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Research Article

Local Exponential Regulation of Nonholonomic Systems in Approximate Chained Form with Applications to Off-Axle Tractor-Trailers

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Most of drift-less nonholonomic systems cannot be exactly converted to an nonholonomic chained form, a wealth of design tools developed for the control of nonholonomic chained form are thus not directly applicable to such systems. Nevertheless, there exists a class of systems that may be locally approximated by the nonholonomic chained form around certain equilibrium points. In this work, we propose a discontinuous and a smooth time-varying control laws respectively for the approximated nonholonomic chained form, guaranteeing local exponential convergence of state to the desired equilibrium point. An tractor towing off-axle trailers is taken as an example to illustrate the approaches.

1. Introduction

The so-called *nonholonomic chained form* (NCF) has motivated many research activities for about twenty years [1]. Several features such as flatness [2, 3], homogeneity, and nilpotency make the NCF especially attractive to work with. These properties have been used for designing control laws to achieve several control objects such as point stabilization and trajectory tracking. Concerning the point stabilization problem of NCF, which is difficult due to Brokett's well-known obstruction [4], a number of approaches have been developed, which may be roughly classified into discontinuous time-invariant feedback [5–7], continuous time-varying feedback [8–10], and hybrid feedback [11, 12]. The stabilization problems of NCF with parameter uncertainties and perturbation terms have also been attacked in recent years [13–17]; however, most of these researches require that the perturbation terms satisfy certain cascaded conditions, which may be very restrictive and thus rule out many interesting examples such as the tractor-trailers with off-axle hitching [18] and the ball-plate systems [19]. It is also mentioned that the dynamics of many nonholonomic driftless systems can be approximated by NCF locally around certain equilibrium points. In [18], a time-varying continuous stabilizing scheme was proposed for such approximate NCF, achieving local exponential stability of the closed-loop system around the selected equilibrium point.

In this paper, we consider the local exponential regulation problem of a class of nonholonomic systems convertible to the approximate NCF. By employing a discontinuous and/or a smooth time-varying coordinate transformations, the approximate NCF is converted to linear perturbed ones with the perturbation terms being second or higher orders of the converted states; then a discontinuous time-invariant and/ or a smooth time-varying control laws are derived respectively, guaranteeing that the state of the approximate NCF converges to zero exponentially, provided the norm of an initial state is sufficiently small. Compared with the control law presented in [18] which is continuous but not differentiable, the time-varying control law proposed in this paper is smooth and can be easily extended to deal with input dynamics.

The paper is organized as follows. Section 2 defines a class of systems that can be approximated by NCF. In Section 3, a discontinuous time-invariant and a smooth time-varying controllers are developed to stabilize the approximate NCF. In Section 4, a tractor-trailer with off-axle hitching is taken as an example to illustrate the effectiveness of the proposed controllers. Section 5 concludes the paper.

2. A Class of Approximated Chained Forms

Consider the following nonlinear system represented by

$$\dot{x}_0 = u_0, \tag{1}$$

$$\dot{x} = g_0(x)u_0 + g_1(x)u_1, \tag{2}$$

where $x_0 \in \mathfrak{R}$, $x \in \mathfrak{R}^n$ are state variables and $u_0 \in \mathfrak{R}$, $u_1 \in \mathfrak{R}$ are control inputs. The control vector fields $g_0(x) \in \mathfrak{R}^n$, $g_1(x) \in \mathfrak{R}^n$ are supposed to have the following forms:

$$g_0(x) = Ax + R_2(x), \qquad g_1(x) = b + R_1(x), \tag{3}$$

where

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{4}$$

$R_1(x) \in \mathfrak{R}^n$ denotes the first-or higher-order residual term of x and $R_2(x) \in \mathfrak{R}^n$ the second or higher-order residual term of x in the state domain D; or, say more precisely, there exist three positive constants r, r_1, and r_2 such that $R_1(x), R_2(x)$ are bounded by

$$\|R_1(x)\|_2 \le r_1\|x\|_2, \qquad \|R_2(x)\|_2 \le r_2\|x\|_2^2 \tag{5}$$

in the compact set $\Omega = \{x : \|x\|_2 \le r\} \in D$.

System (1)-(2) is called the approximate NCF if (3)–(5) are satisfied.

Remark 1. Without loss of generality, it is specially assumed in (4) that $\{A, b\}$ is in the canonical controllable form. For the controllable pair $\{A, b\}$ not in this form, one can always find a linear state transformation to convert it to this form.

Remark 2. It is noted that the approximate NCF (1)-(2) is not flat with certain defects [2] and thus difficult to control.

The approximate NCF represents a large class of nonholonomic systems that cannot be converted to NCF in which $R(x) = 0$. The examples of approximate NCF include tractor-trailers with off-axle hitching [18] and the ball-plate systems [19].

3. Local Exponential Regulation of the Approximate NCF

In this section, a discontinuous and a smooth time-varying control laws are derived to solve the local exponential regulation problem of the approximate NCF defined in (1)–(5).

