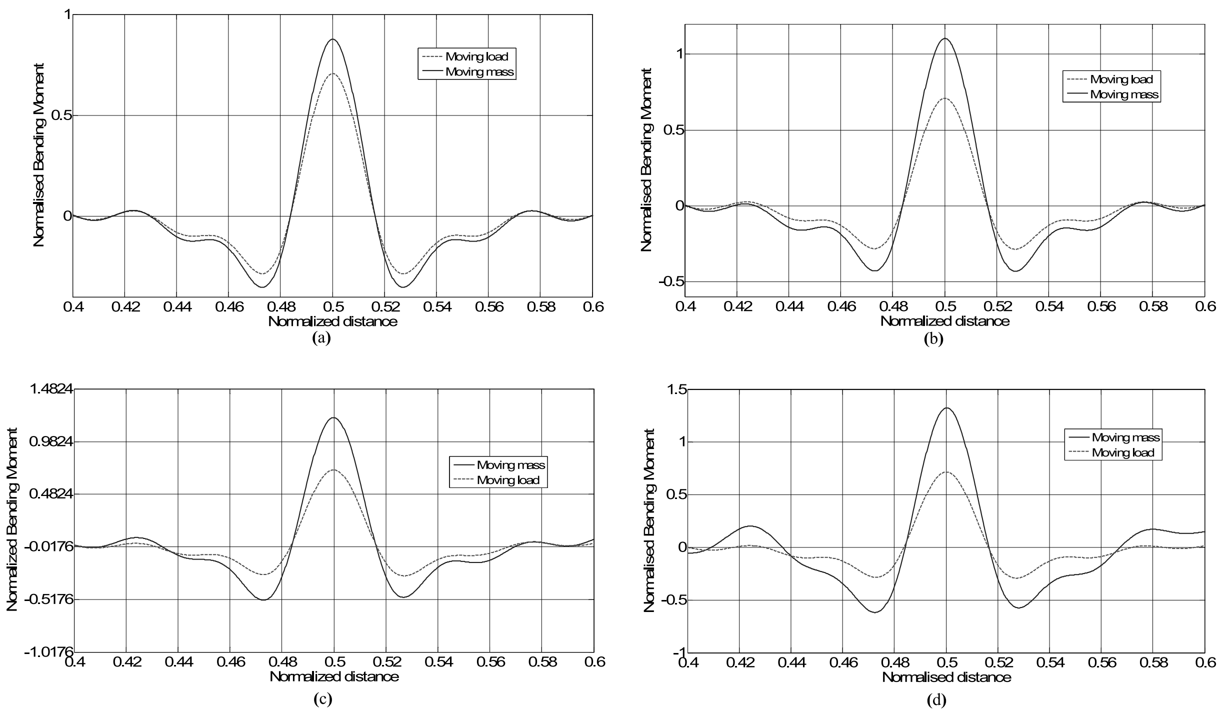
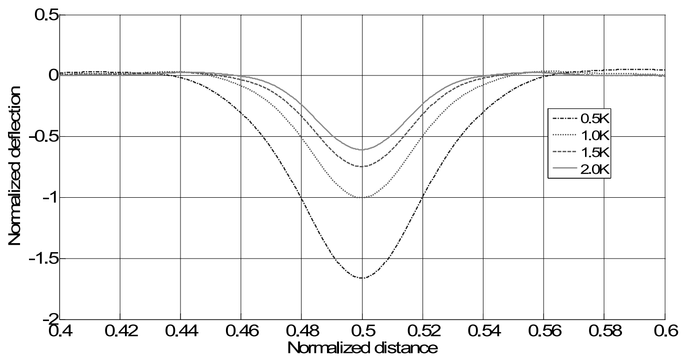


Fig. 6. Normalized bending moment responses with different velocity ratios; (a) velocity ratio = 0.025; (b) velocity ratio = 0.05; (c) velocity ratio = 0.075; (d) velocity ratio = 0.1.

