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*Research Article*

# Discontinuous Sturm-Liouville Problems and Associated Sampling Theories

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This paper investigates the sampling analysis associated with discontinuous Sturm-Liouville problems with eigenvalue parameters in two boundary conditions and with transmission conditions at the point of discontinuity. We closely follow the analysis derived by Fulton (1977) to establish the needed relations for the derivations of the sampling theorems including the construction of Green's function as well as the eigenfunction expansion theorem. We derive sampling representations for transforms whose kernels are either solutions or Green's functions. In the special case, when our problem is continuous, the obtained results coincide with the corresponding results in the work of Annaby and Tharwat (2006).

## 1. Introduction

The recovery of entire functions from a discrete sequence of points is an important problem from mathematical and practical points of view. For instance, in signal processing it is needed to reconstruct (recover) a signal (function) from its values at a sequence of samples. If this aim is achieved, then an analog (continuous) signal can be transformed into a digital (discrete) one and then it can be recovered by the receiver. If the signal is band limited, the sampling process can be done via the celebrated Whittaker, Shannon, and Kotel'nikov (WKS) sampling theorem [1–3]. By a band-limited signal with band width $\sigma$, $\sigma > 0$, that is, the signal contains no frequencies higher than $\sigma/2\pi$ cycles per second (cps), we mean a function in the Paley-Wiener space $PW_\sigma^2$ of entire functions of exponential type at most $\sigma$

which are $L^2(\mathbb{R})$-functions when restricted to $\mathbb{R}$. This space is characterized by the following relation which is due to Paley and Wiener [4, 5]:

$$f(t) \in PW_\sigma^2 \Longleftrightarrow f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} e^{ixt} g(x)dx, \quad \text{for some function } g(\cdot) \in L^2(-\sigma, \sigma). \tag{1.1}$$

Now WKS [6, 7] sampling theorem states the following.

**Theorem 1.1** (WKS). *If $f(t) \in PW_\sigma^2$, then it is completely determined from its values at the points $t_k = k\pi/\sigma$, $k \in \mathbb{Z}$, by means of the formula*

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \operatorname{sinc} \sigma(t - t_k), \quad t \in \mathbb{C}, \tag{1.2}$$

*where*

$$\operatorname{sinc} t = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases} \tag{1.3}$$

*The sampling series* (1.2) *is absolutely and uniformly convergent on compact subsets of $\mathbb{C}$, uniformly convergent on $\mathbb{R}$ and converges in the norm of $L^2(\mathbb{R})$, see [6, 8, 9].*

The WKS sampling theorem has been generalized in many different ways. Here we are interested in two extensions. The first is concerned with replacing the equidistant sampling points by more general ones, which is important from practical and theoretical point of view. The following theorem which is known in some literature as Paley-Wiener theorem, [5] gives a sampling theorem with a more general class of sampling points. Although the theorem in its final form may be attributed to Levinson [10] and Kadec [11], it could be named after Paley and Wiener who first derive the theorem in a more restrictive form, see [6, 7] for more details.

**Theorem 1.2** (Paley and Wiener). *Let $\{t_k\}$, $k \in \mathbb{Z}$ be a sequence of real numbers satisfying*

$$D := \sup_{k \in \mathbb{Z}} \left| t_k - \frac{k\pi}{\sigma} \right| < \frac{\pi}{4\sigma}, \tag{1.4}$$

*and let $G(t)$ be the entire function defined by the canonical product*

$$G(t) := (t - t_0) \prod_{k=1}^{\infty} \left(1 - \frac{t}{t_k}\right)\left(1 - \frac{t}{t_{-k}}\right). \tag{1.5}$$

*Then, for any $f \in PW_\sigma^2$*

$$f(t) = \sum_{k \in \mathbb{Z}} f(t_k) \frac{G(t)}{G'(t_k)(t - t_k)}, \quad t \in \mathbb{C}. \tag{1.6}$$

*The series* (1.6) *converges uniformly on compact subsets of* $\mathbb{C}$.

The WKS sampling theorem is a special case of this theorem because if we choose $t_k = k\pi/\sigma = -t_{-k}$, then

$$G(t) = t\prod_{k=1}^{\infty}\left(1-\frac{t}{t_k}\right)\left(1+\frac{t}{t_k}\right) = t\prod_{k=1}^{\infty}\left(1-\frac{(t\sigma/\pi)^2}{k^2}\right) = \frac{\sin t\sigma}{\sigma}, \qquad G'(t_k) = (-1)^k. \tag{1.7}$$

Expansion (1.6) is of Lagrange-type interpolation.

The second extension of WKS sampling theorem is the theorem of Kramer [12]. In this theorem sampling representations were given for integral transforms whose kernels are more general than $\exp(ixt)$.

**Theorem 1.3** (Kramer). *Let $I$ be a finite closed interval, $K(\cdot,t) : I \times \mathbb{C} \to \mathbb{C}$ a function continuous in $t$ such that $K(\cdot,t) \in L^2(I)$ for all $t \in \mathbb{C}$, and let $\{t_k\}_{k\in\mathbb{Z}}$ be a sequence of real numbers such that $\{K(\cdot,t_k)\}_{k\in\mathbb{Z}}$ is a complete orthogonal set in $L^2(I)$. Suppose that*

$$f(t) = \int_I K(x,t)g(x)dx, \quad g(\cdot) \in L^2(I). \tag{1.8}$$

*Then*

$$f(t) = \sum_{k\in\mathbb{Z}} f(t_k)\frac{\int_I K(x,t)\overline{K(x,t_k)}dx}{\|K(\cdot,t_k)\|^2_{L^2(I)}}. \tag{1.9}$$

*Series* (1.9) *converges uniformly wherever $\|K(\cdot,t)\|_{L^2(I)}$ as a function of $t$ is bounded.*

Again Kramer's theorem is a generalization of WKS theorem. If we take $K(x,t) = e^{itx}$, $I = [-\sigma,\sigma]$, $t_k = k\pi/\sigma$, then (1.9) will be (1.2).

The relationship between both extensions of WKS sampling theorem has been investigated extensively. Starting from a function theory approach, cf. [13], it is proved in [14] that if $K(x,t)$, $x \in I$, $t \in \mathbb{C}$ satisfies some analyticity conditions, then Kramer's sampling formula (1.9) turns out to be a Lagrange interpolation one, see also [15–17]. In another direction, it is shown that Kramer's expansion (1.9) could be written as a Lagrange-type interpolation formula if $K(\cdot,t)$ and $t_k$ are extracted from ordinary differential operators, see the survey [18] and the references cited therein. The present work is a continuation of the second direction mentioned above. We prove that integral transforms associated with second-order eigenvalue problems with an eigenparameter appearing in the boundary conditions and also with an internal point of discontinuity can also be reconstructed in a sampling form of Lagrange interpolation type. We would like to mention that works in direction of sampling associated with eigenproblems with an eigenparameter in the boundary conditions are few, see, for example, [19, 20]. Also papers in sampling with discontinuous eigenproblems are few, see [21–23]. However sampling theories associated with eigenproblems, which contain eigenparameter in the boundary conditions and have at the same time discontinuity conditions, do not exist as for as we know. Our investigation will be the first in that direction, introducing a good example. To achieve our aim we will briefly study the spectral analysis of the problem. Then we derive two sampling theorems using solutions and Green's function, respectively.

## 2. The Eigenvalue Problem

In this section we define our boundary value problem and state some of its properties. Consider the boundary value problem

$$\ell(y) := -r(x)u''(x) + q(x)u(x) = \lambda u(x), \quad x \in [-1,0) \cup (0,1], \tag{2.1}$$

with boundary conditions

$$U_1(u) := (\alpha_1'\lambda - \alpha_1)u(-1) - (\alpha_2'\lambda - \alpha_2)u'(-1) = 0, \tag{2.2}$$

$$U_2(u) := (\beta_1'\lambda + \beta_1)u(1) - (\beta_2'\lambda + \beta_2)u'(1) = 0, \tag{2.3}$$

and transmission conditions

$$\begin{aligned} U_3(u) &:= \gamma_1 u(-0) - \delta_1 u(+0) = 0, \\ U_4(u) &:= \gamma_2 u'(-0) - \delta_2 u'(+0) = 0, \end{aligned} \tag{2.4}$$

where $\lambda$ is a complex spectral parameter; $r(x) = r_1^2$ for $x \in [-1,0)$, $r(x) = r_2^2$ for $x \in (0,1]$; $r_1 > 0$ and $r_2 > 0$ are given real number; $q(x)$ is a given real-valued function, which is continuous in $[-1,0)$ and $(0,1]$ and has a finite limit $q(\pm 0) = \lim_{x\to\pm 0} q(x)$; $\gamma_i,\ \delta_i,\ \alpha_i, \beta_i, \alpha_i', \beta_i'$ $(i = 1,2)$ are real numbers; $\gamma_i \neq 0$, $\delta_i \neq 0$ $(i = 1,2)$; $\rho$ and $\gamma$ are given by

$$\rho := \det\begin{pmatrix} \alpha_1' & \alpha_1 \\ \alpha_2' & \alpha_2 \end{pmatrix} > 0, \qquad \gamma := \det\begin{pmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{pmatrix} > 0. \tag{2.5}$$

In some literature conditions (2.4) are called compatability conditions, see, for example, [24]. To formulate a theoretic approach to problem (2.1)–(2.4) we define the Hilbert space $\mathbf{H} := L^2(-1,1) \oplus \mathbb{C}^2$ with an inner product

$$\langle \mathbf{f}(\cdot), \mathbf{g}(\cdot)\rangle_{\mathbf{H}} := \frac{1}{r_1^2}\int_{-1}^{0} f(x)\overline{g}(x)dx + \frac{1}{r_2^2}\int_{0}^{1} f(x)\overline{g}(x)dx + \frac{1}{\rho}f_1\overline{g_1} + \frac{1}{\gamma}f_2\overline{g_2}, \tag{2.6}$$

where

$$\mathbf{f}(x) = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix}, \qquad \mathbf{g}(x) = \begin{pmatrix} g(x) \\ g_1 \\ g_2 \end{pmatrix} \in \mathbf{H}, \tag{2.7}$$

$f(\cdot), g(\cdot) \in L^2(-1,1)$ and $f_i, g_i \in \mathbb{C}$, $i = 1,2$. For convenience we put

$$\begin{pmatrix} R_{-1}(u) & R_1(u) \\ R'_{-1}(u) & R'_1(u) \end{pmatrix} := \begin{pmatrix} \alpha_1 u(-1) - \alpha_2 u'(-1) & \beta_1 u(1) - \beta_2 u'(1) \\ \alpha'_1 u(-1) - \alpha'_2 u'(-1) & \beta'_1 u(1) - \beta'_2 u'(1) \end{pmatrix}. \tag{2.8}$$

For function $f(x)$, which is defined on $[-1,0) \cup (0,1]$ and has finite limit $f(\pm 0) := \lim_{x\to\pm 0} f(x)$, by $f_{(1)}(x)$ and $f_{(2)}(x)$ we denote the functions

$$f_{(1)}(x) = \begin{cases} f(x), & x \in [-1,0), \\ f(-0), & x = 0, \end{cases} \qquad f_{(2)}(x) = \begin{cases} f(x), & x \in (0,1], \\ f(+0), & x = 0, \end{cases} \tag{2.9}$$

which are defined on $I_1 := [-1,0]$ and $I_2 := [0,1]$, respectively.

In the following we will define the minimal closed operator in $\mathbf{H}$ associated with the differential expression $\ell$, cf. [25, 26].

Let $\mathfrak{D}(\mathbf{A}) \subseteq \mathbf{H}$ be the set of all

$$\mathbf{f}(x) = \begin{pmatrix} f(x) \\ R'_{-1}(f) \\ R'_1(f) \end{pmatrix} \in \mathbf{H} \tag{2.10}$$

such that $f_{(i)}(\cdot)$, $f'_{(i)}(\cdot)$ are absolutely continuous in $I_i$, $i = 1,2$, $\ell(f) \in L^2(-1,0) \oplus L^2(0,1)$ and $U_3(f) = U_4(f) = 0$. Define the operator $\mathbf{A} : \mathfrak{D}(\mathbf{A}) \to \mathbf{H}$ by

$$\mathbf{A}\begin{pmatrix} f(x) \\ R'_{-1}(f) \\ R'_1(f) \end{pmatrix} = \begin{pmatrix} \ell(f) \\ R_{-1}(f) \\ -R_1(f) \end{pmatrix}, \quad \begin{pmatrix} f(x) \\ R'_{-1}(f) \\ R'_1(f) \end{pmatrix} \in \mathfrak{D}(\mathbf{A}). \tag{2.11}$$

The eigenvalues and the eigenfunctions of the problem (2.1)–(2.4) are defined as the eigenvalues and the first components of the corresponding eigenelements of the operator $\mathbf{A}$, respectively.