3.1. Local Exponential Regulation of the Approximate NCF for $x_0(0) \neq 0$. The control law for the first control input is designed as

$$u_0 = -k_0 x_0, \tag{6}$$

with $k_0 > 0$, so that $x_0(t) = x_0(0)e^{-k_0 t} \neq 0 (\forall x_0(0) \neq 0, 0 \le t < \infty)$.

Substituting (6) into (2) results

$$\dot{x} = -k_0 x_0 (Ax + R_2(x)) + (b + R_1(x))u_1. \tag{7}$$

Inspired by the well-known σ-process [5], we introduce the following discontinuous state transformation:

$$y = T^{-1}(x_0)x, \qquad x = T(x_0)y \tag{8}$$

with

$$\begin{aligned} T(x_0) &= x_0^m \operatorname{diag}\{1, x_0, x_0^2, \dots, x_0^{n-1}\}, \\ T^{-1}(x_0) &= x_0^{-m} \operatorname{diag}\left\{1, x_0^{-1}, x_0^{-2}, \dots, x_0^{-(n-1)}\right\}, \end{aligned} \tag{9}$$

and m a positive integer to be determined.

Remark 3. The discontinuous coordinate transformation (8)-(9) is a generalization of the ordinary $\sigma-$process proposed in [5] with $m = 0$ for NCF. It is seen in what follows that the term x_0^m with $m > 0$ is crucial for the controller design of the approximate NCF.

The transformation matrix $T(x_0)$ is clearly nonsingular for $x_0(0) \neq 0$, $0 \le t < \infty$.

The dynamics of the transformed state y can be derived as

$$\begin{aligned} \dot{y} &= T^{-1}(x_0)\dot{x} + \frac{d}{dt}(T^{-1}(x_0))x \\ &= -k_0 x_0 T^{-1}(x_0)AT(x_0)y + T^{-1}(x_0)bu_1 \\ &\quad + T^{-1}(x_0)(-k_0 x_0 R_2 + R_1 u_1) + \frac{d}{dt}(T^{-1}(x_0))T(x_0)y. \end{aligned} \tag{10}$$

Direct calculation reveals that

$$\begin{aligned} &T^{-1}(x_0)b = x_0^{-m}b, \\ &x_0 T^{-1}(x_0)AT(x_0) = A, \\ &\frac{d}{dt}(T^{-1}(x_0))T(x_0) \\ &\qquad = k_0 \operatorname{diag}\{m, m+1, m+2, \dots, m+n-1\}. \end{aligned} \tag{11}$$

Substituting the above identities into (10) results in

$$\dot{y} = A_1 y + x_0^{-m} b u_1 + T^{-1}(x_0)(-k_0 x_0 R_2 + R_1 u_1), \tag{12}$$

where

$$A_1 = k_0(-A + \operatorname{diag}\{m, m+1, m+2, \dots, m+n-1\}). \tag{13}$$

Remark 4. As $\{A, b\}$ is controllable, so is $\{A_1, b\}$; hence, the eigenvalues of $A_1 - bK$ can be arbitrarily assigned by selecting the control gain K.

The second control input is designed as

$$u_1 = -x_0^m K y = -x_0^m K T^{-1}(x_0)x, \tag{14}$$

where $K = [k_1, k_2, \ldots, k_n]$ is a control gain row vector such that $A_1 - bK$ is Hurwitz.

The closed-loop system of (12) and (14) becomes

$$\begin{aligned}\dot{y} &= (A_1 - bK)y - T^{-1}(x_0)(k_0 x_0 R_2 + R_1 x_0^m K y) \\ &= (A_1 - bK)y + R,\end{aligned} \tag{15}$$

where

$$R = -T^{-1}(x_0)(k_0 x_0 R_2 + K y x_0^m R_1). \tag{16}$$

System (15) is a linear stable one perturbed by a residual term R. If R can be shown to be second or higher order of $\|y\|$, then (15) is locally exponential stable.

In view of (5), the converted residual term R is bounded by

$$\begin{aligned}\|R\|_2 &\le \|T^{-1}\|_2 \|k_0 x_0 R_2 + K x_0^m y R_1\|_2 \\ &\le \|T^{-1}\|_2 \left(r_2 k_0 |x_0| \|x\|_2^2 + r_1 \|K\|_2 |x_0|^m \|x\|_2 \|y\|_2 \right) \\ &\le \Big(r_2 k_0 |x_0| \|T^{-1}\|_2 \|T\|_2^2 \\ &\quad + r_1 \|K\|_2 |x_0|^m \|T^{-1}\|_2 \|T\|_2 \Big) \|y\|_2^2 \\ &= h(|x_0|) \|y\|_2^2\end{aligned} \tag{17}$$

with $h(|x_0|)$ defined as

$$\begin{aligned}h(|x_0|) \cong\, & r_2 k_0 \max\left\{|x_0|^{m+1}, |x_0|^{m-n+2}\right\} \max\left\{1, |x_0|^{2(n-1)}\right\} \\ & + r_1 \|K\|_2 \max\left\{|x_0|^m, |x_0|^{m-(n-1)}\right\} \\ & \times \max\left\{1, |x_0|^{n-1}\right\}.\end{aligned} \tag{18}$$

As $|x_0| \le |x_0(0)|$, $h(|x_0|)$ is thus bounded uniformly with t provided $m - (n-1) \ge 0$. In view of the facts that $A_1 - bK$ is Hurwitz and $\lim_{\|y\|_2 \to 0}(\|R\|_2/\|y\|_2) = 0$, system (15) is thus locally exponential stable by Lyapunov indirect approach [20].