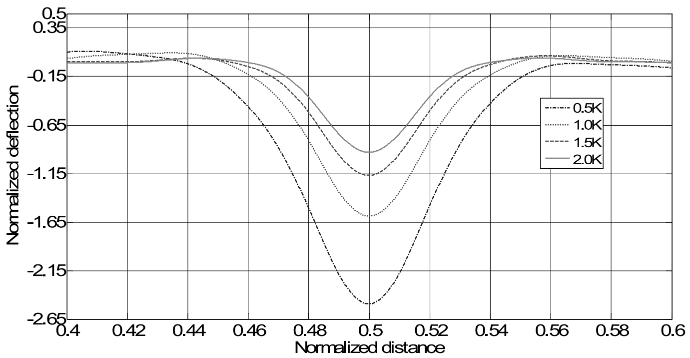
4.1.1. Comparison between moving load and moving mass responses

Figure 5 shows normalized deflection responses for both moving load and moving mass systems for different velocity ratios. The damping ratio of the beam is kept at 0.0005. The figure shows that the maximum deflections occur, for all the given conditions, at the midpoint of the beam. However, the maximum deflection is always higher in case of moving mass, which is attributed to the inertia of the mass. The maximum normalized deflections for moving mass problem are 1.248, 1.582, 1.656, and 1.725 for velocities of 44.12, 88.25, 132.38, and 176.50 km/h, respectively. The maximum normalized deflections for moving load problem are 1.002, 1.003, 1.005, and 1.007 for velocities of 44.12, 88.25, 132.38, and 176.50 km/h, respectively. These results show that the variations in peak normalized deflections for moving mass problem are significant with variations in speed for the given speed range. This variation is not significant for moving load problem. These variations, however, can be significant for higher velocity ratios as shown in Fig. 4(a).

The variations in normalized bending moment response for both moving load and moving mass systems with different velocity ratios are shown in Fig. 6. Similar to the deflection response, the maximum bending moments occur beneath the load for all the given conditions. Figure 6 shows that the maximum bending moment is always higher for moving mass system irrespective of the speed. The maximum normalized bending moments for moving mass problem are 0.8767, 1.104, 1.211, and 1.326 for velocities of 44.12, 88.25, 132.38, and 176.50 km/h, respectively. In case of moving load problem, the maximum normalized bending moments are 0.7063, 0.7096, 0.7127, and 0.7157 for velocities of 44.12, 88.25, 132.38, and 176.50 km/h, respectively. The figures further show that the peak bending moments increase significantly with increase in the speeds in case of moving mass problem for the considered speed range. Thus, simplification of the moving vehicle as a moving load in the analysis of dynamic vehicle-track interactions can result in lower prediction of the deflections and bending moments of the track. In the case of moving load, the increment is not so pronounced which can be attributed to the considered speed ratios. For higher speed ratios, as shown in Fig. 4(b), this increment is significant. It can be concluded that considering the moving load instead of moving mass essentially neglects the inertia of the mass, centrifugal force, velocity of the moving mass which in turn gives inaccurate prediction of the deflections, bending moments and contact forces that arise in between the contact points.



(a)



(b)

Fig. 7. Effect of foundation stiffness on dynamic deflection responses at a constant speed of 88.25 km/h; (a) moving load (b) moving mass.

4.1.2. Effect of foundation stiffness

Foundation stiffness has significant influence on dynamic deflection response under both moving load and moving mass conditions, as shown in Fig. 7. For both cases, decreasing the foundation stiffness increases significantly the dimensionless peak deflections, while the speed is kept constant at 88.25 km/h. However, the rate of increment is slightly higher for moving load problem. When the value of the foundation stiffness decreases from $2.0k$ to $0.5k$, the normalized maximum deflection increases from 0.607 to 1.667 for moving load, while it is 0.9298 to 2.493 for moving mass.

The effect of foundation stiffness on normalized dynamic bending moment response for both moving load and moving mass conditions is shown in Fig. 8. Similar to the deflection response, for a constant speed, decreasing the foundation stiffness increases the peak of the dimensionless bending moment. The rate of increment is slightly higher for the moving load problem. In the case of moving load, the maximum normalized bending moment increases from 0.5541 to .8907, while for moving mass it increases from 0.8545 to 1.362, when the value of the foundation stiffness decreases from $2.0k$ to $0.5k$.