**Theorem 2.1.** *Let $\gamma_1\gamma_2 = \delta_1\delta_2$. Then, the operator $\mathbf{A}$ is symmetric.*

*Proof.* For $\mathbf{f}(\cdot), \mathbf{g}(\cdot) \in \mathfrak{D}(\mathbf{A})$

$$\begin{aligned} \langle \mathbf{A}\mathbf{f}(\cdot), \mathbf{g}(\cdot)\rangle_{\mathbf{H}} &= \frac{1}{r_1^2}\int_{-1}^{0} \ell(f(x))\overline{g}(x)dx + \frac{1}{r_2^2}\int_0^1 \ell(f(x))\overline{g}(x)dx \\ &\quad + \frac{1}{\rho} R_{-1}(f) R'_{-1}(\overline{g}) - \frac{1}{\gamma} R_1(f) R'_1(\overline{g}). \end{aligned} \tag{2.12}$$

By two partial integration we obtain

$$\begin{aligned}\langle \mathbf{Af}(\cdot),\mathbf{g}(\cdot)\rangle_{\mathbf{H}} &= \langle \mathbf{f}(\cdot),\mathbf{Ag}(\cdot)\rangle_{\mathbf{H}} + W(f,\overline{g};-0) - W(f,\overline{g};-1) + W(f,\overline{g};1) - W(f,\overline{g};+0)\\ &\quad + \frac{1}{\rho}\left(R_{-1}(f)R'_{-1}(\overline{g}) - R'_{-1}(f)R_{-1}(\overline{g})\right) + \frac{1}{\gamma}\left(R'_{1}(f)R_{1}(\overline{g}) - R_{1}(f)R'_{1}(\overline{g})\right),\end{aligned} \tag{2.13}$$

where, as usual, by $W(f,g;x)$ we denote the Wronskian of the functions $f$ and $g$

$$W(f,g;x) := f(x)g'(x) - f'(x)g(x). \tag{2.14}$$

Since $f(x)$ and $\overline{g}(x)$ are satisfied the boundary condition (2.2)-(2.3) and transmission conditions (2.4) we get

$$\begin{aligned} R_{-1}(f)R'_{-1}(\overline{g}) - R'_{-1}(f)R_{-1}(\overline{g}) &= \rho W(f,\overline{g};-1),\\ R'_{1}(f)R_{1}(\overline{g}) - R_{1}(f)R'_{1}(\overline{g}) &= -\gamma W(f,\overline{g};1),\\ \gamma_1\gamma_2 W(f,\overline{g};-0) &= \delta_1\delta_2 W(f,\overline{g};+0).\end{aligned} \tag{2.15}$$

Finally substituting (2.15) in (2.13) then we have

$$\langle \mathbf{Af}(\cdot),\mathbf{g}(\cdot)\rangle_{\mathbf{H}} = \langle \mathbf{f}(\cdot),\mathbf{Ag}(\cdot)\rangle_{\mathbf{H}}, \quad \mathbf{f}(\cdot),\mathbf{g}(\cdot) \in \mathfrak{D}(\mathbf{A}), \tag{2.16}$$

thus, the operator $\mathbf{A}$ is Hermitian. The symmetry of $\mathbf{A}$ arises from the well-known fact that $\mathfrak{D}(\mathbf{A})$ is dense in $\mathbf{H}$ see, for example, [24]. □

**Corollary 2.2.** *All eigenvalues of the problem* (2.1)–(2.4) *are real.*

We can now assume that all eigenfunctions of the problem (2.1)–(2.4) are real valued.

**Corollary 2.3.** *Let $\lambda_1$ and $\lambda_2$ be two different eigenvalues of the problem* (2.1)–(2.4)*. Then the corresponding eigenfunctions $u_1$ and $u_2$ of this problem are orthogonal in the sense of*

$$\frac{1}{r_1^2}\int_{-1}^{0} u_1(x)u_2(x)dx + \frac{1}{r_2^2}\int_{0}^{1} u_1(x)u_2(x)dx + \frac{1}{\rho}R'_{-1}(u_1)R'_{-1}(u_2) + \frac{1}{\gamma}R'_{1}(u_1)R'_{1}(u_2) = 0. \tag{2.17}$$

*Proof.* Formula (2.17) follows immediately from the orthogonality of corresponding eigenelements

$$\mathbf{u}_1(x) = \begin{pmatrix} u_1(x)\\ R'_{-1}(u_1)\\ R'_{1}(u_1)\end{pmatrix}, \qquad \mathbf{u}_2(x) = \begin{pmatrix} u_2(x)\\ R'_{-1}(u_2)\\ R'_{1}(u_2)\end{pmatrix} \tag{2.18}$$

in the Hilbert space $\mathbf{H}$. □

Now, we will construct a special fundamental system of solutions of the equation (2.1) for $\lambda$ not being an eigenvalue. Let us consider the next initial value problem:

$$-r_1^2 u''(x) + q(x)u(x) = \lambda u(x), \quad x \in (-1,0), \tag{2.19}$$

$$u(-1) = \lambda\alpha_2' - \alpha_2, \qquad u'(-1) = \lambda\alpha_1' - \alpha_1. \tag{2.20}$$

By virtue of Theorem 1.5 in [27] this problem has a unique solution $u = \varphi_1(x) = \varphi_{1\lambda}(x)$, which is an entire function of $\lambda \in \mathbb{C}$ for each fixed $x \in [-1,0]$. Similarly, employing the same method as in proof of Theorem 1.5 in [27], we see that the problem

$$-r_2^2 u''(x) + q(x)u(x) = \lambda u(x), \quad x \in (0,1), \tag{2.21}$$

$$u(1) = \lambda\beta_2' + \beta_2, \qquad u'(1) = \lambda\beta_1' + \beta_1, \tag{2.22}$$

has a unique solution $u = \chi_2(x) = \chi_{2\lambda}(x)$ which is an entire function of parameter $\lambda$ for each fixed $x \in [0,1]$.

Now the functions $\varphi_{2\lambda}(x)$ and $\chi_{1\lambda}(x)$ are defined in terms of $\varphi_{1\lambda}(x)$ and $\chi_{2\lambda}(x)$ as follows: the initial-value problem,

$$-r_2^2 u''(x) + q(x)u(x) = \lambda u(x), \quad x \in (0,1), \tag{2.23}$$

$$u(0) = \frac{\gamma_1}{\delta_1}\varphi_{1\lambda}(0), \qquad u'(0) = \frac{\gamma_2}{\delta_2}\varphi_{1\lambda}'(0), \tag{2.24}$$

which contains the entire functions of eigenparameter $\lambda$ (in the right-hand side), has unique solution $u = \varphi_{2\lambda}(x)$ for each $\lambda \in \mathbb{C}$.

Similarly, the following problem also has a unique solution $u = \chi_1(x) = \chi_{1\lambda}(x)$:

$$-r_1^2 u''(x) + q(x)u(x) = \lambda u(x), \quad x \in (-1,0), \tag{2.25}$$

$$u(0) = \frac{\delta_1}{\gamma_1}\chi_{2\lambda}(0), \qquad u'(0) = \frac{\delta_2}{\gamma_2}\chi_{2\lambda}'(0). \tag{2.26}$$

Since the Wronskians $W(\varphi_{i\lambda}, \chi_{i\lambda}; x)$ are independent on variable $x \in I_i$ $(i = 1,2)$ and $\varphi_{i\lambda}(x)$ and $\chi_{i\lambda}(x)$ are the entire functions of the parameter $\lambda$ for each $x \in I_i$ $(i = 1,2)$, then the functions

$$\omega_i(\lambda) := W(\varphi_{i\lambda}, \chi_{i\lambda}; x), \quad x \in I_i, \; i = 1,2, \tag{2.27}$$

are the entire functions of parameter $\lambda$.

**Lemma 2.4.** *If the condition $\gamma_1\gamma_2 = \delta_1\delta_2$ is satisfied, then the equality $\omega_1(\lambda) = \omega_2(\lambda)$ holds for each $\lambda \in \mathbb{C}$.*

*Proof.* Taking into account (2.24) and (2.26), a short calculation gives $\gamma_1\gamma_2 W(\varphi_{1\lambda}, \chi_{1\lambda}; 0) = \delta_1\delta_2 W(\varphi_{2\lambda}, \chi_{2\lambda}; 0)$, so $\omega_1(\lambda) = \omega_2(\lambda)$ for each $\lambda \in \mathbb{C}$. □

**Corollary 2.5.** *The zeros of the functions $\omega_1(\lambda)$ and $\omega_2(\lambda)$ coincide.*

Let us construct two basic solutions of (2.1) as

$$\varphi_\lambda(x) = \begin{cases} \varphi_{1\lambda}(x), & x \in [-1,0), \\ \varphi_{2\lambda}(x), & x \in (0,1], \end{cases} \qquad \chi_\lambda(x) = \begin{cases} \chi_{1\lambda}(x), & x \in [-1,0), \\ \chi_{2\lambda}(x), & x \in (0,1]. \end{cases} \tag{2.28}$$

By virtue of (2.24) and (2.26) these solutions satisfy both transmission conditions (2.4).

Now we may introduce to the consideration the characteristic function $\omega(\lambda)$ as

$$\omega(\lambda) := \omega_1(\lambda) = \omega_2(\lambda). \tag{2.29}$$

**Theorem 2.6.** *The eigenvalues of the problem* (2.1)–(2.4) *are coincided zeros of the function $\omega(\lambda)$.*

*Proof.* Let $\omega(\lambda_0) = 0$. Then $W(\varphi_{1\lambda_0}, \chi_{1\lambda_0}; x) = 0$, and so the functions $\varphi_{1\lambda_0}(x)$ and $\chi_{1\lambda_0}(x)$ are linearly dependent, that is,

$$\chi_{1\lambda_0}(x) = k\varphi_{1\lambda_0}(x), \quad x \in [-1,0], \text{ for some } k \neq 0. \tag{2.30}$$

Consequently, $\chi_{\lambda_0}(x)$ satisfied the boundary condition (2.3), so the function $\chi_{\lambda_0}(x)$ is an eigenfunction of the problem (2.1)–(2.4) corresponding to the eigenvalue $\lambda_0$.

Now let $u_0(x)$ be any eigenfunction corresponding to the eigenvalue $\lambda_0$, but $\omega(\lambda_0) \neq 0$. Then the functions $\varphi_{i\lambda_0}(x)$, $\chi_{i\lambda_0}(x)$ are linearly independent on $I_i$, $i = 1,2$. Thus, $u_0(x)$ may be represented as in the form

$$u_0(x) = \begin{cases} c_1\varphi_{1\lambda_0}(x) + c_2\chi_{1\lambda_0}(x), & x \in [-1,0), \\ c_3\varphi_{2\lambda_0}(x) + c_4\chi_{2\lambda_0}(x), & x \in (0,1], \end{cases} \tag{2.31}$$

where at least one of the constants $c_i$, $i = 1,2,3,4$, is not zero.

Consider the equations

$$U_i(u_0(x)) = 0, \quad i = 1,2,3,4 \tag{2.32}$$

as the homogenous system of linear equations of the variables $c_i$, $i = 1,2,3,4$, and taking into account (2.24) and (2.26), it follows that the determinant of this system is

$$\begin{vmatrix} 0 & -\omega_1(\lambda_0) & 0 & 0 \\ 0 & 0 & \omega_2(\lambda_0) & 0 \\ \gamma_1\varphi_{1\lambda_0}(0) & \gamma_1\chi_{1\lambda_0}(0) & -\delta_1\varphi_{2\lambda_0}(0) & -\delta_1\chi_{2\lambda_0}(0) \\ \gamma_2\varphi'_{1\lambda_0}(0) & \gamma_2\chi'_{1\lambda_0}(0) & -\delta_2\varphi'_{2\lambda_0}(0) & -\delta_2\chi'_{2\lambda_0}(0) \end{vmatrix} = \delta_1\delta_2\omega_1(\lambda_0)\omega_2^2(\lambda_0) \neq 0. \tag{2.33}$$

Thus, the system (2.32) has only trivial solution $c_i = 0$, $i = 1,2,3,4$, and so we get contradiction which completes the proof. □

**Lemma 2.7.** *If $\lambda = \lambda_0$ is an eigenvalue, then $\varphi_{\lambda_0}(x)$ and $\chi_{\lambda_0}(x)$ are linearly dependent.*

*Proof.* Since $\lambda_0$ is an eigenvalue, then from Theorem 2.6 we have $W(\varphi_{i\lambda_0}, \chi_{i\lambda_i}; x) = \omega(\lambda_0) = 0$, $i = 1, 2$. Therefore

$$\chi_{i\lambda_0}(x) = k_i \varphi_{i\lambda_0}(x), \quad i = 1, 2, \tag{2.34}$$

for some $k_i \neq 0$, $i = 1, 2$. Now, we must show that $k_1 = k_2$. Suppose if possible that $k_1 \neq k_2$. Taking into account the definitions of solution $\varphi_{i\lambda_0}(x)$ and $\varphi_{i\lambda_0}(x)$, $i = 1, 2$, from the equalities (2.34) we get

$$\begin{aligned} U_3(\chi_{\lambda_0}) &= \gamma_1 \chi_{\lambda_0}(-0) - \delta_1 \chi_{\lambda_0}(+0) \\ &= \gamma_1 \chi_{1\lambda_0}(0) - \delta_1 \chi_{2\lambda_0}(0) \\ &= \gamma_1 k_1 \varphi_{1\lambda_0}(0) - \delta_1 k_2 \varphi_{2\lambda_0}(0) \\ &= \delta_1 k_1 \varphi_{2\lambda_0}(0) - \delta_1 k_2 \varphi_{2\lambda_0}(0) \\ &= \delta_1 (k_1 - k_2) \varphi_{2\lambda_0}(0). \end{aligned} \tag{2.35}$$

Since $U_3(\chi_{\lambda_0}) = 0$, $\delta_1 \neq 0$, and $k_1 - k_2 \neq 0$ it follows that

$$\varphi_{2\lambda_0}(0) = 0. \tag{2.36}$$

By the same procedure from the equality $U_4(\chi_{\lambda_0}) = 0$ we can derive that

$$\varphi'_{2\lambda_0}(0) = 0. \tag{2.37}$$

From the fact that $\varphi_{2\lambda_0}(x)$ is a solution of (2.1) on $[0, 1]$ and satisfied the initial conditions (2.36) and (2.37) it follows that $\varphi_{2\lambda_0}(x) = 0$ identically on $[0, 1]$, because of the well-known existence and uniqueness theorem for the initial value problems of the ordinary linear differential equations.

