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Research Article

Specific Mathematical Aspects of Dynamics Generated by Coherence Functions

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This study presents specific aspects of dynamics generated by the coherence function (acting in an integral manner). It is considered that an oscillating system starting to work from initial nonzero conditions is commanded by the coherence function between the output of the system and an alternating function of a certain frequency. For different initial conditions, the evolution of the system is analyzed. The equivalence between integrodifferential equations and integral equations implying the same number of state variables is investigated; it is shown that integro-differential equations of second order are far more restrictive regarding the initial conditions for the state variables. Then, the analysis is extended to equations of evolution where the coherence function is acting under the form of a multiple integral. It is shown that for the coherence function represented under the form of an nth integral, some specific aspects as multiscale behaviour suitable for modelling transitions in complex systems (e.g., quantum physics) could be noticed when n equals 4, 5, or 6.

1. Introduction

Nonlinear phenomena generated by a sequence of external pulses can be noticed at a macroscopic scale as the breaking effect of a set of medium-power shocks (e.g., applied as transverse force upon a beam fixed at both ends) when the time interval between these pulses is large enough so as the final effect not be considered a superposition of individual effects of each pulse. It is well known from practice that workers using traditional tools have to apply some medium-power shocks at certain intervals upon a beam fixed at both ends, before a final great-power shock to be applied for breaking the material. Each medium-power shock generates specific damping vibrations inside the material medium, and the subsequent shock has to be applied right before the annihilation of these damping vibrations by the fluctuations of the external medium (the noise). Thus, a certain degree of coherence for the effects of external pulses can be achieved, for the maximum possible value of the amplitude of fundamental harmonic corresponding to envelope of generated vibrations (if the time interval between external pulses is shortened, the final value of envelope function at the end of this interval is no more equal to zero and the difference between extreme values of envelope function decreases—thus the amplitude of fundamental harmonic component decreases also). This implies the use of some non-Markov aspects, while the memory of previous similar events should be involved (similar to [1]).

Similar aspects can be noticed at microscopic (quantum) scale by considering that the wave functions with certain frequencies associated to particles can be represented as a sequence of pulses (each pulse corresponding to an oscillation); they generate transitions after a certain number of oscillations for the wave-trains of particles involved in interaction, as a nonlinear effect. Quantum transitions cannot be considered as instantaneous phenomena, because the frequency (an important physical quantity in quantum dynamics theory) requires a certain time interval for an estimation.

For such phenomena to be noticed, a low level for additional noise as related to phenomena generated by previous events is required. Usually this can be achieved: (i) by extreme-low values of temperature (macroscopic quantum phenomena at ultra-low temperature being strongly related to Bose-Einstein condensation) and (ii) for very fast high-intensity cooperative phenomena; see the NIST experiment recently performed where an ultrafast laser pulse was used to excite special crystals so as to create a form of light known as a squeezed vacuum (this was the first quantum superposition of states with opposite properties to be made by detecting three photons at once and is one of the largest and most well-defined cat states ever made from light) [2, 3].

Linear differential equations are not suitable for modelling such aspects. Better qualitative results were obtained using dynamical equations able to generate practical test functions (similar to wavelets) and delayed pulses (when a free term which corresponds to an external action is added) [4] for justifying fracture phenomena appearing in a certain material medium [5]. It has been considered that an external action (described by a short wavelength sine function multiplied by a gaussian function) acts in a certain area. Using a specific differential equation (able to generate symmetrical functions for a null free term) for describing the generation of the corresponding effect along an axis inside the material medium, it has been shown that a significant effect could appear at a certain distance. The model can be also applied by considering the time axis instead of spatial axis (thus a significant delayed effect appears). However, this model cannot explain the effect of a sequence of external pulses when the time interval between these pulses is large enough so as the final effect not be considered a superposition of individual effects of each pulse.

For this reason, some specific differential equations based on the coherence function between the generated deformation and the alternating input have been taken into consideration [6]. Since this coherence function vanishes if the output equals zero, the initial condition should be set at a small nonzero value. This model has given good qualitative results for modelling the generation of oscillations with different local maximum/minimum values (for second-order differential equations) and multiscale behaviour (for higher-order differential equations). However, this model has used an equivalence between integrodifferential equations and differential equations which requires a deeper analysis. The correspondence between simulation results and basic aspects of Floquet-Lyapunov theorem for differential equations with periodical coefficients should be also investigated.