4.1.3. Effect of shear modulus

The effect of shear modulus on dynamic normalized deflection responses for both moving load and moving mass conditions is shown in Fig. 9. The figure shows that, for a constant speed, increasing the shear modulus decreases the peak of the dimensionless deflection. The rate of decrease is slightly higher for moving load problem. When the value of the shear modulus decreases from $3.0k_1$ to $0.1k_1$, the maximum normalized deflection increases by about 17.71% and 15.22% for moving load and for moving mass, respectively.

Figure 10 shows the effect of shear modulus on normalized bending moment responses for both moving load and moving mass conditions at a constant speed of 88.25 km/h. the figure clearly shows that decrease in shear modulus increases the normalized bending moment for given conditions. when the value of the shear modulus decreases from

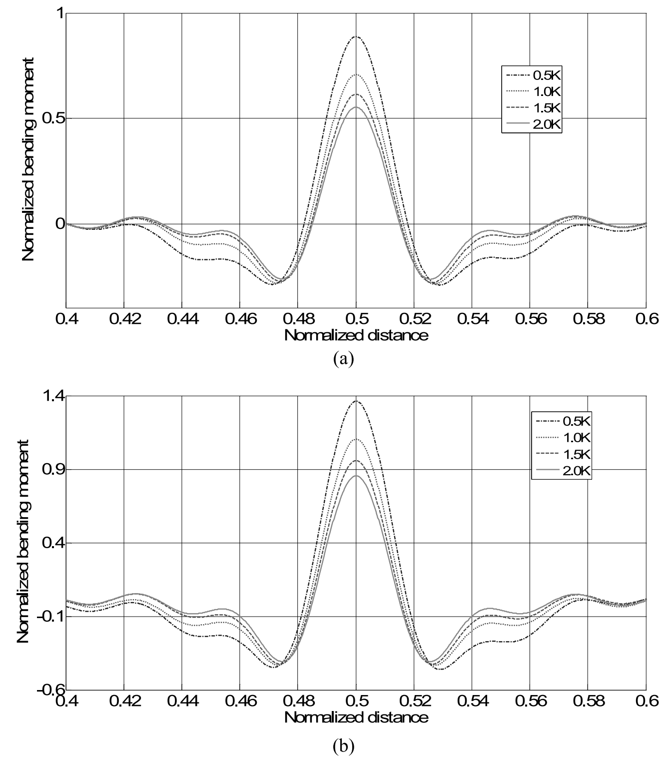


Fig. 8. Effect of foundation stiffness on dynamic bending moment responses at a constant speed of 88.25 km/h; (a) moving load (b) moving mass.

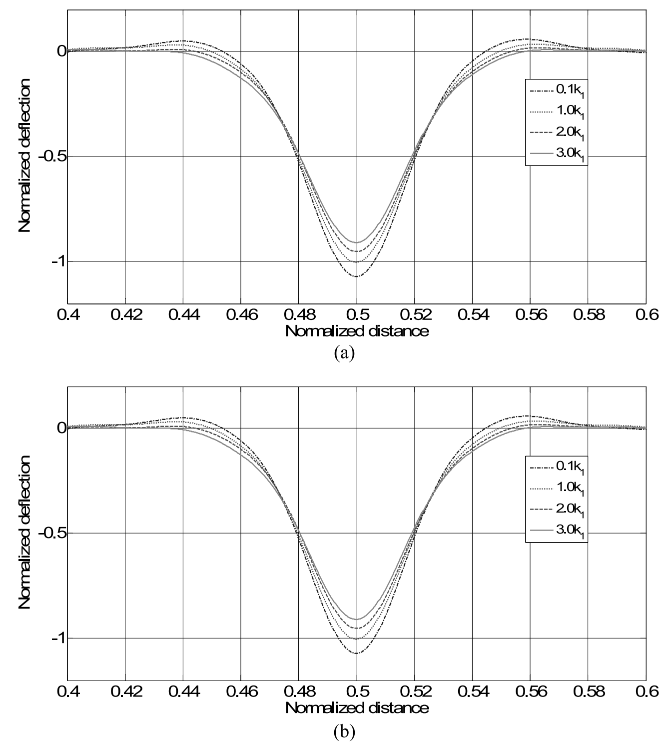


Fig. 9. Effect of shear modulus on dynamic deflection responses at a constant speed of 88.25 km/h; (a) moving load (b) moving mass.