By using (2.24), (2.36), and (2.37) we may also find

$$\varphi_{1\lambda_0}(0) = \varphi'_{1\lambda_0}(0) = 0. \tag{2.38}$$

For latter discussion for $\varphi_{2\lambda_0}(x)$, it follows that $\varphi_{1\lambda_0}(x) = 0$ identically on $[-1, 0]$. Therefore $\varphi_{\lambda_0}(x) = 0$ identically on $[-1, 0) \cup (0, 1]$. But this is contradicted with (2.20), which completes the proof. □

**Corollary 2.8.** *If $\lambda = \lambda_0$ is an eigenvalue, then both $\varphi_{\lambda_0}(x)$ and $\chi_{\lambda_0}(x)$ are eigenfunctions corresponding to this eigenvalue.*

**Lemma 2.9.** *If the condition $\gamma_1 \gamma_2 = \delta_1 \delta_2$ is satisfied, then all eigenvalues $\lambda_n$ are simple zeros of $\omega(\lambda)$.*

*Proof.* Since

$$\begin{aligned}
&\frac{1}{r_1^2}\int_{-1}^{0}\ell(\varphi_\lambda(x))\varphi_{\lambda_0}(x)dx+\frac{1}{r_2^2}\int_{0}^{1}\ell(\varphi_\lambda(x))\varphi_{\lambda_0}(x)dx\\
&\quad=\frac{1}{r_1^2}\int_{-1}^{0}\varphi_\lambda(x)\ell(\varphi_{\lambda_0}(x))dx+\frac{1}{r_2^2}\int_{0}^{1}\varphi_\lambda(x)\ell(\varphi_{\lambda_0}(x))dx+W(\varphi_\lambda,\varphi_{\lambda_0};1)-W(\varphi_\lambda,\varphi_{\lambda_0};-1),
\end{aligned}\tag{2.39}$$

then

$$(\lambda-\lambda_0)\left[\frac{1}{r_1^2}\int_{-1}^{0}\varphi_\lambda(x)\varphi_{\lambda_0}(x)dx+\frac{1}{r_2^2}\int_{0}^{1}\varphi_\lambda(x)\varphi_{\lambda_0}(x)dx\right]=W(\varphi_\lambda,\varphi_{\lambda_0};1)-(\lambda-\lambda_0)\rho,\tag{2.40}$$

for any $\lambda$. Since

$$\chi_{\lambda_0}(x)=k_0\varphi_{\lambda_0}(x),\quad x\in[-1,0)\cup(0,1],\tag{2.41}$$

for some $k_0\neq 0$, then

$$\begin{aligned}
W(\varphi_\lambda,\varphi_{\lambda_n};1)&=\frac{1}{k_n}W(\varphi_\lambda,\chi_{\lambda_n};1)\\
&=\frac{1}{k_n}(\lambda_nR_1'(\varphi_\lambda)+R_1(\varphi_\lambda))\\
&=\frac{1}{k_n}[\omega(\lambda)-(\lambda-\lambda_n)R_1'(\varphi_\lambda)]\\
&=(\lambda-\lambda_n)\frac{1}{k_n}\left[\frac{\omega(\lambda)}{\lambda-\lambda_n}-R_1'(\varphi_\lambda)\right].
\end{aligned}\tag{2.42}$$

Substituting (2.42) in (2.40) and letting $\lambda\to\lambda_n$ we get

$$\frac{1}{r_1^2}\int_{-1}^{0}(\varphi_{\lambda_n}(x))^2dx+\frac{1}{r_2^2}\int_{0}^{1}(\varphi_{\lambda_0}(x))^2dx=\frac{1}{k_n}[\omega'(\lambda_n)-R_1'(\varphi_{\lambda_n})]-\rho.\tag{2.43}$$

Now putting

$$R_1'(\varphi_{\lambda_n})=\frac{1}{k_n}R_1'(\chi_{\lambda_n})=\frac{\gamma}{k_n}\tag{2.44}$$

in (2.43) it yields $\omega'(\lambda_n)\neq 0$, which completes the proof. □

If $\lambda_n$, $n = 0, 1, 2, \ldots$ denote the zeros of $\omega(\lambda)$, then the three-component vectors

$$\mathbf{\Phi}_n(x) := \begin{pmatrix} \varphi_{\lambda_n}(x) \\ R'_{-1}(\varphi_{\lambda_n}) \\ R'_1(\varphi_{\lambda_n}) \end{pmatrix} \tag{2.45}$$

are the corresponding eigenvectors of the operator $\mathbf{A}$ satisfying the orthogonality relation

$$\langle \mathbf{\Phi}_n(\cdot), \mathbf{\Phi}_m(\cdot) \rangle_{\mathbf{H}} = 0 \quad \text{for } n \neq m. \tag{2.46}$$

Here $\{\varphi_{\lambda_n}(\cdot)\}_{n=0}^{\infty}$ will be the sequence of eigenfunctions of (2.1)–(2.4) corresponding to the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$. We denote by $\mathbf{\Psi}_n(\cdot)$ the normalized eigenvectors

$$\mathbf{\Psi}_n(x) := \frac{\mathbf{\Phi}_n(x)}{\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}} = \begin{pmatrix} \psi_n(x) \\ R'_{-1}(\psi_n) \\ R'_1(\psi_n) \end{pmatrix}. \tag{2.47}$$

Because of simplicity of the eigenvalues, we find nonzeros constants $k_n$ such that

$$\chi_{\lambda_n}(x) := k_n \varphi_{\lambda_n}(x), \quad x \in [-1, 0) \cup (0, 1], \ n = 0, 1, \ldots. \tag{2.48}$$

To study the completeness of the eigenvectors of $\mathbf{A}$, and hence the completeness of the eigenfunctions of (2.1)–(2.4), we construct the resolvent of $\mathbf{A}$ as well as Green's function of problem (2.1)–(2.4). We assume without any loss of generality that $\lambda = 0$ is not an eigenvalue of $\mathbf{A}$. Otherwise, from discreteness of eigenvalues, we can find a real number $c$ such that $c \neq \lambda_n$ for all $n$ and replace the eigenparameter $\lambda$ by $\lambda - c$. Now let $\lambda \in \mathbb{C}$ not be an eigenvalue of $\mathbf{A}$ and consider the inhomogeneous problem

$$(\lambda I - \mathbf{A})\mathbf{u}(x) = \mathbf{f}(x), \quad \text{for } \mathbf{f}(x) = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix} \in \mathbf{H}, \ \mathbf{u}(x) = \begin{pmatrix} u(x) \\ R'_{-1}(u) \\ R'_1(u) \end{pmatrix} \in \mathfrak{D}(\mathbf{A}), \tag{2.49}$$

and $I$ is the identity operator. Since

$$(\lambda I - \mathbf{A})\mathbf{u}(x) = \lambda \begin{pmatrix} u(x) \\ R'_{-1}(u) \\ R'_1(u) \end{pmatrix} - \begin{pmatrix} \ell(u(x)) \\ R_{-1}(u) \\ -R_1(u) \end{pmatrix} = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix}, \tag{2.50}$$

then we have

$$(\lambda - \ell)\mathbf{u}(x) = \mathbf{f}(x), \quad x \in [-1, 0) \cup (0, 1], \tag{2.51}$$

$$f_1 = \lambda R'_{-1}(u) - R_{-1}(u), \qquad f_2 = \lambda R'_1(u) + R_1(u). \tag{2.52}$$

Now, we can represent the general solution of (2.51) in the following form:

$$u(x,\lambda)=\begin{cases}A_1\varphi_{1\lambda}(x)+B_1\chi_{1\lambda}(x), & x\in[-1,0),\\ A_2\varphi_{2\lambda}(x)+B_2\chi_{2\lambda}(x), & x\in(0,1].\end{cases}\tag{2.53}$$

Applying the method of variation of the constants to (2.53), thus, the functions $A_1(x,\lambda)$, $B_1(x,\lambda)$ and $A_2(x,\lambda)$, $B_2(x,\lambda)$ satisfy the linear system of equations

$$\begin{gathered}A_1'(x,\lambda)\varphi_{1\lambda}(x)+B_1'(x,\lambda)\chi_{1\lambda}(x)=0,\quad A_1'(x,\lambda)\varphi_{1\lambda}'(x)+B_1'(x,\lambda)\chi_{1\lambda}'(x)=\frac{f(x)}{r_1^2},\quad x\in[-1,0),\\ A_2'(x,\lambda)\varphi_{2\lambda}(x)+B_2'(x,\lambda)\chi_{2\lambda}(x)=0,\quad A_2'(x,\lambda)\varphi_{2\lambda}'(x)+B_2'(x,\lambda)\chi_{2\lambda}'(x)=\frac{f(x)}{r_2^2},\quad x\in(0,1].\end{gathered}\tag{2.54}$$

Since $\lambda$ is not an eigenvalue and $W(\varphi_{i\lambda}(x),\chi_{i\lambda}(x);x)\neq0$, $i=1,2$, each of the linear systems in (2.54) has a unique solution which leads

$$\begin{gathered}A_1(x,\lambda)=\frac{1}{r_1^2\omega(\lambda)}\int_x^0 f(\xi)\chi_{1\lambda}(\xi)d\xi+A_1,\qquad B_1(x,\lambda)=\frac{1}{r_1^2\omega(\lambda)}\int_{-1}^x f(\xi)\varphi_{1\lambda}(\xi)d\xi+B_1,\\ x\in[-1,0),\\ A_2(x,\lambda)=\frac{1}{r_2^2\omega(\lambda)}\int_x^1 f(\xi)\chi_{2\lambda}(\xi)d\xi+A_2,\qquad B_2(x,\lambda)=\frac{1}{r_2^2\omega(\lambda)}\int_0^x f(\xi)\varphi_{2\lambda}(\xi)d\xi+B_2,\\ x\in(0,1],\end{gathered}\tag{2.55}$$

where $A_1, A_2, B_1$, and $B_2$ are arbitrary constants. Substituting (2.55) into (2.53), we obtain the solution of (2.51)

$$u(x,\lambda)=\begin{cases}\dfrac{\varphi_{1\lambda}(x)}{r_1^2\omega(\lambda)}\displaystyle\int_x^0 f(\xi)\chi_{1\lambda}(\xi)d\xi+\dfrac{\chi_{1\lambda}(x)}{r_1^2\omega(\lambda)}\displaystyle\int_{-1}^x f(\xi)\varphi_{1\lambda}(\xi)d\xi+A_1\varphi_{1\lambda}(x)+B_1\chi_{1\lambda}(x),\\ \qquad x\in[-1,0),\\ \dfrac{\varphi_{2\lambda}(x)}{r_2^2\omega(\lambda)}\displaystyle\int_x^1 f(\xi)\chi_{2\lambda}(\xi)d\xi+\dfrac{\chi_{2\lambda}(x)}{r_2^2\omega(\lambda)}\displaystyle\int_0^x f(\xi)\varphi_{2\lambda}(\xi)d\xi+A_2\varphi_{2\lambda}(x)+B_2\chi_{2\lambda}(x),\\ \qquad x\in(0,1].\end{cases}\tag{2.56}$$

Then from (2.52) and the transmission conditions (2.4) we get

$$A_1 = \frac{1}{r_2^2 \omega(\lambda)} \int_0^1 f(\xi)\chi_{2\lambda}(\xi)d\xi + \frac{f_2}{\omega(\lambda)}, \qquad B_1 = -\frac{f_1}{\omega(\lambda)},$$
$$A_2 = \frac{f_2}{\omega(\lambda)}, \qquad B_2 = \frac{1}{r_1^2 \omega(\lambda)} \int_{-1}^0 f(\xi)\varphi_{1\lambda}(\xi)d\xi - \frac{f_1}{\omega(\lambda)}. \tag{2.57}$$