2. Mathematical Model for Generating Amplitude-Modulated Deformations

As was shown in [6], a differential equation modelling aspects similar to quantum phenomena should be based on the use of coherence function as free term. The external command (corresponding to the sequence of external pulses) has been considered as a superposition of cosine functions and it has been analyzed just for the output generated by a certain cosine function (with the period set ot the value $T = 1$). The free term of the differential equation was represented by the coherence function

$$\mathrm{Ch}(x) = \int_{x_{\mathrm{in}}}^{x} y(t)\cos\omega t\, dt = \int_{x_{\mathrm{in}}}^{x} y(t)\cos 2\pi t\, dt, \tag{2.1}$$

where variable x corresponds to time. By choosing an undamped second-order system with time constant $T_0 = 1$ (the period being $2\pi \approx 6T$ so as very weak resonance aspects appear), the following equation has been obtained:

$$y'' = -\frac{1}{T_0^2}y + \int_0^x y(t)\cos\omega t\, dt = -y + \int_0^x y(t)\cos 2\pi t\, dt. \tag{2.2}$$

At first sight, this integrodifferential equation is equivalent to the differential equation of third order

$$y^{(3)} = -\frac{1}{T_0^2}y^{(1)} + y(x)\cos\omega x = -y^{(1)} + y(x)\cos 2\pi x \tag{2.3}$$

(obtained by differentiating the initial integrodifferential equation). By setting the initial value y_0 of the output function $y(x)$ to a small nonzero value $y_0 = 0.3$ at the initial time moment $x_0 = 0$ and for null initial conditions for $y^{(1)}, y^{(2)}$, the simulation performed in MATLAB (based on Runge-Kutta functions) has generated for the derivative of the output $y'(x)$ (denoted as $z(x)$) the function represented in Figure 1. The continuous line corresponds to function $z(x) = y'(x)$; the discontinuous line corresponds to the function $f(x) = -0.1\cos 2\pi x$ necessary for studying the correlation between $z(x)$ and $f(x)$.

This mathematical model of the third-order differential equation with initial nonzero value for $y(x)$ and initial null values for y', y'' has important oscillatory properties: the local maximum values and the local minimum values of each oscillation of period $T = 1$ are amplitude modulated by a periodical signal with a time period six times greater than the period T of the external command. For three successive oscillations, the local maximum values increase, and for next three successive oscillations the local maximum values decrease (the peak-to-peak value for each oscillation being the same). This aspect could be put in correspondence with an increasing velocity of particles vibrating in a bar under the influence of an external alternating force, generating fracture phenomena after a few oscillations (when the velocity becomes higher than a certain threshold value). It can be also put in correspondence with quantum aspects, where a transition can appear just after a certain number of oscillations for the wave trains of particles involved in interaction.

However, we must take care that the previous two equations (the integrodifferential equations and the third-order differential equation) are not equivalent. The initial value of the

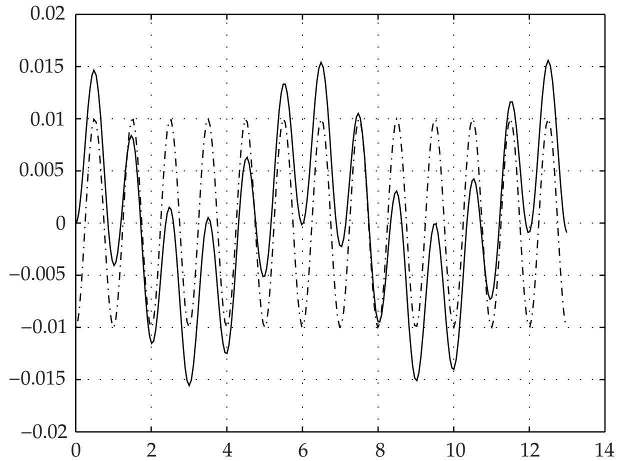

Figure 1: Output generated for second-order system by cosine function.

coherence function is zero (corresponding to an integral from zero to zero); as a consequence, the initial value of y'' should be

$$y''(0) = -y(0) + \mathrm{Ch}(0) = -y(0) \tag{2.4}$$

according to the integrodifferential equation. So, we cannot set the initial value of $y(t)$ at a nonzero value and the initial value of y'' at a zero value.