The above analysis is summarized as the following proposition.

Proposition 1. *Suppose that* $0 < |x_0(0)| < \infty$, $k_0 > 0$, $m \ge n-1$, *K is selected such that $A_1 - bK$ is Hurwitz then the following control law*

$$u_0 = -k_0 x_0, \qquad u_1 = -x_0^m K y = -x_0^m K T^{-1}(x_0) x \tag{19}$$

guarantees that the states $x_0(t)$, $u_0(t)$ globally converge to zero and $x(t)$, $u_1(t)$ converge to zero exponentially for a sufficiently small $\|y(0)\|_2$.

Proof. It is obvious that $x_0(t) = x_0(0)e^{-k_0 t}$, $u_0(t) = -k_0 x_0(t)$ globally converge to zero exponentially. As $A_1 - bK$ is Hurwitz and $\|R\|_2 \le h(|x_0|)\|y\|_2^2$ with $h(|x_0|)$ uniformly bounded with t, the closed-loop system (15) is locally exponential stable, implying that $y(t), x(t) = T(x_0(t))y(t)$ and $u_1(t) = -x_0^m(t)Ky(t)$ are all convergent to zero exponentially for a sufficiently small $\|y(0)\|_2$. □

Proposition 1 is only applicable for $x_0(0) \ne 0$. In the case of $x_0(0) = 0$, the proposed control law fails to work as the transformation matrix $T(x_0)$ becomes singular. This problem may be solved by introducing a switching mechanism that first drives x_0 away from zero in finite time and then switches to the control law (19) to achieve local exponential regulation for an arbitrarily $x_0(0)$ and a sufficiently small $\|x(0)\|_2$. However, such switching control law is discontinuous and may cause problems when the velocity input dynamics is included in the model since the discontinuities of velocity inputs lead to infinite accelerations.

In the next subsection, the controller (19) is modified to be smooth and time varying for an arbitrary $x_0(0)$ so that the acceleration signals are bounded.

3.2. Local Exponential Regulation of the Approximate NCF for an Arbitrary $x_0(0)$. The control law for the first control input is designed as

$$u_0 = -k_0 \alpha(t) - \bar{k}_0 (x_0 - \alpha(t)) \tag{20}$$

with $\alpha(t) = \alpha_0 e^{-k_0 t}$, $\alpha_0 \ne 0$, $\bar{k}_0 > k_0 > 0$.

Let $e_0(t) = x_0(t) - \alpha(t)$; then $\dot{e}_0(t) = u_0(t) - \dot{\alpha}(t) = -\bar{k}_0 e_0(t)$, so that $e_0(t) = e_0(0)e^{-\bar{k}_0 t}$, $x_0(t) = \alpha(t) + e_0(t) = \alpha(t) + e_0(0)e^{-\bar{k}_0 t}$, $u_0(t) = -k_0\alpha(t) - \bar{k}_0 e_0(0)e^{-\bar{k}_0 t}$, and $e_0(t)/\alpha(t) = (e_0(0)/\alpha_0)e^{-(\bar{k}_0 - k_0)t}$ are all globally convergent to zero exponentially.

Now we introduce the following smooth time-varying state transformation:

$$y = T^{-1}(\alpha)x, \qquad x = T(\alpha)y \tag{21}$$

with

$$\begin{aligned}T(\alpha) &= \alpha^m \operatorname{diag}\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}, \\ T^{-1}(\alpha) &= \alpha^{-m} \operatorname{diag}\left\{1, \alpha^{-1}, \alpha^{-2}, \ldots, \alpha^{-(n-1)}\right\},\end{aligned} \tag{22}$$

and m a positive integer to be determined.

As $\alpha_0 \ne 0$, the transformation matrix $T(\alpha)$ is clearly nonsingular for all $0 \le t < \infty$.

