ON THE STRETCHING OF MAXWELL'S EQUATIONS IN GENERAL ORTHOGONAL COORDINATE SYSTEMS AND THE PERFECTLY MATCHED LAYER

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Abstract It is shown that the complex coordinate stretching and diagonal anisotropy formulations of the perfectly matched layer are equivalent in a general orthogonal coordinate system setting. The results are obtained by taking advantage of the tensorial invariance of the line element.
Keywords Absorbing boundary condition, perfectly matched layer, electromagnetic scattering.

1 INTRODUCTION

In [1] Berenger proposed the perfectly matched layer (PML) to truncate computational domains for use in the numerical solution of Maxwell's equations, without introducing reflections. The original split-field approach of Berenger was reformulated by Chew and Weedon [2] in terms of complex coordinate stretching and by Sacks et al. [3] in terms of perfectly matched anisotropic absorbers. Recently, efforts have been directed, by using a blend of split-field, complex stretching and anisotropic methods, to extend the rectangular PML to cylindrical and spherical coordinate systems [4]-[6] or to more general coordinate systems [7].

In this paper we show that the complex coordinate stretching and diagonal anisotropy formulations are equivalent in a general orthogonal coordinate system setting. The derivations are based on the tensorial invariance of the line element [8], and the different representations of the curl operator in orthogonal coordinate systems. Examples include the classic PML's (cartesian, cylindrical, spherical) and the PML in elliptic cylinder coordinates.

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2 MAIN RESULT

In a general coordinate system, the line element ds and the fundamental tensor g_{ij} [8] are related by

$$ds^2 = \sum_{i,j} g_{ij} dq_i dq_j \tag{1}$$

In a three-dimensional orthogonal system, where the fundamental tensor is diagonal, this can be written as

$$ds^2 = h_1(\mathbf{q})^2 dq_1^2 + h_2(\mathbf{q})^2 dq_2^2 + h_3(\mathbf{q})^2 dq_3^2 \tag{2}$$

It is known that the curl in an orthogonal system is given by

$$\mathrm{curl}_3 \mathbf{V} = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial q_1}(h_2 V_2) - \frac{\partial}{\partial q_2}(h_1 V_1)\right] \tag{3}$$

with similar expressions for the other components, obtained by cyclic permutation. Hence, denoting a diagonal square matrix as

$$\{a, b, c\} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \tag{4}$$

the curl of a vector in any orthogonal coordinate system can be written as

$$\mathrm{curl}\, \mathbf{V} = \left\{\frac{1}{h_2 h_3}, \frac{1}{h_1 h_3}, \frac{1}{h_1 h_2}\right\} \cdot \mathrm{carl}_q(\{h_1, h_2, h_3\} \cdot \mathbf{V}) \tag{5}$$

where carl_q is the cartesian curl operator

$$\mathrm{carl}_q = \begin{pmatrix} 0 & -\partial/\partial q_3 & \partial/\partial q_2 \\ \partial/\partial q_3 & 0 & -\partial/\partial q_1 \\ -\partial/\partial q_2 & \partial/\partial q_1 & 0 \end{pmatrix} \tag{6}$$

If we stretch the coordinates by means of continuously differentiable functions

$$Q_i = Q_i(q_i) \qquad i = 1, 2, 3 \tag{7}$$

orthogonality is maintained, since the line element — which is the fundamental invariant — can be written as

$$ds^2 = h_1(\mathbf{q})^2 \frac{dq_1^2}{dQ_1^2} dQ_1^2 + h_2(\mathbf{q})^2 \frac{dq_2^2}{dQ_2^2} dQ_2^2 + h_3(\mathbf{q})^2 \frac{dq_3^2}{dQ_3^2} dQ_3^2 \tag{8}$$