Then (2.56) can be written as

$$u(x,\lambda) = \frac{f_2}{\omega(\lambda)}\varphi_\lambda(x) - \frac{f_1}{\omega(\lambda)}\chi_\lambda(x) + \frac{\chi_\lambda(x)}{\omega(\lambda)} \int_{-1}^x \frac{f(\xi)}{r(\xi)}\varphi_\lambda(\xi)d\xi + \frac{\varphi_\lambda(x)}{\omega(\lambda)} \int_x^1 \frac{f(\xi)}{r(\xi)}\chi_\lambda(\xi)d\xi,$$
$$x,\xi \in [-1,0) \cup (0,1]. \tag{2.58}$$

Hence, we have

$$\mathbf{u}(x) = (\lambda I - \mathbf{A})^{-1}\mathbf{f}(x) = \begin{pmatrix} \dfrac{f_2}{\omega(\lambda)}\varphi_\lambda(x) - \dfrac{f_1}{\omega(\lambda)}\chi_\lambda(x) + \displaystyle\int_{-1}^1 G(x,\xi,\lambda)\frac{f(\xi)}{r(\xi)}d\xi \\ R'_{-1}(u) \\ R'_1(u) \end{pmatrix}, \tag{2.59}$$

where

$$G(x,\xi,\lambda) = \begin{cases} \dfrac{\chi_\lambda(x)\varphi_\lambda(\xi)}{\omega(\lambda)}, & -1 \le \xi \le x \le 1,\ x \ne 0,\ \xi \ne 0, \\ \dfrac{\chi_\lambda(\xi)\varphi_\lambda(x)}{\omega(\lambda)}, & -1 \le x \le \xi \le 1,\ x \ne 0,\ \xi \ne 0, \end{cases} \tag{2.60}$$

is the unique Green's function of problem (2.1)–(2.4). Obviously $G(x,\xi,\lambda)$ is a meromorphic function of $\lambda$, for every $(x,\xi) \in ([-1,0) \cup (0,1])^2$, which has simple poles only at the eigenvalues. Although Green's function looks as simple as that of Sturm-Liouville problems, cf., for example, [28], it is a rather complicated because of the transmission conditions, see the example at the end of this paper.

**Lemma 2.10.** *The operator* $\mathbf{A}$ *is self-adjoint in* $\mathbf{H}$.

*Proof.* Since $\mathbf{A}$ is a symmetric densely defined operator, then it is sufficient to show that the deficiency spaces are the null spaces and hence $\mathbf{A} = \mathbf{A}^*$. Indeed, if $\mathbf{f}(x) = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix} \in \mathbf{H}$ and $\lambda$ is a nonreal number, then taking

$$\mathbf{u}(x) = \begin{pmatrix} \dfrac{f_2}{\omega(\lambda)}\varphi_\lambda(x) - \dfrac{f_1}{\omega(\lambda)}\chi_\lambda(x) + \displaystyle\int_{-1}^1 G(x,\xi,\lambda)\frac{f(\xi)}{r(\xi)}d\xi \\ R'_{-1}(u) \\ R'_1(u) \end{pmatrix} \tag{2.61}$$

implies that $\mathbf{u} \in \mathfrak{D}(\mathbf{A})$. Since $G(x,\xi,\lambda)$ satisfies conditions (2.2)–(2.4), then $(\mathbf{A}-\lambda I)\mathbf{u}(x) = \mathbf{f}(x)$. Now we prove that the inverse of $(\mathbf{A}-\lambda I)$ exists. If $\mathbf{A}\mathbf{u}(x) = \lambda\mathbf{u}(x)$, then

$$\begin{aligned}\left(\overline{\lambda}-\lambda\right)\langle \mathbf{u}(\cdot),\mathbf{u}(\cdot)\rangle_{\mathbf{H}} &= \langle \mathbf{u}(\cdot),\lambda\mathbf{u}(\cdot)\rangle_{\mathbf{H}} - \langle \lambda\mathbf{u}(\cdot),\mathbf{u}(\cdot)\rangle_{\mathbf{H}} \\ &= \langle \mathbf{u}(\cdot),\mathbf{A}\mathbf{u}(\cdot)\rangle_{\mathbf{H}} - \langle \mathbf{A}\mathbf{u}(\cdot),\mathbf{u}(\cdot)\rangle_{\mathbf{H}} \\ &= 0 \quad (\text{since } \mathbf{A} \text{ is symmetric}).\end{aligned} \tag{2.62}$$

Since $\lambda \notin \mathbb{R}$, we have $\overline{\lambda}-\lambda \neq 0$. Thus $\langle \mathbf{u}(\cdot),\mathbf{u}(\cdot)\rangle_{\mathbf{H}} = 0$, that is, $\mathbf{u} = 0$. Then $R(\lambda;\mathbf{A}) := (\mathbf{A}-\lambda I)^{-1}$, the resolvent operator of $\mathbf{A}$, exists. Thus

$$R(\lambda;\mathbf{A})\mathbf{f} = (\mathbf{A}-\lambda I)^{-1}\mathbf{f} = \mathbf{u}. \tag{2.63}$$

Take $\lambda = \pm i$. The domains of $(\mathbf{A}-iI)^{-1}$ and $(\mathbf{A}+iI)^{-1}$ are exactly $\mathbf{H}$. Consequently the ranges of $(\mathbf{A}-iI)$ and $(\mathbf{A}+iI)$ are also $\mathbf{H}$. Hence the deficiency spaces of $\mathbf{A}$ are

$$\begin{aligned}N_{-i} &:= N(\mathbf{A}^{*}+iI) = R(\mathbf{A}-iI)^{\perp} = \mathbf{H}^{\perp} = \{0\}, \\ N_{i} &:= N(\mathbf{A}^{*}-iI) = R(\mathbf{A}+iI)^{\perp} = \mathbf{H}^{\perp} = \{0\}.\end{aligned} \tag{2.64}$$

Hence $\mathbf{A}$ is self-adjoint. □

The next theorem is an eigenfunction expansion theorem, which is similar to that established by Fulton in [29].

**Theorem 2.11.** (i) *For* $\mathbf{u}(\cdot) \in \mathbf{H}$

$$\|\mathbf{u}(\cdot)\|_{\mathbf{H}}^{2} = \sum_{n=-\infty}^{\infty} |\langle \mathbf{u}(\cdot),\mathbf{\Psi}_n(\cdot)\rangle_{\mathbf{H}}|^{2}. \tag{2.65}$$

(ii) *For* $\mathbf{u}(\cdot) \in \mathfrak{D}(\mathbf{A})$

$$\mathbf{u}(x) = \sum_{n=-\infty}^{\infty} \langle \mathbf{u}(\cdot),\mathbf{\Psi}_n(\cdot)\rangle_{\mathbf{H}}\mathbf{\Psi}_n(x), \tag{2.66}$$

*with the series being absolutely and uniformly convergent in the first component for on* $[-1,0)\cup(0,1]$ *and absolutely convergent in the second component.*

*Proof.* The proof is similar to that in [29, pages 298-299]. □

## 3. Asymptotic Formulas of Eigenvalues and Eigenfunctions

Now we derive first- and second-order asymptotics of the eigenvalues and eigenfunctions similar to the classical techniques of [27, 30] and [29], see also [25, 26]. We begin by proving some lemmas.

**Lemma 3.1.** *Let $\varphi_\lambda(x)$ be the solutions of* (2.1) *defined in Section 2, and let $\lambda = s^2$. Then the following integral equations hold for $k = 0$ and $k = 1$:*

$$\begin{aligned}\frac{d^k}{dx^k}\varphi_{1\lambda}(x) &= \left(-\alpha_2 + s^2\alpha_2'\right)\frac{d^k}{dx^k}\cos\left[\frac{s(x+1)}{r_1}\right] - \left(-\alpha_1 + s^2\alpha_1'\right)\frac{r_1}{s}\frac{d^k}{dx^k}\sin\left[\frac{s(x+1)}{r_1}\right] \\ &\quad + \frac{1}{r_1 s}\int_{-1}^{x}\frac{d^k}{dx^k}\sin\left[\frac{s(x-y)}{r_1}\right]q(y)\varphi_{1\lambda}(y)dy,\end{aligned} \tag{3.1}$$

$$\begin{aligned}\frac{d^k}{dx^k}\varphi_{2\lambda}(x) &= \frac{\gamma_1}{\delta_1}\varphi_{1\lambda}(-0)\frac{d^k}{dx^k}\cos\left[\frac{sx}{r_2}\right] + \frac{r_2\gamma_2}{\delta_2 s}\varphi_{1\lambda}'(-0)\frac{d^k}{dx^k}\sin\left[\frac{sx}{r_2}\right] \\ &\quad + \frac{1}{r_2 s}\int_{0}^{x}\frac{d^k}{dx^k}\sin\left[\frac{s(x-y)}{r_2}\right]q(y)\varphi_{2\lambda}(y)dy.\end{aligned} \tag{3.2}$$

*Proof.* For proving it is enough substitute $s^2\varphi_{1\lambda}(y) + r(y)\varphi_{1\lambda}''(y)$ and $s^2\varphi_{2\lambda}(y) + r(y)\varphi_{2\lambda}''(y)$ instead of $q(y)\varphi_{1\lambda}(y)$ and $q(y)\varphi_{2\lambda}(y)$ in the integral terms of the (3.1) and (3.2), respectively, and integrate by parts twice. □

**Lemma 3.2.** *Let $\lambda = s^2$, $\operatorname{Im} s = t$. Then the functions $\varphi_{i\lambda}(x)$ have the following asymptotic representations for $|\lambda| \to \infty$, which hold uniformly for $x \in I_i$ ($i = 1, 2$):*

$$\frac{d^k}{dx^k}\varphi_{1\lambda}(x) = s^2\alpha_2'\frac{d^k}{dx^k}\cos\left[\frac{s(x+1)}{r_1}\right] + \mathcal{O}\left(|s|^{k+1}e^{|t|((x+1)/r_1)}\right), \quad k = 0, 1, \tag{3.3}$$

$$\begin{aligned}\frac{d^k}{dx^k}\varphi_{2\lambda}(x) &= s^2\alpha_2'\left[\frac{\gamma_1}{\delta_1}\cos\left[\frac{s}{r_1}\right]\frac{d^k}{dx^k}\cos\left[\frac{sx}{r_2}\right] - \frac{r_2\gamma_2}{\delta_2 r_1}\sin\left[\frac{s}{r_1}\right]\frac{d^k}{dx^k}\sin\left[\frac{sx}{r_2}\right]\right] \\ &\quad + \mathcal{O}\left(|s|^{k+1}e^{|t|((r_1x+r_2)/r_1r_2)}\right), \quad k = 0, 1,\end{aligned} \tag{3.4}$$

*if $\alpha_2' \neq 0$,*

$$\frac{d^k}{dx^k}\varphi_{1\lambda}(x) = -sr_1\alpha_1'\frac{d^k}{dx^k}\sin\left[\frac{s(x+1)}{r_1}\right] + \mathcal{O}\left(|s|^{k}e^{|t|((x+1)/r_1)}\right), \quad k = 0, 1, \tag{3.5}$$

$$\begin{aligned}\frac{d^k}{dx^k}\varphi_{2\lambda}(x) &= -s\alpha_1'\left[\frac{\gamma_1 r_1}{\delta_1}\sin\left[\frac{s}{r_1}\right]\frac{d^k}{dx^k}\cos\left[\frac{sx}{r_2}\right] + \frac{r_2\gamma_2}{\delta_2}\cos\left[\frac{s}{r_1}\right]\frac{d^k}{dx^k}\sin\left[\frac{sx}{r_2}\right]\right] \\ &\quad + \mathcal{O}\left(|s|^{k}e^{|t|((r_1x+r_2)/r_1r_2)}\right), \quad k = 0, 1,\end{aligned} \tag{3.6}$$

*if $\alpha_2' = 0$.*

*Proof.* Since the proof of the formulae for $\varphi_{1\lambda}(x)$ is identical to Titchmarshs proof of similar results for $\varphi_\lambda(x)$ (see [27, Lemma 1.7 page 9-10]), we may formulate them without proving them here. Therefore we will prove only the formulas for $\varphi_{2\lambda}(x)$. Let $\alpha_2 \neq 0$. Then according to (3.3)

$$\begin{aligned}\varphi_{1\lambda}(-0) &= s^2\alpha_2' \cos\left[\frac{s}{r_1}\right] + \mathcal{O}\left(|s|e^{|t|/r_1}\right),\\ \varphi_{1\lambda}'(-0) &= -\frac{s^3\alpha_2'}{r_1}\sin\left[\frac{s}{r_1}\right] + \mathcal{O}\left(|s|^2 e^{|t|/r_1}\right).\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.2) (for $k = 0$), we get

$$\begin{aligned}\varphi_{2\lambda}(x) = {}& \frac{\gamma_1 s^2\alpha_2'}{\delta_1}\cos\left[\frac{s}{r_1}\right]\cos\left[\frac{sx}{r_2}\right] - \frac{r_2 s^2\alpha_2'\gamma_2}{\delta_2 r_1}\sin\left[\frac{s}{r_1}\right]\sin\left[\frac{sx}{r_2}\right]\\ &+ \frac{1}{r_2 s}\int_0^x \sin\left[\frac{s(x-y)}{r_2}\right] q(y)\varphi_{2\lambda}(y)\,dy + \mathcal{O}\left(|s|e^{|t|((r_1x+r_2)/r_1r_2)}\right).\end{aligned} \tag{3.8}$$