The dynamics of the transformed state y can be derived as

$$\begin{aligned}\dot{y} &= T^{-1}(\alpha)\dot{x} + \frac{d}{dt}(T^{-1}(\alpha))x \\ &= u_0 T^{-1}(\alpha)AT(\alpha)y + T^{-1}(\alpha)bu_1 \\ &\quad + T^{-1}(\alpha)(u_0R_2 + R_1u_1) + \frac{d}{dt}(T^{-1}(\alpha))T(\alpha)y \\ &= -k_0\alpha\left(1+\frac{\overline{k}_0e_0}{k_0\alpha}\right)T^{-1}(\alpha)AT(\alpha)y + T^{-1}(\alpha)bu_1 \\ &\quad + T^{-1}(\alpha)\left(-k_0\alpha\left(1+\frac{\overline{k}_0e_0}{k_0\alpha}\right)R_2 + R_1u_1\right) \\ &\quad + \frac{d}{dt}(T^{-1}(\alpha))T(\alpha)y.\end{aligned} \tag{23}$$

Simple calculation reveals that

$$\begin{aligned}&T^{-1}(\alpha)b = \alpha^{-m}b, \\ &\alpha T^{-1}(\alpha)AT(\alpha) = A, \\ &\frac{d}{dt}(T^{-1}(\alpha))T(\alpha) \\ &\qquad = k_0 \operatorname{diag}\{m, m+1, m+2, \ldots, m+n-1\}.\end{aligned} \tag{24}$$

Substituting the above identities into (23) results in

$$\begin{aligned}\dot{y} &= A_1\left(1+\frac{\overline{k}_0e_0}{k_0\alpha}\right)y + \alpha^{-m}bu_1 \\ &\quad + T^{-1}(x_0)\left(-k_0\alpha\left(1+\frac{\overline{k}_0e_0}{k_0\alpha}\right)R_2 + R_1u_1\right),\end{aligned} \tag{25}$$

where A_1 is defined in (13).

The second control input is designed as

$$u_1 = -\alpha^m Ky = -\alpha^m KT^{-1}(\alpha)x, \tag{26}$$

where $K = [k_1, k_2, \ldots, k_n]$ is a control gain row vector selected such that $A_1 - bK$ is Hurwitz.

The closed-loop system of (25) and (26) becomes

$$\begin{aligned}\dot{y} &= \left(A_1 - bK + \frac{\overline{k}_0e_0}{k_0\alpha}A_1\right)y \\ &\quad - T^{-1}(\alpha)\left(k_0\alpha\left(1+\frac{\overline{k}_0e_0}{k_0\alpha}\right)R_2 + R_1\alpha^m Ky\right) \\ &= \left(A_1 - bK + \frac{\overline{k}_0e_0}{k_0\alpha}A_1\right)y + R^*,\end{aligned} \tag{27}$$

where

$$R^* = -T^{-1}(\alpha)\left(k_0\alpha\left(1+\frac{\overline{k}_0e_0}{k_0\alpha}\right)R_2 + R_1\alpha^m Ky\right). \tag{28}$$

In view of (5), the converted residual term R^* can be shown to be bounded by

$$\begin{aligned}\|R^*\|_2 &\le \|T^{-1}\|_2 k_0|\alpha|\left|1+\frac{\overline{k}_0e_0}{k_0\alpha}\right|\|R_2\|_2 \\ &\quad + \|T^{-1}\|_2\|K\|_2|\alpha|^m\|y\|_2\|R_1\|_2 \\ &\le \|T^{-1}\|_2 r_2k_0|\alpha|\left|1+\frac{\overline{k}_0e_0}{k_0\alpha}\right|\|x\|_2^2 \\ &\quad + \|T^{-1}\|_2 r_1\|K\|_2|\alpha|^m\|x\|_2\|y\|_2 \\ &\le r_2k_0\left|1+\frac{\overline{k}_0e_0}{k_0\alpha}\right||\alpha|\|T^{-1}\|_2\|T\|_2^2\|y\|_2^2 \\ &\quad + r_1\|K\|_2|\alpha|^m\|T^{-1}\|_2\|T\|_2\|y\|_2^2 \\ &= h(\alpha, e_0)\|y\|_2^2\end{aligned} \tag{29}$$

with $h(\alpha, e_0)$ defined as

$$\begin{aligned}&h(\alpha, e_0) \\ &\cong r_2k_0\left|1+\frac{\overline{k}_0e_0}{k_0\alpha}\right|\max\left\{|\alpha|^{m+1}, |\alpha|^{m-n+2}\right\} \\ &\quad \times \max\left\{1, |\alpha|^{2(n-1)}\right\} \\ &\quad + r_1\|K\|_2\max\left\{|\alpha|^m, |\alpha|^{m-(n-1)}\right\}\max\left\{1, |\alpha|^{n-1}\right\}.\end{aligned} \tag{30}$$

As α, $e_0/\alpha = (e_0/\alpha_0)e^{-(\overline{k}_0-k_0)t}$ are both bounded uniformly with t, and $h(x_0, e_0)$ is thus uniformly bounded provided $m-(n-1) \ge 0$. Since $A_1 - bK$ is Hurwitz and e_0/α converges to zero exponentially, system $\dot{y} = (A_1 - bK + (\overline{k}_0e_0/k_0\alpha)A_1)y$ is globally exponential stable, and hence the perturbed system $\dot{y} = (A_2 + (\overline{k}_0e_0/k_0\alpha)A_1)y + R^*$ is locally exponential stable by Lyapunov indirect approach [20].