Putting

$$h'_i = h_i(\mathbf{q})\,\frac{dq_i}{dQ_i} \qquad i = 1, 2, 3 \tag{9}$$

it is clear that the curl in the stretched coordinate system can be written as

$$\mathrm{curl}\,\mathbf{V} = \left\{\frac{1}{h'_2 h'_3}, \frac{1}{h'_1 h'_3}, \frac{1}{h'_1 h'_2}\right\} \cdot \mathrm{carl}_Q(\{h'_1, h'_2, h'_3\} \cdot \mathbf{V}) \tag{10}$$

where of course carl_Q is

$$\mathrm{carl}_Q = \begin{pmatrix} 0 & -\partial/\partial Q_3 & \partial/\partial Q_2 \\ \partial/\partial Q_3 & 0 & -\partial/\partial Q_1 \\ -\partial/\partial Q_2 & \partial/\partial Q_1 & 0 \end{pmatrix} \tag{11}$$

Next suppose that we do not have any notion that coordinate stretching has occurred. If overnight the $q-$system had changed into the $Q-$system, then the (pseudo) squared line element would have been assumed to be

$$ds_Q^2 = h_1(\mathbf{Q})^2 dQ_1^2 + h_2(\mathbf{Q})^2 dQ_2^2 + h_3(\mathbf{Q})^2 dQ_3^2 \tag{12}$$

with of course $ds_Q^2 \neq ds^2$ in general. However, the pseudo curl with respect to the orthogonal coordinate system defined by the line element (12) can still be defined as

$$\mathrm{CURL}\,\mathbf{V} = \left\{\frac{1}{\tilde{h}_2 \tilde{h}_3}, \frac{1}{\tilde{h}_1 \tilde{h}_3}, \frac{1}{\tilde{h}_1 \tilde{h}_2}\right\} \cdot \mathrm{carl}_Q(\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\} \cdot \mathbf{V}) \tag{13}$$

where $\tilde{h}_i = h_i(\mathbf{Q})$. It is straightforward to show that the curl and the pseudo curl are related by

$$\mathrm{curl}\,\mathbf{V} = \left\{\frac{1}{\alpha_2\alpha_3}, \frac{1}{\alpha_1\alpha_3}, \frac{1}{\alpha_1\alpha_2}\right\} \cdot \mathrm{CURL}\,(\{\alpha_1, \alpha_2, \alpha_3\} \cdot \mathbf{V}) \tag{14}$$

where

$$\alpha_i = h'_i/\tilde{h}_i = \frac{h_i(\mathbf{q})dq_i}{h_i(\mathbf{Q})dQ_i} \qquad i = 1, 2, 3 \tag{15}$$

Note that when no stretching occurs, curl and CURL are identical, as it should be.

To exploit this, we write down Maxwell's equations with $e^{i\omega t}$ time dependence in a non-linear diagonally anisotropic medium as

$$\mathrm{curl}\,\mathbf{E} = -i\omega\{\mu_1, \mu_2, \mu_3\} \cdot \mathbf{H} \tag{16}$$

$$\mathrm{curl}\,\mathbf{H} = i\omega\{\epsilon_1, \epsilon_2, \epsilon_3\} \cdot \mathbf{E} \tag{17}$$

where the μ_i, ϵ_i are complex-valued functions of the coordinates $\mathbf{q}$. Introducing the new field quantities

$$\mathbf{K} = \{\alpha_1, \alpha_2, \alpha_3\} \cdot \mathbf{E} \quad (18)$$

$$\mathbf{L} = \{\alpha_1, \alpha_2, \alpha_3\} \cdot \mathbf{H}, \quad (19)$$

Maxwell's equations can be transformed into the stretched coordinate system with the help of formula (14) to yield

$$\mathrm{CURL}\,\mathbf{K} = -i\omega\{\mu_1\alpha_2\alpha_3/\alpha_1, \mu_2\alpha_1\alpha_3/\alpha_2, \mu_3\alpha_1\alpha_2/\alpha_3\} \cdot \mathbf{L} \quad (20)$$

$$\mathrm{CURL}\,\mathbf{L} = i\omega\{\epsilon_1\alpha_2\alpha_3/\alpha_1, \epsilon_2\alpha_1\alpha_3/\alpha_2, \epsilon_3\alpha_1\alpha_2/\alpha_3\} \cdot \mathbf{K} \quad (21)$$