Multiplying (3.8) by $|s|^{-2}e^{-|t|((r_1x+r_2)/r_1r_2)}$ and denoting

$$F_\lambda(x) = |s|^{-2}e^{-|t|((r_1x+r_2)/r_1r_2)}\varphi_{2\lambda}(x) \tag{3.9}$$

we get

$$\begin{aligned}F_\lambda(x) = {}& |s|^{-2}e^{-|t|((r_1x+r_2)/r_1r_2)}\left\{\frac{s^2\alpha_2'\gamma_1}{\delta_1}\cos\left[\frac{s}{r_1}\right]\cos\left[\frac{sx}{r_2}\right] - \frac{r_2 s^2\alpha_2'\gamma_2}{\delta_2 r_1}\sin\left[\frac{s}{r_1}\right]\sin\left[\frac{sx}{r_2}\right]\right\}\\ &+ \frac{1}{r_2 s}\int_0^x \sin\left[\frac{s(x-y)}{r_2}\right] q(y)e^{-|t|((x-y)/r_2)}F_\lambda(y)\,dy + \mathcal{O}\left(|s|^{-1}\right).\end{aligned} \tag{3.10}$$

Denoting $M(\lambda) := \max_{0\le x\le 1}|F_\lambda(x)|$ from the last formula, it follows that

$$M(\lambda) \le \frac{|\gamma_1||\alpha_2'|}{|\delta_1|} + \frac{r_2|\alpha_2'||\gamma_2|}{|\delta_2|r_1} + \frac{M(\lambda)}{r_2|s|}\int_0^1 |q(y)|\,dy + \frac{M_0}{|s|} \tag{3.11}$$

for some $M_0 > 0$. From this, it follows that $M(\lambda) = \mathcal{O}(1)$ as $\lambda \to \infty$, so

$$\varphi_{2\lambda}(x) = \mathcal{O}\left(|s|^2 e^{|t|((r_1x+r_2)/r_1r_2)}\right). \tag{3.12}$$

Substituting (3.12) into the integral on the right of (3.8) yields (3.4) for $k = 0$. The case $k = 1$ of (3.4) follows by applying the same procedure as in the case $k = 0$. The case $\alpha_2' = 0$ is proved analogically. □

**Lemma 3.3.** *Let $\lambda = s^2$, $\operatorname{Im} s = t$. Then the characteristic function $\omega(\lambda)$ has the following asymptotic representations.*

*Case 1.* If $\beta_2' \neq 0$ and $\alpha_2' \neq 0$, then

$$\omega(\lambda) = \alpha_2'\beta_2' s^5 \left[\frac{\gamma_1}{r_2\delta_1}\cos\left[\frac{s}{r_1}\right]\sin\left[\frac{s}{r_2}\right] + \frac{\gamma_2}{\delta_2 r_1}\sin\left[\frac{s}{r_1}\right]\cos\left[\frac{s}{r_2}\right]\right] + \mathcal{O}\left(|s|^4 e^{|t|((r_1+r_2)/r_1r_2)}\right). \quad (3.13)$$

*Case 2.* If $\beta_2' \neq 0$ and $\alpha_2' = 0$, then

$$\omega(\lambda) = \alpha_1'\beta_2' s^4 \left[-\frac{\gamma_1 r_1}{r_2\delta_1}\sin\left[\frac{s}{r_1}\right]\sin\left[\frac{s}{r_2}\right] + \frac{\gamma_2}{\delta_2}\cos\left[\frac{s}{r_1}\right]\cos\left[\frac{s}{r_2}\right]\right] + \mathcal{O}\left(|s|^3 e^{|t|((r_1+r_2)/r_1r_2)}\right). \quad (3.14)$$

*Case 3.* If $\beta_2' = 0$ and $\alpha_2' \neq 0$, then

$$\omega(\lambda) = \beta_1'\alpha_2' s^4 \left[\frac{\gamma_1}{\delta_1}\cos\left[\frac{s}{r_1}\right]\cos\left[\frac{s}{r_2}\right] - \frac{r_2\gamma_2}{\delta_2 r_1}\sin\left[\frac{s}{r_1}\right]\sin\left[\frac{s}{r_2}\right]\right] + \mathcal{O}\left(|s|^3 e^{|t|((r_1+r_2)/r_1r_2)}\right). \quad (3.15)$$

*Case 4.* If $\beta_2' = 0$ and $\alpha_2' = 0$, then

$$\omega(\lambda) = -\beta_1'\alpha_1' s^3 \left[\frac{\gamma_1 r_1}{\delta_1}\sin\left[\frac{s}{r_1}\right]\cos\left[\frac{s}{r_2}\right] + \frac{r_2\gamma_2}{\delta_2}\cos\left[\frac{s}{r_1}\right]\sin\left[\frac{s}{r_2}\right]\right] + \mathcal{O}\left(|s|^2 e^{|t|((r_1+r_2)/r_1r_2)}\right). \quad (3.16)$$

*Proof.* The proof is immediate by substituting (3.4) and (3.6) into the representation

$$\omega(\lambda) = \left(\lambda\beta_1' + \beta_1\right)\varphi_{2\lambda}(1) - \left(\lambda\beta_2' + \beta_2\right)\varphi_{2\lambda}'(1). \quad (3.17)$$

□

**Corollary 3.4.** *The eigenvalues of the problem* (2.1)–(2.4) *are bounded below.*

*Proof.* Putting $s = it$ $(t > 0)$ in the above formulae, it follows that $\omega(-t^2) \to \infty$ as $t \to \infty$. Hence, $\omega(\lambda) \neq 0$ for $\lambda$ negative and sufficiently large. □

Now we can obtain the asymptotic approximation formula for the eigenvalues of the considered problem (2.1)–(2.4). Since the eigenvalues coincide with the zeros of the entire function $\omega(\lambda)$, it follows that they have no finite limit. Moreover, we know from Corollaries 2.2 and 3.4 that all eigenvalues are real and bounded below. Therefore, we may renumber them as $\lambda_0 \leq \lambda_1 \leq \lambda_2, \ldots$, listed according to their multiplicity.

**Theorem 3.5.** *The eigenvalues $\lambda_n = s_n^2$, $n = 0, 1, 2, \ldots$, of the problem* (2.1)–(2.4) *have the following asymptotic representation for $n \to \infty$, with $\gamma_1\delta_2 r_1 - \gamma_2\delta_1 r_2 = 0$.*

*Case 1.* If $\beta_2' \neq 0$ and $\alpha_2' \neq 0$, then

$$s_n = \frac{r_1 r_2}{r_1 + r_2}(n-1)\pi + \mathcal{O}\left(n^{-1}\right). \quad (3.18)$$

*Case 2.* If $\beta_2' \neq 0$ and $\alpha_2' = 0$, then

$$s_n = \frac{r_1 r_2}{r_1 + r_2}\left(n - \frac{1}{2}\right)\pi + \mathcal{O}\left(n^{-1}\right). \tag{3.19}$$

*Case 3.* If $\beta_2' = 0$ and $\alpha_2' \neq 0$, then

$$s_n = \frac{r_1 r_2}{r_1 + r_2}\left(n - \frac{1}{2}\right)\pi + \mathcal{O}\left(n^{-1}\right). \tag{3.20}$$

*Case 4.* If $\beta_2' = 0$ and $\alpha_2' = 0$, then

$$s_n = \frac{r_1 r_2}{r_1 + r_2} n\pi + \mathcal{O}\left(n^{-1}\right). \tag{3.21}$$

*Proof.* We will only consider the first case. From (3.13) we have

$$\omega(\lambda) = \frac{\alpha_2'\beta_2'\gamma_2 s^5}{\delta_2 r_1}\sin\left[\frac{r_1 + r_2}{r_1 r_2}s\right] + \mathcal{O}\left(|s|^4 e^{|t|((r_1+r_2)/r_1 r_2)}\right). \tag{3.22}$$

We will apply the well-known Rouche theorem, which asserts that if $f(\lambda)$ and $g(\lambda)$ are analytic inside and on a closed contour $C$ and $|g(\lambda)| < |f(\lambda)|$ on $C$, then $f(\lambda)$ and $f(\lambda) + g(\lambda)$ have the same number of zeros inside $C$, provided that each zero is counted according to its multiplicity. It follows that $\omega(\lambda)$ has the same number of zeros inside the contour as the leading term in (3.22). If $\lambda_0 \leq \lambda_1 \leq \lambda_2, \ldots,$ are the zeros of $\omega(\lambda)$ and $\lambda_n = s_n^2$, we have

$$s_n = \frac{r_1 r_2}{r_1 + r_2}(n-1)\pi + \delta_n, \tag{3.23}$$

for sufficiently large $n$, where $|\delta_n| < \pi/4$, for sufficiently large $n$. By putting in (3.22) we have $\delta_n = \mathcal{O}(n^{-1})$, so the proof is completed for Case 1. The proof for the other cases is similar. □

Then from (3.3)–(3.6) (for $k = 0$) and the above theorem, the asymptotic behavior of the eigenfunctions

$$\varphi_{\lambda_n}(x) = \begin{cases} \varphi_{1\lambda_n}(x), & x \in [-1, 0), \\ \varphi_{2\lambda_n}(x), & x \in (0, 1], \end{cases} \tag{3.24}$$

of (2.1)–(2.4) is given by, $\gamma_1\delta_2 r_1 - \gamma_2\delta_1 r_2 = 0$,

$$\varphi_{\lambda_n}(x)=\begin{cases}\begin{cases}\alpha_2'\cos\left(\dfrac{r_2(n-1)\pi}{r_1+r_2}(x+1)\right)+\mathcal{O}\left(n^{-1}\right), & x\in[-1,0),\\ \dfrac{\gamma_1\alpha_2'}{\delta_1}\cos\left(\dfrac{(n-1)\pi}{r_1+r_2}(r_1x+r_2)\right)+\mathcal{O}\left(n^{-1}\right), & x\in(0,1],\end{cases} & \text{if } \beta_2'\neq 0,\ \alpha_2'\neq 0,\\[2ex] \begin{cases}-r_1\alpha_1'\sin\left(\dfrac{r_2(n-1/2)\pi}{r_1+r_2}(x+1)\right)+\mathcal{O}\left(n^{-1}\right), & x\in[-1,0),\\ -\dfrac{\gamma_1 r_1\alpha_1'}{\delta_1}\sin\left(\dfrac{(n-1/2)\pi}{r_1+r_2}(r_1x+r_2)\right)+\mathcal{O}\left(n^{-1}\right), & x\in(0,1],\end{cases} & \text{if } \beta_2'\neq 0,\ \alpha_2'=0,\\[2ex] \begin{cases}\alpha_2'\cos\left(\dfrac{r_2(n-1/2)\pi}{r_1+r_2}(x+1)\right)+\mathcal{O}\left(n^{-1}\right), & x\in[-1,0),\\ \dfrac{\gamma_1\alpha_2'}{\delta_1}\cos\left(\dfrac{(n-1/2)\pi}{r_1+r_2}(r_1x+r_2)\right)+\mathcal{O}\left(n^{-1}\right), & x\in(0,1],\end{cases} & \text{if } \beta_2'=0,\ \alpha_2'\neq 0,\\[2ex] \begin{cases}-r_1\alpha_1'\sin\left(\dfrac{r_2 n\pi}{r_1+r_2}(x+1)\right)+\mathcal{O}\left(n^{-1}\right), & x\in[-1,0),\\ -\dfrac{\gamma_1 r_1\alpha_1'}{\delta_1}\sin\left(\dfrac{n\pi}{r_1+r_2}(r_1x+r_2)\right)+\mathcal{O}\left(n^{-1}\right), & x\in(0,1],\end{cases} & \text{if } \beta_2'=0,\ \alpha_2'=0.\end{cases} \tag{3.25}$$

All these asymptotic formulae hold uniformly for $x$.

## 4. The Sampling Theorem

In this section we derive two sampling theorems associated with problem (2.1)–(2.4). For convenience we may assume that the eigenvectors of $\mathbf{A}$ are real valued.