Based on the above analysis, we arrive at the following results.

Proposition 2. *Suppose that $\alpha = \alpha(t) = \alpha_0 e^{-k_0t}$, $\alpha_0 \neq 0$, $\overline{k}_0 > k_0 > 0$, $m \ge n-1$, K is selected such that $A_1 - bK$ is Hurwitz, then the following control law*

$$u_0 = -k_0\alpha - \overline{k}_0(x_0 - \alpha), \qquad u_1 = -\alpha^m Ky \tag{31}$$

guarantees that the states $x_0(t)$, $u_0(t)$ globally converge to zero exponentially and $x(t)$, $u_1(t)$ converge to zero exponentially for a sufficiently small $\|y(0)\|_2$.

Proof. It is obvious that $x_0(t), u_0(t)$ globally converge to zero exponentially. As $A_1 - bK$ is Hurwitz and $\|R^*\|_2 \le h(x_0, e_0)\|y\|_2^2$ with $h(x_0, e_0)$ uniformly bounded with t, the closed-loop system (27) is locally exponential stable, implying that $y(t), x(t) = T(\alpha(t))y(t)$ and $u_1(t) = -\alpha^m(t)Ky(t)$ are all convergent to zero exponentially for a sufficiently small $\|y(0)\|_2$. □

Remark 5. Compared with the approach presented in [18] where the control law is continuous but not differentiable,

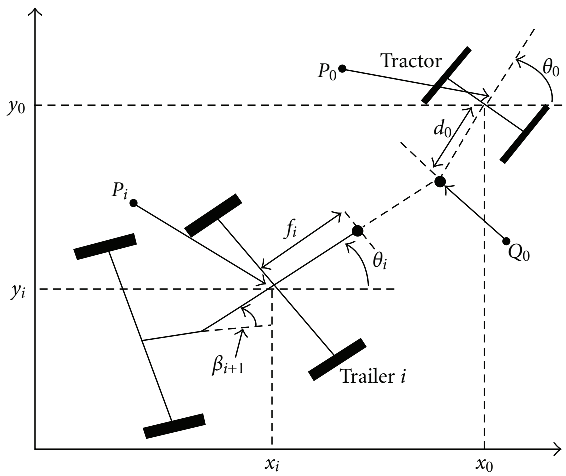

FIGURE 1: A tractor towing n trailer with off-axle hitching.

the proposed control law (31) in this paper is smooth time varying and hence can be easily extended to include input dynamics of the approximate NCF (1)-(2) by one-step back-steeping.

4. An Example: Local Exponential Regulation of an Off-Axle Tractor-Trailer

Consider a tractor-trailer with a wheeled mobile tractor towing n off-axle wheeled trailers shown in Figure 1, where (x_i, y_i, θ_i) denote the position and orientation of body i $(i = 0, 1, 2, \ldots, n)$, $(v_i, \omega_i = \dot{\theta}_i)$ denote the linear and angular velocities of body i $(i = 0, 1, 2, \ldots, n)$, $\beta_i = \theta_{i-1} - \theta_i$ $(i = 1, 2, 3, \ldots, n)$ represent the difference of orientation angles between body i and body $i-1$. P_i $(i = 0, 1, 2, \ldots)$ is the center point on the wheel axle of body i and Q_{i-1} $(i = 1, 2, \ldots, n)$ the connection point of body i and body $i-1$. The distance between P_i and Q_i is d_i, and the distance between P_i and Q_{i-1} is f_i.

The kinematic equation of the tractor is

$$\dot{x}_0 = v_0 \cos\theta_0, \qquad \dot{y}_0 = v_0 \sin\theta_0, \qquad \dot{\theta}_0 = \omega_0. \tag{32}$$

The kinematic relations of trailer i can be derived as

$$\begin{aligned} v_i &= v_{i-1}\cos\beta_i + d_{i-1}\dot{\theta}_{i-1}\sin\beta_i, \\ \dot{x}_i &= v_i \cos\theta_i, \\ \dot{y}_i &= v_i \sin\theta_i, \\ \dot{\theta}_i &= \frac{1}{f_i}\left(v_{i-1}\sin\beta_i - d_{i-1}\dot{\theta}_{i-1}\cos\beta_i\right). \end{aligned} \tag{33}$$