Now if we take

$$\mu_i = \frac{\mu_0\alpha_i^2}{\alpha_1\alpha_2\alpha_3}, \quad \epsilon_i = \frac{\epsilon_0\alpha_i^2}{\alpha_1\alpha_2\alpha_3} \qquad i = 1, 2, 3 \quad (22)$$

then equations (20)-(21) simply turn out to be

$$\mathrm{CURL}\,\mathbf{K} = -i\omega\mu_0\mathbf{L} \quad (23)$$

$$\mathrm{CURL}\,\mathbf{L} = i\omega\epsilon_0\mathbf{K} \quad (24)$$

Hence the original Maxwell equations in the non-linear diagonally anisotropic medium have been transformed to a linear isotropic problem. Of course, the non-linear character of the medium has been absorbed in the $q \to Q$ stretching transformation. Note that equations (23)-(24) admit field decompositions in stretched plane waves

$$e^{i\omega t - ik_1Q_1(q_1) - ik_2Q_2(q_2) - ik_3Q_3(q_3)} \quad (25)$$

where

$$k_1^2 + k_2^2 + k_3^2 = k_0^2 = \omega^2\mu_0\epsilon_0 \quad (26)$$

As a final remark, it should be stressed that the stretched variables Q_j may be taken to be complex functions of the q_j.

3 PERFECTLY MATCHED LAYERS

3.1 CARTESIAN COORDINATES WITH STRETCHING IN z

In cartesian coordinates we have

$$(q_1, q_2, q_3) = (x, y, z) \qquad (h_1, h_2, h_3) = (1, 1, 1) \quad (27)$$

In the stretched coordinate system we have

$$(Q_1, Q_2, Q_3) = (X, Y, Z) = (x, y, Z(z)) \tag{28}$$

From equation (15) we obtain

$$(\alpha_1, \alpha_2, \alpha_3) = (1, 1, dz/dZ) \tag{29}$$

Formula (22) yields

$$\{\mu_1, \mu_2, \mu_3\} = \mu_0\{dZ/dz, dZ/dz, dz/dZ\} \tag{30}$$

and similarly for the dielectric constants. If we take

$$Z(z) = z - i\sigma(z) \tag{31}$$

where

$$\begin{cases} \sigma(z) = 0 & z \leq 0 \\ \sigma(z) > 0 & z > 0 \end{cases} \tag{32}$$

with $\sigma(0) = \sigma'(0) = 0$, we obtain the classical cartesian PML [3], [5] at the interface $z = 0$. From equation (25) the stretched plane wave can be written as

$$e^{i\omega t - ik_1x - ik_2y - ik_3z} e^{-k_3\sigma(z)} \tag{33}$$

exhibiting consistent damping in the half-space $z > 0$, provided $k_3 > 0$.

3.2 CYLINDRICAL COORDINATES WITH STRETCHING IN r

In cylindrical coordinates we have

$$(q_1, q_2, q_3) = (r, \theta, z) \qquad (h_1, h_2, h_3) = (1, r, 1) \tag{34}$$

In the stretched coordinate system we have

$$(Q_1, Q_2, Q_3) = (R, \Theta, Z) = (R(r), \theta, z) \tag{35}$$

From equation (15) we obtain

$$(\alpha_1, \alpha_2, \alpha_3) = (dr/dR, r/R, 1) \tag{36}$$

Formula (22) yields

$$\{\mu_1, \mu_2, \mu_3\} = \mu_0\{Rdr/rdR, rdR/Rdr, RdR/rdr\} \tag{37}$$

and similarly for the dielectric constants. If we take

$$R(r) = r - i\sigma(r) \tag{38}$$

where

$$\begin{cases} \sigma(r) = 0 & r \leq a \\ \sigma(r) > 0 & r > a \end{cases} \tag{39}$$

with $\sigma(a) = \sigma'(a) = 0$, we obtain a cylindrical PML at the interface $r = a$. Applied to the outgoing cylindrical wave

$$\frac{i}{4} e^{i\omega t} H_0^{(2)}(k_0 r) \tag{40}$$

we obtain the stretched cylindrical wave

$$\frac{i}{4} e^{i\omega t} H_0^{(2)}(k_0 r - ik_0 \sigma(r)) \tag{41}$$

exhibiting consistent damping in the region $r > a$.