**Theorem 4.1.** *Consider the boundary value problem* (2.1)–(2.4)*, and let*

$$\varphi_\lambda(x)=\begin{cases}\varphi_{1\lambda}(x), & x\in[-1,0),\\ \varphi_{2\lambda}(x), & x\in(0,1],\end{cases} \tag{4.1}$$

*be the solution defined above. Let* $g(\cdot)\in L^2(-1,1)$ *and*

$$F(\lambda)=\frac{1}{r_1^2}\int_{-1}^{0} g(x)\varphi_{1\lambda}(x)dx+\frac{1}{r_2^2}\int_{0}^{1} g(x)\varphi_{2\lambda}(x)dx. \tag{4.2}$$

*Then $F(\lambda)$ is an entire function of exponential type 2 that can be reconstructed from its values at the points $\{\lambda_n\}_{n=0}^{\infty}$ via the sampling formula*

$$F(\lambda) = \sum_{n=0}^{\infty} F(\lambda_n) \frac{\omega(\lambda)}{(\lambda - \lambda_n)\omega'(\lambda_n)}. \tag{4.3}$$

*The series* (4.3) *converges absolutely on $\mathbb{C}$ and uniformly on compact subset of $\mathbb{C}$. Here $\omega(\lambda)$ is the entire function defined in* (2.29).

*Proof.* Relation (4.2) can be rewritten as an inner product of $\mathbf{H}$ as follows

$$F(\lambda) = \langle \mathbf{g}(\cdot), \mathbf{\Phi}_\lambda(\cdot) \rangle_{\mathbf{H}} = \frac{1}{r_1^2} \int_{-1}^{0} g(x)\varphi_{1\lambda}(x)dx + \frac{1}{r_2^2} \int_{0}^{1} g(x)\varphi_{2\lambda}(x)dx, \tag{4.4}$$

where

$$\mathbf{g}(x) = \begin{pmatrix} g(x) \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{\Phi}_\lambda(x) = \begin{pmatrix} \varphi_\lambda(x) \\ R'_{-1}(\phi\lambda) \\ R'_1(\phi\lambda) \end{pmatrix} \in \mathbf{H}. \tag{4.5}$$

Both $\mathbf{g}(\cdot)$ and $\mathbf{\Phi}_\lambda(\cdot)$ can be expanded in terms of the orthogonal basis on eigenfunctions, that is,

$$\mathbf{g}(x) = \sum_{n=0}^{\infty} \widehat{\mathbf{g}}(n) \frac{\mathbf{\Phi}_n(x)}{\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}^2}, \qquad \mathbf{\Phi}_\lambda(x) = \sum_{n=0}^{\infty} \widehat{\mathbf{\Phi}}_\lambda(n) \frac{\mathbf{\Phi}_n(x)}{\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}^2}, \tag{4.6}$$

where $\widehat{\mathbf{g}}(n)$ and $\widehat{\mathbf{\Phi}}_\lambda(n)$ are the fourier coefficients

$$\begin{aligned} \widehat{\mathbf{g}}(n) = \langle \mathbf{g}(\cdot), \mathbf{\Phi}_n(\cdot) \rangle_{\mathbf{H}} &= \frac{1}{r_1^2} \int_{-1}^{0} g(x)\varphi_{1\lambda_n}(x)dx + \frac{1}{r_2^2} \int_{0}^{1} g(x)\varphi_{2\lambda_n}(x)dx \\ &= F(\lambda_n). \end{aligned} \tag{4.7}$$

Applying Parseval's identity to (4.4) and using (4.7), we obtain

$$F(\lambda) = \sum_{n=0}^{\infty} F(\lambda_n) \frac{\langle \mathbf{\Phi}_n(\cdot), \mathbf{\Phi}_\lambda(\cdot) \rangle_{\mathbf{H}}}{\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}^2}. \tag{4.8}$$

Now we calculate $\widehat{\mathbf{\Phi}}_\lambda(n) = \langle \mathbf{\Phi}_n(\cdot), \mathbf{\Phi}_\lambda(\cdot) \rangle_{\mathbf{H}}$ and $\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}$. Let $\lambda \in \mathbb{C}$ not be an eigenvalue and $n \in \mathbb{N}$. To prove (4.3) we need to show that

$$\frac{\langle \mathbf{\Phi}_n(\cdot), \mathbf{\Phi}_\lambda(\cdot) \rangle_{\mathbf{H}}}{\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}^2} = \frac{\omega(\lambda)}{(\lambda - \lambda_n)\omega'(\lambda)}, \quad n = 0, 1, 2, \ldots. \tag{4.9}$$

By the definition of the inner product of $\mathbf{H}$, we have

$$\begin{aligned}\langle \boldsymbol{\Phi}_\lambda(\cdot), \boldsymbol{\Phi}_n(\cdot)\rangle_{\mathbf{H}} &= \frac{1}{r_1^2}\int_{-1}^{0} \varphi_{1\lambda}(x)\varphi_{1\lambda_n}(x)dx + \frac{1}{r_2^2}\int_{0}^{1} \varphi_{2\lambda}(x)\varphi_{2\lambda_n}(x)dx \\ &\quad + \frac{1}{\rho}R'_{-1}(\varphi_\lambda)R'_{-1}(\varphi_{\lambda_n}) + \frac{1}{\gamma}R'_1(\varphi_\lambda)R'_1(\varphi_{\lambda_n}).\end{aligned} \tag{4.10}$$

Since

$$\begin{aligned}&\frac{1}{r_1^2}\int_{-1}^{0} \ell(\varphi_{1\lambda}(x))\varphi_{1\lambda_n}(x)dx + \frac{1}{r_2^2}\int_{0}^{1} \ell(\varphi_{2\lambda}(x))\varphi_{2\lambda_n}(x)dx \\ &\quad = \frac{1}{r_1^2}\int_{-1}^{0} \varphi_{1\lambda}(x)\ell(\varphi_{1\lambda_n}(x))dx + \frac{1}{r_2^2}\int_{0}^{1} \varphi_{2\lambda}(x)\ell(\varphi_{2\lambda_n}(x))dx + W(\varphi_{1\lambda}, \varphi_{1\lambda_n}; -0) \\ &\quad - W(\varphi_{1\lambda}, \varphi_{1\lambda_n}; -1) - W(\varphi_{2\lambda}, \varphi_{2\lambda_n}; +0) + W(\varphi_{2\lambda}, \varphi_{2\lambda_n}; 1),\end{aligned} \tag{4.11}$$

then, from (2.20) and (2.24), (4.11) becomes

$$(\lambda - \lambda_n)\left[\frac{1}{r_1^2}\int_{-1}^{0} \varphi_{1\lambda}(x)\varphi_{1\lambda_n}(x)dx + \frac{1}{r_2^2}\int_{0}^{1} \varphi_{2\lambda}(x)\varphi_{2\lambda_n}(x)dx\right] = W(\varphi_{2\lambda}, \varphi_{2\lambda_n}; 1) - (\lambda - \lambda_n)\rho. \tag{4.12}$$

Thus

$$\frac{1}{r_1^2}\int_{-1}^{0} \varphi_{1\lambda}(x)\varphi_{1\lambda_n}(x)dx + \frac{1}{r_2^2}\int_{0}^{1} \varphi_{2\lambda}(x)\varphi_{2\lambda_n}(x)dx = \frac{W(\varphi_{2\lambda}, \varphi_{2\lambda_n}; 1)}{\lambda - \lambda_n} - \rho. \tag{4.13}$$

From (2.48), (2.22), and (2.8), the Wronskian of $\varphi_{2\lambda_n}$ and $\varphi_{2\lambda}$ at $x = 1$ will be

$$\begin{aligned}W(\varphi_{2\lambda}, \varphi_{2\lambda_n}; 1) &= \varphi_{2\lambda}(1)\varphi'_{2\lambda_n}(1) - \varphi'_{2\lambda}(1)\varphi_{2\lambda_n}(1) \\ &= k_n^{-1}\left[\chi'_{2\lambda_n}(1)\varphi_{2\lambda}(1) - \chi_{2\lambda_n}(1)\varphi'_{2\lambda}(1)\right] \\ &= k_n^{-1}\left[(\beta'_1\lambda_n + \beta_1)\varphi_{2\lambda}(1) - (\beta'_2\lambda_n + \beta_2)\varphi'_{2\lambda}(1)\right] \\ &= k_n^{-1}\left[\omega(\lambda) + (\lambda_n - \lambda)R'_1(\varphi_\lambda)\right].\end{aligned} \tag{4.14}$$

Relations (2.48) and $R'_1(\chi_{\lambda_n}) = \gamma$ and the linearity of the boundary conditions yield

$$\frac{1}{\gamma}R'_1(\varphi_\lambda)R'_1(\varphi_{\lambda_n}) = \frac{k_n^{-1}}{\gamma}R'_1(\varphi_\lambda)R'_1(\chi_{\lambda_n}) = k_n^{-1}R'_1(\varphi_\lambda). \tag{4.15}$$

Substituting from (4.13), (4.14), (4.15), and $R'_{-1}(\varphi_\lambda) = R'_{-1}(\varphi_{\lambda_n}) = -\rho$ into (4.10), we get

$$\langle \mathbf{\Phi}_\lambda(\cdot), \mathbf{\Phi}_n(\cdot)\rangle_{\mathbf{H}} = k_n^{-1}\frac{\omega(\lambda)}{\lambda - \lambda_n}. \tag{4.16}$$

Letting $\lambda \to \lambda_n$ in (4.16) and since the zeros of $\omega(\lambda)$ are simple, we have

$$\langle \mathbf{\Phi}_n(\cdot), \mathbf{\Phi}_n(\cdot)\rangle_{\mathbf{H}} = \|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}^2 = k_n^{-1}\omega'(\lambda_n). \tag{4.17}$$

Therefore from (4.16) and (4.17) we establish (4.9). Since $\lambda$ and $n$ are arbitrary, then (4.3) is proved with a pointwise convergence on $\mathbb{C}$, since the case $\lambda = \lambda_n$ is trivial.

Now we investigate the convergence of (4.3). First we prove that it is absolutely convergent on $\mathbb{C}$. Using Cauchy-Schwarz' inequality for $\lambda \in \mathbb{C}$,

$$\sum_{n=0}^{\infty}\left|F(\lambda_n)\frac{\omega(\lambda)}{(\lambda-\lambda_n)\omega'(\lambda_n)}\right| \le \left(\sum_{n=0}^{\infty}\frac{\left|\langle \mathbf{g}(\cdot), \mathbf{\Phi}_n(\cdot)\rangle_{\mathbf{H}}\right|^2}{\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}^2}\right)^{1/2}\left(\sum_{n=0}^{\infty}\frac{\left|\langle \mathbf{\Phi}_n(\cdot), \mathbf{\Phi}_\lambda(\cdot)\rangle_{\mathbf{H}}\right|^2}{\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}^2}\right)^{1/2}. \tag{4.18}$$

Since $\mathbf{g}(\cdot)$, $\mathbf{\Phi}_\lambda(\cdot) \in \mathbf{H}$, then both series in the right-hand side of (4.18) converge. Thus series (4.3) converges absolutely on $\mathbb{C}$. For uniform convergence let $M \subset \mathbb{C}$ be compact. Let $\lambda \in M$ and $N > 0$. Define $\sigma_N(\lambda)$ to be

$$\sigma_N(\lambda) := \left|F(\lambda) - \sum_{n=0}^{N}F(\lambda_n)\frac{\omega(\lambda)}{(\lambda-\lambda_n)\omega'(\lambda_n)}\right|. \tag{4.19}$$

Using the same method developed above

$$\sigma_N(\lambda) \le \left(\sum_{n=N+1}^{\infty}\frac{\left|\langle \mathbf{g}(\cdot), \mathbf{\Phi}_n(\cdot)\rangle_{\mathbf{H}}\right|^2}{\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}^2}\right)^{1/2}\left(\sum_{n=N+1}^{\infty}\frac{\left|\langle \mathbf{\Phi}_n(\cdot), \mathbf{\Phi}_\lambda(\cdot)\rangle_{\mathbf{H}}\right|^2}{\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}^2}\right)^{1/2}. \tag{4.20}$$

Therefore

$$\sigma_N(\lambda) \le \|\mathbf{\Phi}_\lambda(\cdot)\|_{\mathbf{H}}\left(\sum_{n=N+1}^{\infty}\frac{\left|\langle \mathbf{g}(\cdot), \mathbf{\Phi}_n(\cdot)\rangle_{\mathbf{H}}\right|^2}{\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}^2}\right)^{1/2}. \tag{4.21}$$

Since $[-1,1] \times M$ is compact, then, cf., for example, [31, page 225], we can find a positive constant $C_M$ such that

$$\|\mathbf{\Phi}_\lambda(\cdot)\|_{\mathbf{H}} \le C_M, \quad \forall\lambda \in M. \tag{4.22}$$