If the initial conditions are adjusted by taking into account that $y''(0) = -y(0) = -0.3$ (for $y'(0) = 0$), it results for the derivative of the output $y'(x)$ (denoted as $z(x)$) the function represented in Figure 2. The continuous line corresponds to function $z(x) = y'(x)$; the discontinuous line corresponds to the function $f(x) = -0.1\cos 2\pi x$, necessary for studying the correlation between $z(x)$ and $f(x)$. It can be noticed that oscillations with period 1 (the period of the cosine function) are not present in this case—only proper oscillations with period $2\pi T_0$ can be observed. The amplitude of $z(x) = y'(x)$ is increasing as a monotone function for the time interval $2\pi T_0/2 = \pi$ (corresponding to the half-period of proper oscillations of second-order system) and is decreasing also as a monotone function on the next half-period of proper oscillations, without any influence of external cosine function.

Similar aspects can be noticed if the external command is considered as a sequence of rectangular pulses similar to $\delta'(x-k)$. For nonzero initial condition $y = 0.3$, null initial conditions for y', y'' at the initial moment of time $x_{\mathrm{in}} = -1.2$, and for an external command function represented by a sequence of step pulses $s(x)$:

$$\begin{aligned} s &= 5 \quad \text{for } x \in [-1.2,-1] \cup [-0.2,0] \cup [0.8,1] \cup [1.8,2] \cup [2.8,3] \cdots, \\ s &= -5 \quad \text{for } x \in [-1,-0.8] \cup [0,0.2] \cup [1,1.2] \cup [2,2.2] \cup [3,3.2] \cdots, \end{aligned} \tag{2.5}$$

the simulation performed in MATLAB (based on Runge-Kutta functions) has generated for the derivative of the output $y'(x)$ (denoted as $z(x)$) the function represented in Figure 3 (see [6]). The continuous line corresponds to function $z(x) = y'(x)$; the discontinuous line corresponds to the function $f(x) = 0.02s(x)$ necessary for studying the correlation between

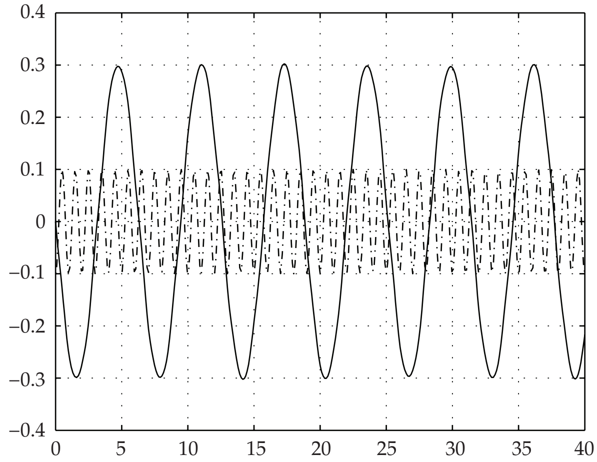

Figure 2: Output generated for second-order system by cosine-function-adjusted initial conditions.

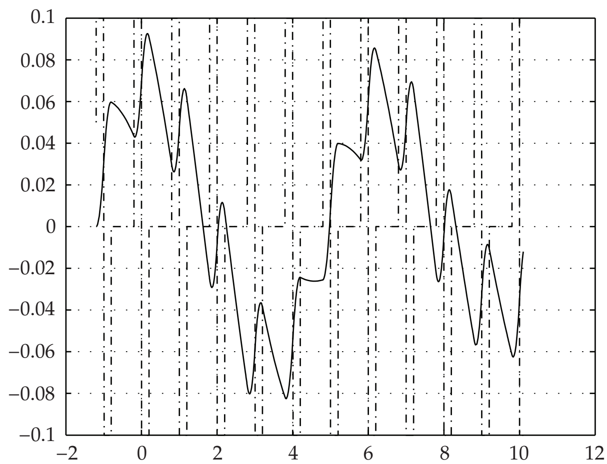

Figure 3: Output generated for second-order system by sequence of alternating pulses.