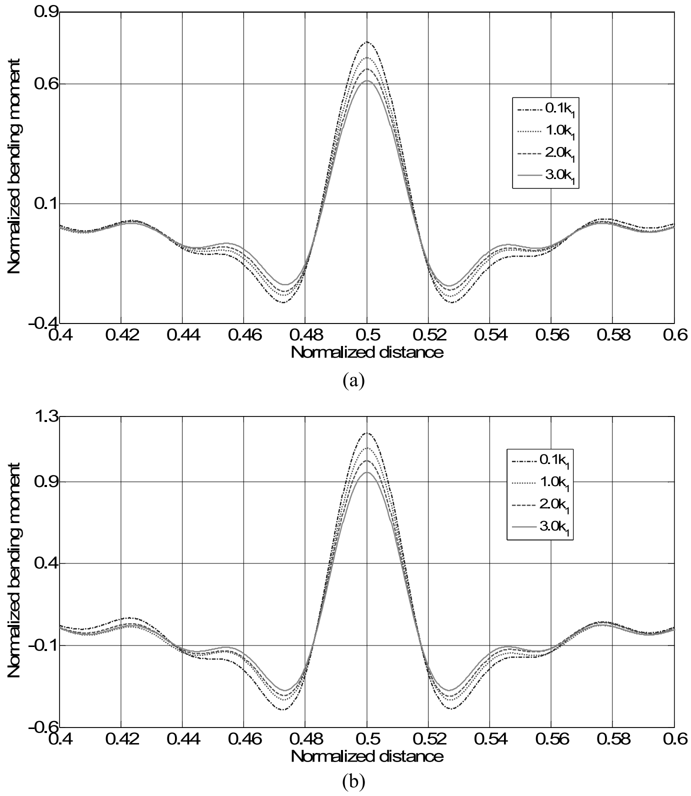


Fig. 10. Effect of shear modulus on dynamic bending moment responses at a constant speed of 88.25 km/h; (a) moving load (b) moving mass.

$3.0k_1$ to $0.1k_1$, the maximum normalized bending moment increases by about 26.44% and 25.40% for moving load and for moving mass, respectively.

5. Conclusions

This paper investigates the dynamic response of an Euler-Bernoulli beam supported on two-parameter Pasternak foundation and subjected to a moving load as well as moving mass. Fourier transform technique is employed to find the exact analytical solution of the governing partial differential equation of an infinite beam subjected to a moving load. Modal analysis is employed to convert the partial differential equation into a series of ordinary differential equations for both moving load and moving mass conditions. Numerical calculations have been performed to analyze the displacement and bending moment responses of the beam on the Pasternak foundation subjected to both moving load and moving mass with different velocity ratios. The study reveals that the results obtained from modal analysis are comparable to those obtained from exact analytical method. This study further reveals that moving mass has significantly higher effect on dynamic responses of the beam over the moving load. The study further shows that increase in speed increases both displacements and bending moments of the beam. In the case of moving load, the study shows that the deflections are symmetric about the contact point and maximum deflections occur beneath the load for all the selected velocity ratios. The study further shows that deflection and bending moment responses are always higher for moving mass condition than those for the moving load. Detailed analyses are also performed to investigate the effect of various parameters such as foundation stiffness and shear modulus on dynamic deflection and bending moment responses under a moving load as well as moving mass. The study shows that the maximum deflection and bending moment of the beam increases significantly with the reduced shear modulus and stiffness of the foundation for a constant velocity. For a given velocity, this increment is always higher for moving load than the moving mass. For the given range of shear modulus, the normalized maximum deflection increases about 26.44% and 25.40% for moving load and moving mass, respectively.

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