Then

$$\sigma_N(\lambda) \le C_M\left(\sum_{n=N+1}^{\infty}\frac{\left|\langle \mathbf{g}(\cdot), \mathbf{\Phi}_n(\cdot)\rangle_{\mathbf{H}}\right|^2}{\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}^2}\right)^{1/2} \tag{4.23}$$

uniformly on $M$. In view of Parseval's equality,

$$\left(\sum_{n=N+1}^{\infty}\frac{\left|\langle \mathbf{g}(\cdot),\mathbf{\Phi}_n(\cdot)\rangle_{\mathbf{H}}\right|^2}{\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}^2}\right)^{1/2}\longrightarrow 0 \quad \text{as } N\longrightarrow\infty. \tag{4.24}$$

Thus $\sigma_N(\lambda) \to 0$ uniformly on $M$. Hence (4.3) converges uniformly on $M$. Thus $F(\lambda)$ is analytic on compact subsets of $\mathbb{C}$ and hence it is entire. From the relation

$$|F(\lambda)|\le\frac{1}{r_1^2}\int_{-1}^{0}|g(x)||\varphi_{1\lambda}(x)|dx+\frac{1}{r_2^2}\int_{0}^{1}|g(x)||\varphi_{2\lambda}(x)|dx \tag{4.25}$$

and the fact that $\varphi_{1\lambda}(x)$ and $\varphi_{2\lambda}(x)$ are entire function of exponential type 2, we conclude that $F(\lambda)$ is also of exponential type 2. □

*Remark 4.2.* To see that expansion (4.3) is a Lagrange-type interpolation, we may replace $\omega(\lambda)$ by the canonical product

$$\widetilde{\omega}(\lambda)=\begin{cases}\displaystyle\prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right), & \text{if none of the eigenvalues is zero;}\\ \displaystyle\lambda\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right), & \text{if one of the eigenvalues, say } \lambda_0=0.\end{cases} \tag{4.26}$$

From Hadamard's factorization theorem, see [4], $\omega(\lambda) = h(\lambda)\widetilde{\omega}(\lambda)$, where $h(\lambda)$ is an entire function with no zeros. Thus,

$$\frac{\omega(\lambda)}{\omega'(\lambda_n)}=\frac{h(\lambda)\widetilde{\omega}(\lambda)}{h(\lambda_n)\widetilde{\omega}'(\lambda_n)} \tag{4.27}$$

and (4.2), (4.3) remain valid for the function $F(\lambda)/h(\lambda)$. Hence

$$F(\lambda)=\sum_{n=0}^{\infty}F(\lambda_n)\frac{h(\lambda)\widetilde{\omega}(\lambda)}{h(\lambda_n)\widetilde{\omega}'(\lambda_n)(\lambda-\lambda_n)}. \tag{4.28}$$

We may redefine (4.2) by taking kernel $\varphi_\lambda(\cdot)/h(\lambda) = \widetilde{\varphi}_\lambda(\cdot)$ to get

$$\widetilde{F}(\lambda)=\frac{F(\lambda)}{h(\lambda)}=\sum_{n=0}^{\infty}\widetilde{F}(\lambda_n)\frac{\widetilde{\omega}(\lambda)}{(\lambda-\lambda_n)\widetilde{\omega}'(\lambda_n)}. \tag{4.29}$$

The next theorem is devoted to give interpolation sampling expansions associated with problem (2.1)–(2.4) for integral transforms whose kernels defined in terms of Green's function. There are many results concerning the use of Green's function in sampling theory, cf., for example, [22, 32–34]. As we see in (2.60), Green's function $G(x,\xi,\lambda)$ of problem (2.1)–(2.4) has simple poles at $\{\lambda_n\}_{n=0}^{\infty}$. Define the function $G(x,\lambda)$ to be $G(x,\lambda) := \omega(\lambda)G(x,\xi_0,\lambda)$, where $\xi_0 \in [-1,0)\cup(0,1]$ is a fixed point and $\omega(\lambda)$ is the function defined in (2.29) or it is the canonical product (4.26).

**Theorem 4.3.** *Let $g(\cdot) \in L^2(-1,1)$ and $F(\lambda)$ the integral transform*

$$F(\lambda) = \frac{1}{r_1^2}\int_{-1}^{0} G(x,\lambda)\overline{g}(x)dx + \frac{1}{r_2^2}\int_{0}^{1} G(x,\lambda)\overline{g}(x)dx. \tag{4.30}$$

*Then $F(\lambda)$ is an entire function of exponential type 2 which admits the sampling representation*

$$F(\lambda) = \sum_{n=0}^{\infty} F(\lambda_n)\frac{\omega(\lambda)}{(\lambda-\lambda_n)\omega'(\lambda_n)}. \tag{4.31}$$

*Series* (4.31) *converges absolutely on $\mathbb{C}$ and uniformly on compact subsets of $\mathbb{C}$.*

*Proof.* The integral transform (4.30) can be written as

$$F(\lambda) = \langle \mathcal{G}(\cdot,\lambda), \mathbf{g}(\cdot)\rangle_{\mathbf{H}}, \tag{4.32}$$

$$\mathbf{g}(x) = \begin{pmatrix} g(x) \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{G}(x,\lambda) = \begin{pmatrix} G(x,\lambda) \\ R'_{-1}(G(x,\lambda)) \\ R'_1(G(x,\lambda)) \end{pmatrix} \in \mathbf{H}. \tag{4.33}$$

Applying Parseval's identity to (4.32) with respect to $\{\mathbf{\Phi}_n(\cdot)\}_{n=0}^{\infty}$, we obtain

$$F(\lambda) = \sum_{n=0}^{\infty} \langle \mathcal{G}(\cdot,\lambda), \mathbf{\Phi}_n(\cdot)\rangle_{\mathbf{H}} \frac{\overline{\langle \mathbf{g}(\cdot), \mathbf{\Phi}_n(\cdot)\rangle_{\mathbf{H}}}}{\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}^2}. \tag{4.34}$$

Let $\lambda \neq \lambda_n$. Since each $\mathbf{\Phi}_n(\cdot)$ is an eigenvector of $\mathbf{A}$, then

$$(\lambda I - \mathbf{A})\mathbf{\Phi}_n(x) = (\lambda-\lambda_n)\mathbf{\Phi}_n(x). \tag{4.35}$$

Thus

$$(\lambda I - \mathbf{A})^{-1}\mathbf{\Phi}_n(x) = \frac{1}{\lambda-\lambda_n}\mathbf{\Phi}_n(x). \tag{4.36}$$

From (2.59) and (4.36) we obtain

$$\begin{aligned} &\frac{R'_1(\varphi_{\lambda_n})}{\omega(\lambda)}\varphi_{\lambda}(\xi_0) - \frac{R'_{-1}(\varphi_{\lambda_n})}{\omega(\lambda)}\chi_{\lambda}(\xi_0) + \frac{1}{r_1^2}\int_{-1}^{0} G(x,\xi_0,\lambda)\varphi_{1\lambda_n}(x)dx + \frac{1}{r_2^2}\int_{0}^{1} G(x,\xi_0,\lambda)\varphi_{2\lambda_n}(x)dx \\ &\quad = \frac{1}{\lambda-\lambda_n}\varphi_{\lambda_n}(\xi_0). \end{aligned} \tag{4.37}$$

Using $R'_{-1}(\varphi_{\lambda_n}) = -\rho$, (2.48), and $R'_1(\chi_{\lambda_n}) = \gamma$, (4.37) becomes

$$\begin{aligned}
&\frac{\gamma k_n^{-1}}{\omega(\lambda)}\varphi_\lambda(\xi_0) + \frac{\rho}{\omega(\lambda)}\chi_\lambda(\xi_0) + \frac{1}{r_1^2}\int_{-1}^{0} G(x,\xi_0,\lambda)\varphi_{1\lambda_n}(x)dx + \frac{1}{r_2^2}\int_0^1 G(x,\xi_0,\lambda)\varphi_{2\lambda_n}(x)dx \\
&\qquad = \frac{1}{\lambda-\lambda_n}\varphi_{\lambda_n}(\xi_0).
\end{aligned}\tag{4.38}$$

Hence (4.38) can be rewritten as

$$\gamma k_n^{-1}\varphi_\lambda(\xi_0) + \rho\chi_\lambda(\xi_0) + \frac{1}{r_1^2}\int_{-1}^{0} G(x,\lambda)\varphi_{1\lambda_n}(x)dx + \frac{1}{r_2^2}\int_0^1 G(x,\lambda)\varphi_{2\lambda_n}(x)dx = \frac{\omega(\lambda)}{\lambda-\lambda_n}\varphi_{\lambda_n}(\xi_0).\tag{4.39}$$

From the definition of $\mathcal{G}(\cdot,\lambda)$, we have

$$\begin{aligned}
\langle \mathcal{G}(\cdot,\lambda), \mathbf{\Phi}_n(\cdot)\rangle_{\mathbf{H}} &= \frac{1}{r_1^2}\int_{-1}^{0} G(x,\lambda)\varphi_{1\lambda_n}(x)dx + \frac{1}{r_2^2}\int_0^1 G(x,\lambda)\varphi_{2\lambda_n}(x)dx \\
&\quad + \frac{1}{\rho}R'_{-1}(G(x,\lambda))R'_{-1}(\varphi_{\lambda_n}) + \frac{1}{\gamma}R'_1(G(x,\lambda))R'_1(\varphi_{\lambda_n}).
\end{aligned}\tag{4.40}$$

From formula (2.60), we get

$$R'_{-1}(G(x,\lambda)) = \chi_\lambda(\xi_0)R'_{-1}(\varphi_\lambda), \qquad R'_1(G(x,\lambda)) = \varphi_\lambda(\xi_0)R'_1(\chi_\lambda).\tag{4.41}$$

Combining (4.41), $R'_{-1}(\varphi_\lambda) = R'_{-1}(\varphi_{\lambda_n}) = -\rho$, $R'_1(\chi_\lambda) = R'_1(\chi_{\lambda_n}) = \gamma$, and (2.48) together with (4.40) yields

$$\langle \mathcal{G}(\cdot,\lambda), \mathbf{\Phi}_n(\cdot)\rangle_{\mathbf{H}} = \frac{1}{r_1^2}\int_{-1}^{0} G(x,\lambda)\varphi_{1\lambda_n}(x)dx + \frac{1}{r_2^2}\int_0^1 G(x,\lambda)\varphi_{2\lambda_n}(x)dx + \gamma k_n^{-1}\varphi_\lambda(\xi_0) + \rho\chi_\lambda(\xi_0).\tag{4.42}$$

Substituting from (4.39) and (4.42) gives

$$\langle \mathcal{G}(\cdot,\lambda), \mathbf{\Phi}_n(\cdot)\rangle_{\mathbf{H}} = \frac{\omega(\lambda)}{\lambda-\lambda_n}\varphi_{\lambda_n}(\xi_0).\tag{4.43}$$

As an element of $\mathbf{H}$, $\mathcal{G}(\cdot,\lambda)$ has the eigenvectors expansion

$$\begin{aligned}
\mathcal{G}(x,\lambda) &= \sum_{i=0}^{\infty}\langle \mathcal{G}(\cdot,\lambda), \mathbf{\Phi}_{\mathbf{i}}(\cdot)\rangle_{\mathbf{H}}\frac{\mathbf{\Phi}_i(x)}{\|\mathbf{\Phi}_i(\cdot)\|_{\mathbf{H}}^2} \\
&= \sum_{i=0}^{\infty}\frac{\omega(\lambda)}{(\lambda-\lambda_i)}\varphi_{\lambda_i}(\xi_0)\frac{\mathbf{\Phi}_i(x)}{\|\mathbf{\Phi}_i(\cdot)\|_{\mathbf{H}}^2}.
\end{aligned}\tag{4.44}$$

Taking the limit when $\lambda \to \lambda_n$ in (4.32), we get

$$F(\lambda_n) = \lim_{\lambda \to \lambda_n} \langle \mathcal{G}(\cdot, \lambda), \mathbf{g}(\cdot) \rangle_{\mathbf{H}}. \tag{4.45}$$

The interchange of the limit and summation processes is justified by the uniform convergence of the eigenvector expansion of $\mathcal{G}(x, \lambda)$ on $[-1, 1]$ on compact subsets of $\mathbb{C}$, cf., (2.60), (3.3)–(3.6), and (3.18)–(3.21). Making use of (4.44), we may rewrite (4.45) as

$$\begin{aligned} F(\lambda_n) &= \lim_{\lambda \to \lambda_n} \sum_{i=0}^{\infty} \frac{\omega(\lambda)}{(\lambda - \lambda_i)} \varphi_{\lambda_i}(\xi_0) \frac{\langle \mathbf{\Phi}_i(\cdot), \mathbf{g}(\cdot) \rangle_{\mathbf{H}}}{\|\mathbf{\Phi}_i(\cdot)\|_{\mathbf{H}}^2} \\ &= \omega'(\lambda_n) \varphi_{\lambda_n}(\xi_0) \frac{\langle \mathbf{\Phi}_n(\cdot), \mathbf{g}(\cdot) \rangle_{\mathbf{H}}}{\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}^2}. \end{aligned} \tag{4.46}$$

The interchange of the limit and summation is justified by the asymptotic behavior of $\mathbf{\Phi}_i(x)$ and $\omega(\lambda)$. If $\varphi_{\lambda_n}(\xi_0) \neq 0$, then (4.46) gives

$$\frac{\overline{\langle \mathbf{g}(\cdot), \mathbf{\Phi}_n(\cdot) \rangle}_{\mathbf{H}}}{\|\mathbf{\Phi}_n(\cdot)\|_{\mathbf{H}}^2} = \frac{F(\lambda_n)}{\omega'(\lambda_n) \varphi_{\lambda_n}(\xi_0)}. \tag{4.47}$$

Combining (4.43), (4.47), and (4.34) we get (4.31) under the assumption that $\varphi_{\lambda_n}(\xi_0) \neq 0$ for all $n$. If $\varphi_{\lambda_n}(\xi_0) = 0$, for some $n$, the same expansion holds with $F(\lambda_n) = 0$. The convergence properties as well as the analytic and growth properties can be established as in Theorem 4.1. □

Now, we give an example exhibiting the obtained results.