Select $x = [x_0, y_0, \theta_0, \beta_1, \beta_2, \ldots, \beta_n]^T$ as the state variables, and $u_0 = v_0\cos\theta_0, \omega_0$ as the control inputs, the state equation can be derived from (32)-(33) as

$$\dot{x}_0 = u_0, \tag{34}$$

$$\dot{x} = (A + R_2(x))u_0 + (b + R_1(x))\omega_0, \tag{35}$$

where R_1, R_2 are high-order residual terms satisfying (5) and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & -\dfrac{1}{f_1} & 0 & \ldots & 0 \\ 0 & 0 & \dfrac{1}{f_1}\left(1 + \dfrac{d_1}{f_2}\right) & -\dfrac{1}{f_2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{1,n} & a_{2,n} & \ldots & -\dfrac{1}{f_n} \end{bmatrix}, \tag{36}$$

$$b = \left[0 \quad 1 \quad 1 + \frac{d_0}{f_1} \quad -\frac{d_0}{f_1}\left(1 + \frac{d_1}{f_2}\right) \quad \ldots \quad b_n\right]^T,$$

$$\begin{aligned} a_{1,n} &= (-1)^{n-2}\frac{d_{n-2}\times d_{n-3}\cdots\times d_2\times d_1}{f_{n-1}\times f_{n-2}\cdots f_1}\left(1 + \frac{d_{n-1}}{f_n}\right), \\ a_{2,n} &= (-1)^{n-3}\frac{d_{n-2}\times d_{n-3}\cdots\times d_3\times d_2}{f_{n-1}\times f_{n-2}\cdots f_2}\left(1 + \frac{d_{n-1}}{f_n}\right), \\ b_n &= (-1)^{n-1}\frac{d_{n-2}\times d_{n-3}\cdots\times d_1\times d_0}{f_{n-1}\times f_{n-2}\cdots f_1}\left(1 + \frac{d_{n-1}}{f_n}\right), \end{aligned} \tag{37}$$

The control object can be stated as design control law $u_0(\cdot), \omega_0(\cdot)$ such that the states $(x_0, y_0, \theta_0, \beta_1, \beta_2, \ldots, \beta_n)$ of the closed-loop system (34)-(35) converge to zero exponentially.

To apply Propositions 1 and 2 obtained in Section 3, it is required to verify the controllability of $\{A, b\}$.

Lemma 1. *Suppose that $d_i > 0$ $(i = 0, 1, \ldots, n-1)$ and $f_i > 0$ $(i = 1, 2, \ldots, n)$, then $\{A, b\}$ is a controllable pair.*

Proof. The lemma can be proved by verifying PBH criterion of linear systems and is omitted here for brevity. □

Remark 6. As $\{A, b\}$ is controllable, it can thus be further converted to the canonical controllable form (4) by a linear transformation so that the tractor-trailers system (34)-(35) can be expressed in approximate NCF (1)-(2).

To illustrate the effectiveness of the proposed control approaches, a tractor towing one trailer is taken as a simulation example. The state equation in this special case can be explicitly obtained as

$$\begin{aligned} \dot{x}_0 &= v_0\cos\theta_0, \\ \dot{y}_0 &= v_0\sin\theta_0, \\ \dot{\theta}_0 &= \omega_0, \\ \dot{\beta}_1 &= -c_1 v_0 \sin\beta_1 + (1 + c_2\cos\beta_1)\omega_0, \end{aligned} \tag{38}$$

where $c_1 = 1/f_1$, $c_2 = d_0/f_1$.