$z(x)$ and $f(x)$. It can be noticed that $z(x)$ is a saw-tooth function. The external command $s(x)$ and $z(x)$ are no more in-phase functions, but the main feature of Figure 1 is still present: an alternance of three increasing local maximum values and of three decreasing local maximum values of $z(t)$ can be noticed, the period of these alternances being six times greater than the period of the external command $s(x)$ (represented by a sequence of pulses similar to $\delta'(x-k)$, $k \in -1, 0, 1 \ldots$).

If the initial conditions of the previous third-order differential equation are adjusted as $y' = 0$, $y'' = -y = 0.3$, the simulation performed in MATLAB (based on Runge-Kutta functions) generates for the derivative of the output $y'(x)$ (denoted as $z(x)$) the function represented in Figure 4 (similar to Figure 2). Oscillations with period 1 (the period of the cosine function) have almost disappeared; the amplitude of proper oscillations varies slowly in time.

However, it is hard to consider that the initial conditions for $y(x)$ and $y''(x)$ could be correlated in such a rigorous manner (so as $y''(0) = -y(0)$). Thus the mathematical model of

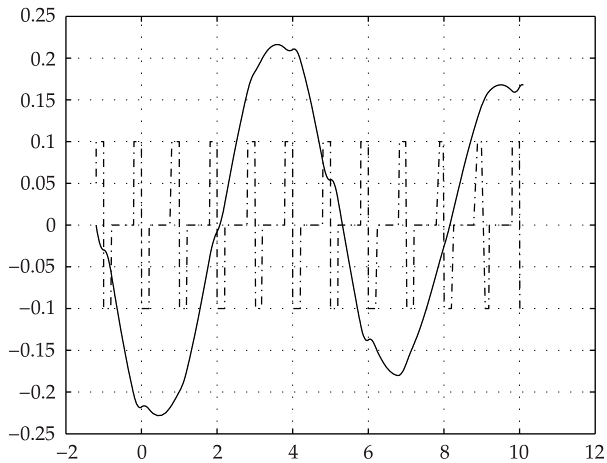

Figure 4: Output generated for second-order system by sequence of alternating pulse-adjusted initial conditions.

the third-order differential equation with initial null conditions for y', y'' and nonzero initial condition for $y(x)$ should be preserved. This problem can be solved (in a formal manner) by replacing the initial integrodifferential equation with

$$\int_0^x y''(t)dt = -\frac{1}{T_0^2}\int_0^x y(t)dt + \int_0^x \int_0^t y(\tau)\cos\omega\tau\, d\tau. \tag{2.6}$$

This integral equation is obtained by integrating the initial integrodifferential equation. It allows null initial values for y', y'', for $y(0) \neq 0$ since all integrals (in left side and right side) are equal to zero for $x = 0$ (they correspond to integrals from zero to zero), and it is more suitable for modelling nonlinear transition phenomena in a robust manner.

3. Analysis of Higher Order Differential Equations Based on Coherence Function

The analysis can be extended by investigating differential equations of higher order. In [6] it has been investigated the integrodifferential equation

$$y^{(6)} = -\frac{1}{T_0^2}y^{(4)} + \int_0^x y(t)\cos\omega t\, dt = -0.1y^{(4)} + \int_0^x y(t)\cos 2\pi t\, dt. \tag{3.1}$$