*Example 4.4.* The boundary value problem,

$$\begin{gathered} -y''(x) + q(x) y(x) = \lambda y(x), \quad x \in [-1, 0) \cup (0, 1], \\ y'(-1) = -\lambda y(-1), \qquad y'(1) = \lambda y(1), \\ 2y(-0) - y(+0) = 0, \qquad y'(-0) - 2y'(+0) = 0, \end{gathered} \tag{4.48}$$

is special case of the problem (2.1)–(2.4) when $\alpha_1 = \alpha_2' = \beta_1 = \beta_2' = 0$, $\beta_2 = \beta_1' = \alpha_2 = \alpha_1' = r_1 = r_2 = 1$, $\gamma_1 = \delta_2 = 2$, $\gamma_2 = \delta_1 = 1$, and

$$q(x) = \begin{cases} -1, & x \in [-1, 0), \\ -2, & x \in (0, 1]. \end{cases} \tag{4.49}$$

Then $\rho = \gamma = 1 > 0$. The solutions $\varphi_\lambda(\cdot)$ and $\chi_\lambda(\cdot)$ are

$$\varphi_\lambda(x) = \begin{cases} \varphi_{1\lambda}(x) = \dfrac{(2\zeta_{1\lambda}^2 - \zeta_{2\lambda}^2) \sin \zeta_{1\lambda}(x+1)}{\zeta_{1\lambda}} - \cos \zeta_{1\lambda}(x+1), \quad x \in [-1, 0), \\ \varphi_{2\lambda}(x) = 2\left( \dfrac{(2\zeta_{1\lambda}^2 - \zeta_{2\lambda}^2) \sin \zeta_{1\lambda}}{\zeta_{1\lambda}} - \cos \zeta_{1\lambda} \right) \cos \zeta_{2\lambda} x \\ \qquad + \dfrac{(2\zeta_{1\lambda}^2 - \zeta_{2\lambda}^2) \cos \zeta_{1\lambda} + \zeta_{1\lambda} \sin \zeta_{1\lambda}}{2\zeta_{2\lambda}} \sin \zeta_{2\lambda} x, \quad x \in (0, 1], \end{cases}$$

$$\chi_\lambda(x)=\begin{cases}\chi_{1\lambda}(x)=\dfrac{1}{2}\left(\cos\zeta_{2\lambda}-\dfrac{2\zeta_{1\lambda}^2-\zeta_{2\lambda}^2}{\zeta_{2\lambda}}\sin\zeta_{2\lambda}\right)\cos\zeta_{1\lambda}x\\ \qquad\qquad +2\left(\dfrac{\zeta_{2\lambda}}{\zeta_{1\lambda}}\sin\zeta_{2\lambda}+\dfrac{2\zeta_{1\lambda}^2-\zeta_{2\lambda}^2}{\zeta_{1\lambda}}\cos\zeta_{2\lambda}\right)\sin\zeta_{1\lambda}x, & x\in[-1,0),\\ \chi_{2\lambda}(x)=\cos\zeta_{2\lambda}(x-1)+\dfrac{2\zeta_{1\lambda}^2-\zeta_{2\lambda}^2}{\zeta_{2\lambda}}\sin\zeta_{2\lambda}(x-1), & x\in(0,1],\end{cases}\tag{4.50}$$

where $\zeta_{1\lambda}:=\sqrt{\lambda+1}$ and $\zeta_{2\lambda}:=\sqrt{\lambda+2}$. Here the characteristic function is

$$\begin{aligned}\omega(\lambda)=\frac{1}{2\zeta_{1\lambda}\zeta_{2\lambda}}\Big\{&\zeta_{1\lambda}\cos\zeta_{1\lambda}\left(-5\left(2\zeta_{1\lambda}^2-\zeta_{2\lambda}^2\right)\zeta_{2\lambda}\cos\zeta_{2\lambda}+\left(\zeta_{2\lambda}^4-8\zeta_{1\lambda}^2-4\right)\sin\zeta_{2\lambda}\right)\\ &+\sin\zeta_{1\lambda}\left(\left[-\zeta_{1\lambda}^2+4\left(2\zeta_{1\lambda}^2-\zeta_{2\lambda}^2\right)^2\zeta_{2\lambda}\right]\cos\zeta_{2\lambda}+\left(2\zeta_{1\lambda}^2-\zeta_{2\lambda}^2\right)\left(4+5\zeta_{2\lambda}^1\right)\sin\zeta_{2\lambda}\right)\Big\}.\end{aligned}\tag{4.51}$$

By Theorem 4.1, the transform

$$\begin{aligned}F(\lambda)=&\int_{-1}^{0}g(x)\left[\frac{(2\zeta_{1\lambda}^2-\zeta_{2\lambda}^2)\sin\zeta_{1\lambda}(x+1)}{\zeta_{1\lambda}}-\cos\zeta_{1\lambda}(x+1)\right]dx\\ &+\int_0^1 g(x)\left[2\left(\frac{(2\zeta_{1\lambda}^2-\zeta_{2\lambda}^2)\sin\zeta_{1\lambda}}{\zeta_{1\lambda}}-\cos\zeta_{1\lambda}\right)\cos\zeta_{2\lambda}x\right.\\ &\left.\qquad+\frac{(2\zeta_{1\lambda}^2-\zeta_{2\lambda}^2)\cos\zeta_{1\lambda}+\zeta_{1\lambda}\sin\zeta_{1\lambda}}{2\zeta_{2\lambda}}\sin\zeta_{2\lambda}x\right]dx\end{aligned}\tag{4.52}$$

has the following expansion:

$$F(\lambda)=\sum_{n=0}^{\infty}F(\lambda_n)\frac{\omega(\lambda)}{\left((2\zeta_{1\lambda}^2-\zeta_{2\lambda}^2)-\left(2\zeta_{1\lambda_n}^2-\zeta_{2\lambda_n}^2\right)\right)\omega'(\lambda_n)},\tag{4.53}$$

$$\begin{aligned}\omega'(\lambda_n):=\frac{1}{4\zeta_{1\lambda_n}^3\zeta_{2\lambda_n}^3}\Big[&\sin\zeta_{1\lambda_n}\Big(2\zeta_{2\lambda_n}\left(-1+24\lambda+34\lambda^2+11\lambda^{11}\right)\cos\zeta_{2\lambda_n}\\ &\qquad+\left(54+104\lambda+59\lambda^2-5\lambda^4\right)\sin\zeta_{2\lambda_n}\Big)\\ &+\zeta_{1\lambda_n}\cos\zeta_{1\lambda_n}\Big(\zeta_{2\lambda_n}\left(-30-45\lambda-6\lambda^2+5\lambda^3\right)\cos\zeta_{2\lambda_n}\\ &\qquad+\left(-8+24\lambda+41\lambda^2+13\lambda^3\right)\sin\zeta_{2\lambda_n}\Big)\Big].\end{aligned}\tag{4.54}$$

The Green's function has the following form:

$$
\begin{aligned}
&G(x,\xi,\lambda)\\
&\quad=\frac{1}{2\zeta_{1\lambda}\zeta_{2\lambda}}\Big\{\zeta_{1\lambda}\cos\zeta_{1\lambda}\Big(-5\Big(2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2}\Big)\zeta_{2\lambda}\cos\zeta_{2\lambda}+\Big(\zeta_{2\lambda}^{4}-8\zeta_{1\lambda}^{2}-4\Big)\sin\zeta_{2\lambda}\Big)\\
&\qquad\qquad+\sin\zeta_{1\lambda}\Big(\Big[-\zeta_{1\lambda}^{2}+4\Big(2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2}\Big)^{2}\zeta_{2\lambda}\Big]\cos\zeta_{2\lambda}+\Big(2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2}\Big)\Big(4+5\zeta_{2\lambda}^{1}\Big)\sin\zeta_{2\lambda}\Big)\Big\}\\
&\times\begin{cases}
\left(\dfrac{(2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2})\sin\zeta_{1\lambda}(x+1)}{\zeta_{1\lambda}}-\cos\zeta_{1\lambda}(x+1)\right)\\
\times\left[\dfrac{1}{2}\left(\cos\zeta_{2\lambda}-\dfrac{2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2}}{\zeta_{2\lambda}}\sin\zeta_{2\lambda}\right)\cos\zeta_{1\lambda}\xi+2\left(\dfrac{\zeta_{2\lambda}}{\zeta_{1\lambda}}\sin\zeta_{2\lambda}+\dfrac{2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2}}{\zeta_{1\lambda}}\cos\zeta_{2\lambda}\right)\sin\zeta_{1\lambda}\xi\right], & -1\le x\le\xi<0,\\
\left(\dfrac{(2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2})\sin\zeta_{1\lambda}(\xi+1)}{\zeta_{1\lambda}}-\cos\zeta_{1\lambda}(\xi+1)\right)\\
\times\left[\dfrac{1}{2}\left(\cos\zeta_{2\lambda}-\dfrac{2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2}}{\zeta_{2\lambda}}\sin\zeta_{2\lambda}\right)\cos\zeta_{1\lambda}x+2\left(\dfrac{\zeta_{2\lambda}}{\zeta_{1\lambda}}\sin\zeta_{2\lambda}+\dfrac{2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2}}{\zeta_{1\lambda}}\cos\zeta_{2\lambda}\right)\sin\zeta_{1\lambda}x\right], & -1\le\xi\le x<0,\\
\left(\dfrac{(2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2})\sin\zeta_{1\lambda}(\xi+1)}{\zeta_{1\lambda}}-\cos\zeta_{1\lambda}(\xi+1)\right)\left[\cos\zeta_{2\lambda}(x-1)+\dfrac{2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2}}{\zeta_{2\lambda}}\sin\zeta_{2\lambda}(x-1)\right], & -1\le\xi<0,\ 0<x\le 1,\\
\left(\dfrac{(2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2})\sin\zeta_{1\lambda}(x+1)}{\zeta_{1\lambda}}-\cos\zeta_{1\lambda}(x+1)\right)\left[\cos\zeta_{2\lambda}(\xi-1)+\dfrac{2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2}}{\zeta_{2\lambda}}\sin\zeta_{2\lambda}(\xi-1)\right], & -1\le x<,\ 0<\xi\le 1,\\
\left[\cos\zeta_{2\lambda}(x-1)+\dfrac{2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2}}{\zeta_{2\lambda}}\sin\zeta_{2\lambda}(x-1)\right]\\
\times\left[2\left(\dfrac{(2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2})\sin\zeta_{1\lambda}}{\zeta_{1\lambda}}-\cos\zeta_{1\lambda}\right)\cos\zeta_{2\lambda}\xi+\dfrac{(2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2})\cos\zeta_{1\lambda}+\zeta_{1\lambda}\sin\zeta_{1\lambda}}{2\zeta_{2\lambda}}\sin\zeta_{2\lambda}\xi\right], & 0<\xi\le x\le 1,\\
\left[\cos\zeta_{2\lambda}(\xi-1)+\dfrac{2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2}}{\zeta_{2\lambda}}\sin\zeta_{2\lambda}(\xi-1)\right]\\
\times\left[2\left(\dfrac{(2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2})\sin\zeta_{1\lambda}}{\zeta_{1\lambda}}-\cos\zeta_{1\lambda}\right)\cos\zeta_{2\lambda}x+\dfrac{(2\zeta_{1\lambda}^{2}-\zeta_{2\lambda}^{2})\cos\zeta_{1\lambda}+\zeta_{1\lambda}\sin\zeta_{1\lambda}}{2\zeta_{2\lambda}}\sin\zeta_{2\lambda}x\right], & 0<x\le\xi\le 1.
\end{cases}
\end{aligned}
\tag{4.55}
$$

Taking $\xi \in [-1,0) \cup (0,1]$, the transform

$$F(\lambda) = \int_{-1}^{0} G(x,\lambda)\overline{g}(x)dx + 4\int_{0}^{1} G(x,\lambda)\overline{g}(x)dx. \tag{4.56}$$

has a sampling representation of the type.

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