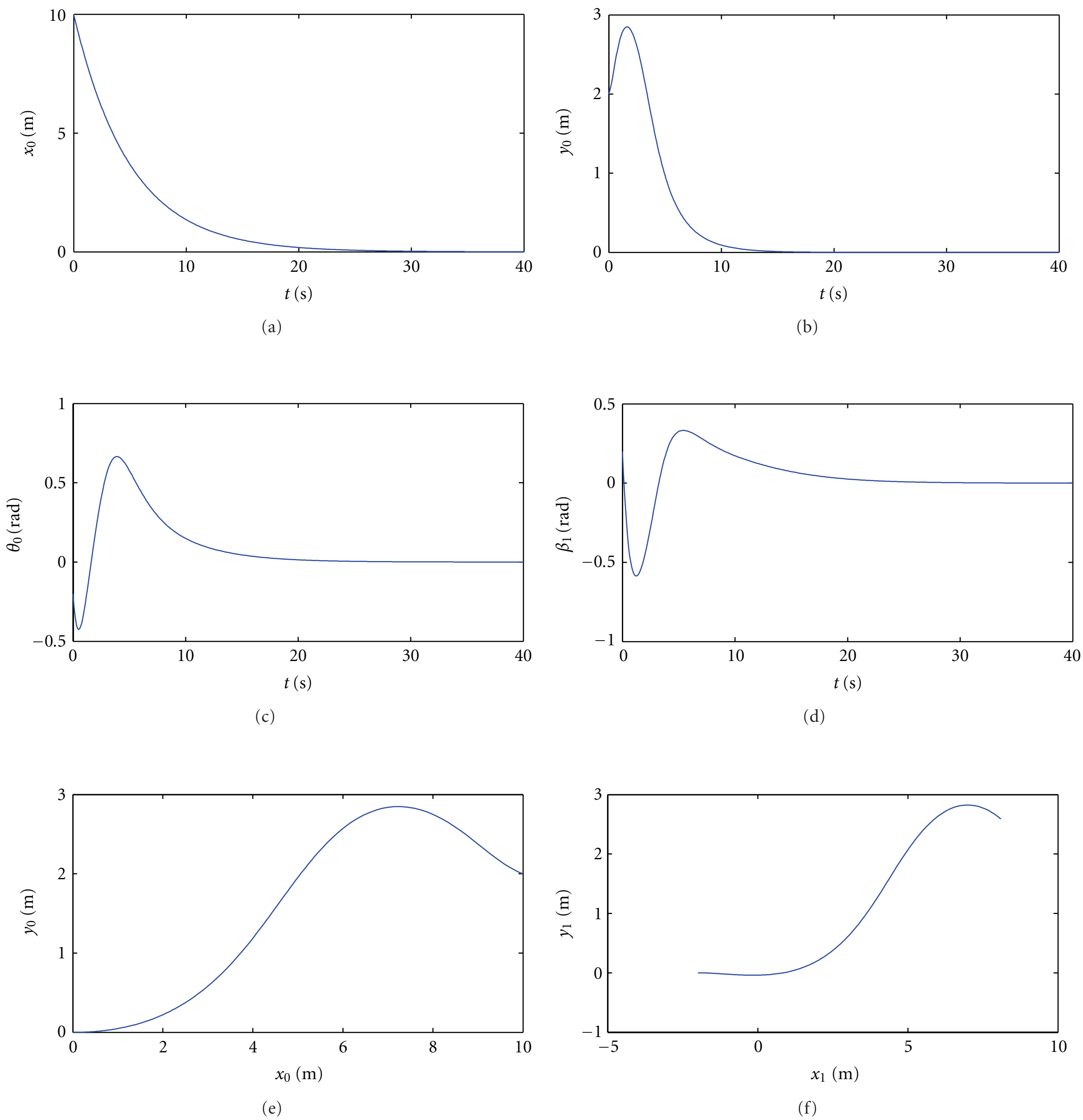

Figure 2: Time trajectories of states and geometric paths of the tractor and the trailer starting from the first initial state.

Under the following coordinate and input transformations:

$$
\begin{aligned}
x_1 &= c_1^2\beta_1, \\
x_2 &= -c_1\beta_1 + c_1(1+c_2)\theta_0, \\
x_3 &= \beta_1 - (1+c_2)\theta_0 + c_1(1+c_2)y_0, \\
u_0 &= v_0\cos\theta_0, \\
u_1 &= c_1^2(-c_1 v_0 \sin\beta_1 + (1+c_2\cos\beta_1)\omega_0).
\end{aligned}
\tag{39}
$$

the state equation (38) is converted to the following form:

$$
\begin{aligned}
\dot{x}_3 &= (x_2 + R_{23})u_0 + R_{13}u_1, \\
\dot{x}_2 &= (x_1 + R_{22})u_0 + R_{12}u_1, \\
\dot{x}_1 &= u_1, \\
\dot{x}_0 &= u_0,
\end{aligned}
\tag{40}
$$