The evolution of $z(x) = y'(x)$ has been simulated in Matlab for $y(0) = 0.3$ and for null initial values for $y', y^{(2)}, y^{(3)}, y^{(4)}$, and $y^{(5)}$ (as in the case of second-order differential equation presented at the beginning of the previous paragraph). The time constant T_0 has been increased $\sqrt{10}$ times from unity value for avoiding any resonance effect. It was shown that z presents a multiscale behaviour in time. Different alternances of $z(t)$ could be noticed, depending on the time interval selected for analysis. The ratio between peak values corresponding to consecutive alternances was very high (about $|Mv_{k+1}/Mv_k| \approx 25\cdots 30$), and the time interval between moments corresponding to peak values for consecutive alternances was about $t_{k+1} - t_k \approx 45 \cdots 50$ (approximately two times greater than the period $2\pi T_0 = 2\sqrt{10}\pi$ of the proper oscillations). A certain delay time (dead time) could be also noticed for each alternance; the difference between delay times for two consecutive alternances was ($td_{k+1} - td_k \approx 40 \cdots 50$). Similar multiscale aspects implying alternances with high ratio between peak values corresponding to consecutive alternances and a delay (dead) time increasing as an arithmetic progression can be also noticed for the 5th order integrodifferential equation

$$y^{(5)} = -\frac{1}{T_0^2} y^{(3)} + \int_0^x y(t) \cos \omega t \, dt = -0.1 y^{(3)} + \int_0^x y(t) \cos 2\pi t \, dt. \tag{3.2}$$

Both previous integrodifferential equations were modelled in Matlab using differential equations obtained by a differentiation of corresponding integrodifferential equations, as

$$y^{(7)} = -\frac{1}{T_0^2} y^{(5)} + y(x) \cos \omega x = -0.1 y^{(5)} + y(x) \cos 2\pi x, \tag{3.3}$$

$$y^{(6)} = -\frac{1}{T_0^2} y^{(4)} + y(x) \cos \omega x = -0.1 y^{(4)} + y(x) \cos 2\pi x, \tag{3.4}$$

respectively. All initial conditions were set to zero, excepting $y(0)$ which was set to $y(0) = 0.3$. These two differential equations used for modelling are equivalent with previous integrodifferential equations, because they allow these sets of initial conditions (an initial nonzero value for $y(x)$ does not imply any nonzero value for any of its derivatives, as in the case of second-order integrodifferential equation presented in previous paragraph). If the variable $y(x)$ is replaced by $Y = y^{(4)}$ (for first equation) or $Y = y^{(3)}$ (for second equation), both can be written under the form of an undamped second order integrodifferential equation by considering the coherence function as a multiple integral:

$$Y^{(2)} = -0.1Y + \int_0^x \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \int_0^{t_4} Y(t) \cos \omega t \, dt \, dt_4 \, dt_3 \, dt_2 \, dt_1 \, dx \tag{3.5}$$

(for first equation), and

$$Y^{(2)} = -0.1Y + \int_0^x \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} Y(t) \cos \omega t \, dt \, dt_3 \, dt_2 \, dt_1 \, dx \tag{3.6}$$

(for second equation).

The analysis of similar integrodifferential equations of order higher than 6 does not reveal multiscale aspects any more. For $n = 7$ and $n = 8$, the simulation of

$$y^{(n)} = -\frac{1}{T_0^2} y^{(n-2)} + \int_0^x y(t) \cos \omega t = -0.1 y^{(n-2)} + \int_0^x y(t) \cos 2\pi t \tag{3.7}$$

on extended time intervals shows just some monotone increasing convex functions. The same aspect can be noticed for such an integrodifferential equation of third order (n equals 3 in previous equation).

An analysis on small time intervals of this kind of equations reveals the existence of an alternating component of higher frequency (with period $T = 1$, the period of the cosine function). This is in accordance with basic aspects of Floquet-Lyapunov theorem for differential equations with periodical coefficients. According to it [7, 8], for the fundamental matrix solution $X(t)$ of the periodic system

$$\frac{dx}{dt} = A(t)x \tag{3.8}$$

with $A(t)$ being a periodic function with period T, there are (i) a periodic matrix function $P(t)$ (with period T) and a matrix B (possibly complex) such that

$$X(t) = P(t) \exp Bt \tag{3.9}$$

and (ii) a real periodic matrix function $Q(t)$ (with period $2T$) and a real matrix R such that

$$X(t) = Q(t) \exp Rt. \tag{3.10}$$

So it is quite normal for the output of the differential system with periodical coefficients (the period being T) to be represented by a sum of oscillating functions of higher frequency (with period T) multiplied by exponential functions with exponent determined mainly by the parameters of the undamped second-order differential system. It was shown that for integrodifferential equation of 6th order, the time interval between two successive peak values for the analysis on an extended time interval was proportional to $2\pi T_0$ (the period of proper oscillations of the undamped second-order system).