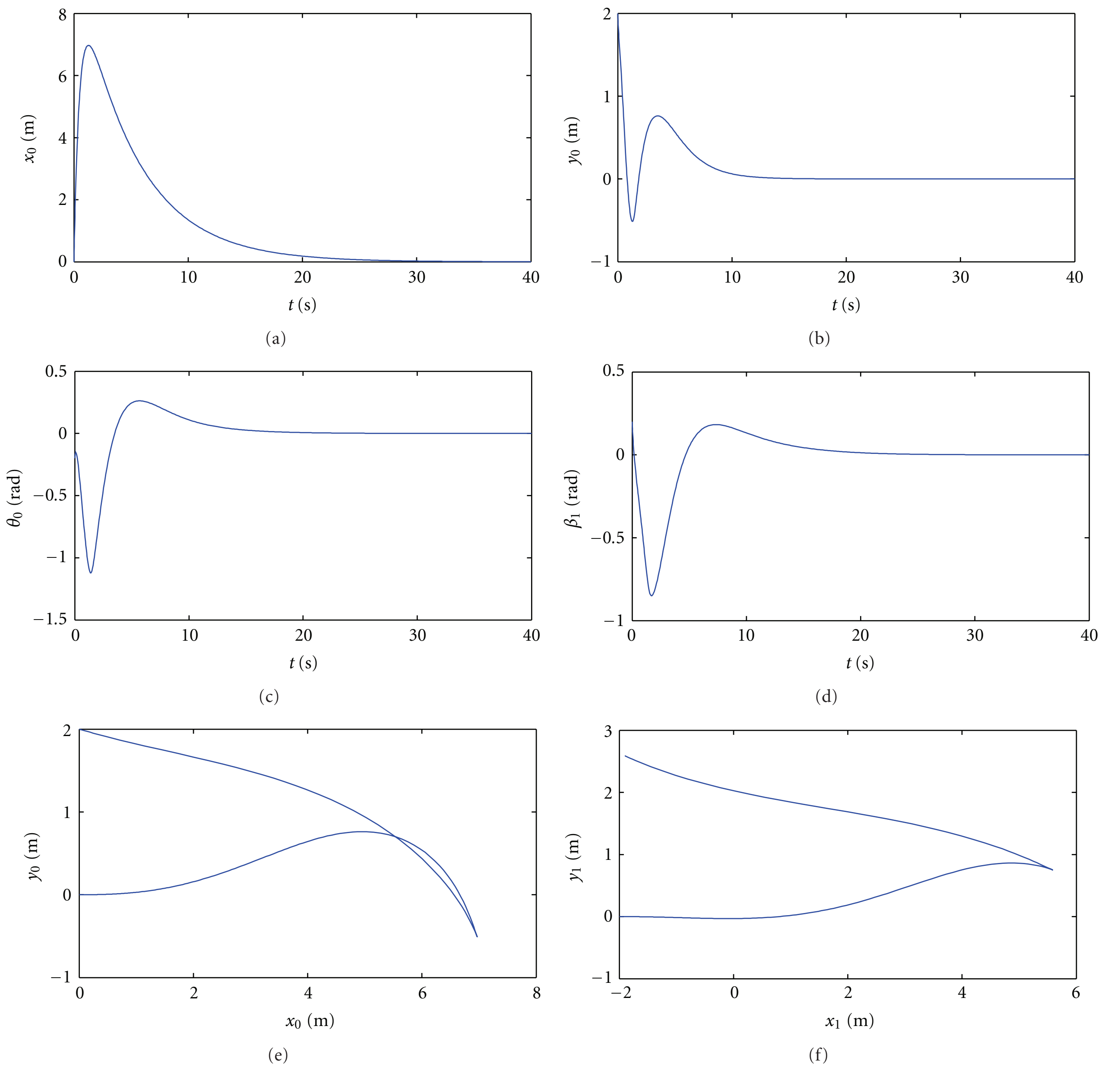

Figure 3: Time trajectories of states and geometric paths of the tractor and the trailer starting from the second initial state.

where

$$\begin{aligned} R_{23} &= c_1\left(-\left(\frac{\sin\beta_1}{\cos\theta_0}-\beta_1\right)+(1+c_2)\left(\frac{\sin\theta_0}{\cos\theta_0}-\theta_0\right)\right) \\ &\quad+\frac{c_1c_2(\cos\beta_1-1)\sin\beta_1}{(1+c_2\cos\beta_1)/\cos\theta_0}, \\ R_{13} &= \frac{c_2(\cos\beta_1-1)}{c_1^2/(1+c_2\cos\beta_1)}, \\ R_{22} &= c_1^2\left(\frac{\sin\beta_1}{\cos\theta_0}-\beta_1\right)+\frac{c_1^2c_2(1-\cos\beta_1)\sin\beta_1}{(1+c_2\cos\beta_1)/\cos\theta_0}, \\ R_{12} &= \frac{c_2}{c_1(1-\cos\beta_1)/(1+c_2\cos\beta_1)}. \end{aligned} \tag{41}$$

In the state region $D = \{(x_0, y_0, \theta_0, \beta_1) : |\theta_0| \leq \theta_{0M} < \pi/2, |\beta_1| \leq \beta_{1M}\}$, $|R_{2j}|$ $(j = 2, 3)$ can be shown to be $O(\|(\theta_0, \beta_1)\|_2^3)$ and R_{1j} $(j = 2, 3)$ to be $O(\|(\theta_0, \beta_1)\|_2^2)$.

The geometric parameters are set to $d_0 = f_1 = 1$. The controller parameters are selected as $m = 2$, $k_0 = 0.2$, $\alpha_0 = 10$, $\overline{k}_0 = 2$, and $K = [1.92, -8.26, 14.81]$ chosen such that the eigenvalues of $A_1 - bK$ are assigned to $-(0.02, 0.04, 0.06)$.

The simulation is implemented for two initial states $(x_0(0), y_0(0), \theta_0(0), \beta_1(0)) = (10, 2, -0.2, 0.2)$ and $(x_0(0), y_0(0), \theta_0(0), \beta_1(0)) = (0, 2, -0.2, 0.2)$. For the first initial state, where $x_0(0) \neq 0$, the control law (19) is applied; for the second initial state where $x_0(0) = 0$, the control law (31) is applied. The time plots of state trajectories and geometric paths of the tractor and the trailer are shown in

Figures 2 and 3 in respect to the two initial states. It is observed that the proposed control laws successfully regulate the state to zero from initial states and produce nice geometric paths for both the tractor and the trailer.

5. Conclusion

In this paper, we propose a discontinuous and a smooth time-varying control schemes for a class of nonlinear driftless systems in the approximated nonholonomic chained form, achieving local exponential convergence of state to the desired equilibrium point. The proposed control laws rely on the discontinuous and the smooth time-varying state transformations that convert the system to linear stable one perturbed by two- or higher-order terms of state. An application example of off-axle tractor-trailers is discussed in detail for illustrating the effectiveness of the proposed control approaches.

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