As a consequence, a higher-order differential equation with a free term corresponding to the coherence function between an external cosine command and the output $y(x)$ generates for $z(x) = y'(x)$ alternances with the same temporal pattern and an increasing amplitude according to a geometrical progression just for fifth-order and sixth-order undamped differential equation driven by the coherence function. These equations are suitable for modeling multiscale phenomena and for explaining multiscale threshold transitions—see also [9, 10].

4. Differential Equations Generating Transitions with Intermediate Levels

A special analysis requires the case of fourth-order differential equation. This means that

$$y^{(4)} = -\frac{1}{T_0^2} y^{(2)} + \int_0^x y(t) \cos \omega t = -0.1 y^{(2)} + \int_0^x y(t) \cos 2\pi t. \tag{4.1}$$

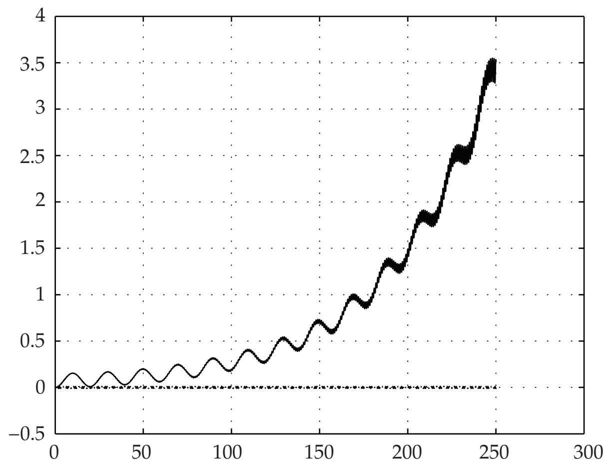

Figure 5: Intermediate levels generated for fourth-order system.

This equation has been simulated in Matlab using the equivalent differential equation

$$y^{(5)} = -0.1y^{(3)} + y(x)\cos 2\pi x. \tag{4.2}$$

The initial condition for y was set to a small nonzero value ($y(0) = 0.3$), and the initial values for its derivatives were set to zero (these initial conditions being allowed by the integrodifferential equations, as can be easily checked).

It can be noticed that for small time values some oscillations with time period ≈ 20 overlapped on an increasing convex monotone function can be noticed and (ii) for time values higher than 180, these oscillations cannot be noticed any more and the behaviour is similar to the evolution of a system reaching some intermediate levels from time to time.

For this reason this fourth-order undamped differential equation driven by the coherence function is highly recommended for modelling transitions where intermediate levels are involved (e.g., as in complex transitions in atoms). Unlike integrodifferential equation of fifth- and sixth-order (suitable for modelling transitions appearing after a certain interaction time under the form of sharp pulses, with amplitude and polarity depending on time scale) the fouth-order undamped differential equation driven by the coherence function is suitable for modelling complex transitions where a discrete sequence of quantum states can be noticed (as in multiphoton phenomena or in higher-order perturbation theory analysis, where Feynman diagram requires the use of intermediate virtual particles).

5. Conclusions

This study has presented specific aspects of dynamics generated by the coherence function (acting in an integral manner). It has been considered that an oscillating system starting to work from initial nonzero conditions is commanded by the coherence function between the output of the system and an alternating function of a certain frequency. For different initial conditions, the evolution of the system was analyzed. The equivalence between integrodifferential equations and integral equations implying the same number of state variables has been investigated; it was shown that integrodifferential equations of second order are far more restrictive regarding the initial conditions for the state variables. Then, the analysis has been extended to equations of evolution where the coherence function is acting under the form of a multiple integral. It was shown that for the coherence function represented under the form of an nth integral some specific aspects as multiscale behaviour suitable for modelling transitions in complex systems (quantum physics, for example) can be noticed when n equals $4, 5$, or 6.

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