3.3 SPHERICAL COORDINATES WITH STRETCHING IN r

In spherical coordinates we have

$$(q_1, q_2, q_3) = (r, \theta, \phi) \qquad (h_1, h_2, h_3) = (1, r, r \sin\theta) \tag{42}$$

In the stretched coordinate system we have

$$(Q_1, Q_2, Q_3) = (R, \Theta, \Phi) = (R(r), \theta, \phi) \tag{43}$$

From equation (15) we obtain

$$(\alpha_1, \alpha_2, \alpha_3) = (dr/dR, r/R, r/R) \tag{44}$$

Formula (22) yields

$$\{\mu_1, \mu_2, \mu_3\} = \mu_0 \{R^2 dr/r^2 dR, dR/dr, dR/dr\} \tag{45}$$

and similarly for the dielectric constants. If we take

$$R(r) = r - i\sigma(r) \tag{46}$$

where

$$\begin{cases} \sigma(r) = 0 & r \leq a \\ \sigma(r) > 0 & r > a \end{cases} \tag{47}$$

with $\sigma(a) = \sigma'(a) = 0$, we obtain a spherical PML at the interface $r = a$. Applied to the outgoing spherical wave

$$\frac{1}{4\pi r} e^{i\omega t - ik_0 r} \tag{48}$$

we obtain the stretched spherical wave

$$\frac{1}{4\pi R} e^{i\omega t - ik_0 r - k_0 \sigma(r)} \tag{49}$$

exhibiting consistent damping in the region $r > a$.

3.4 ELLIPTIC COORDINATES WITH STRETCHING IN u

Elliptic cylinder coordinates (u, v, z) are related to the cartesian coordinates (x, y, z) by the transformatian formulas

$$x = c \cosh u \cos v, \qquad y = c \sinh u \sin v \tag{50}$$

where $u \geq 0$, $0 \leq v \leq 2\pi$ and c is a positive constant. The domain $u \leq u_0$ represents the interior of the elliptic cylinder

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \tag{51}$$

where the major and minor semi-axes a, b of the ellipse and the constants c, u_0 are related by

$$a = c \cosh u_0, \qquad b = c \sinh u_0 \tag{52}$$

In elliptic cylinder coordinates we have

$$(q_1, q_2, q_3) = (u, v, z) \qquad (h_1, h_2, h_3) = (c\Delta(u, v), c\Delta(u, v), 1) \tag{53}$$

where

$$\Delta(u, v) = \sqrt{\cosh^2 u - \cos^2 v} \tag{54}$$

In the stretched coordinate system we have

$$(Q_1, Q_2, Q_3) = (U, V, Z) = (U(u), v, z) \tag{55}$$

From equation (15) we obtain

$$(\alpha_1, \alpha_2, \alpha_3) = (\Delta(u, v)du/\Delta(U, v)dU, \Delta(u, v)/\Delta(U, v), 1) \tag{56}$$

Formula (22) yields

$$\{\mu_1, \mu_2, \mu_3\} = \mu_0\{du/dU, dU/du, \Delta^2(U, v)dU/\Delta^2(u, v)du\} \tag{57}$$

and similarly for the dielectric constants. If we take

$$\cosh U(u) = \cosh u - i\sigma(u) \tag{58}$$

where

$$\begin{cases} \sigma(u) = 0 & u \le u_0 \\ \sigma(u) > 0 & u > u_0 \end{cases} \tag{59}$$

with $\sigma(u_0) = \sigma'(u_0) = 0$, we obtain an elliptic cylinder PML at the interface $u = u_0$. Applied to the outgoing elliptic cylinder wave with radial Mathieu function [9]

$$R^{(2)}_{em\lambda}(u) = \sqrt{\pi/2}\, \sum_n{}' i^{m-n} D^m_n H^{(2)}_n(c\lambda \cosh u) \tag{60}$$

we obtain a stretched elliptic cylinder wave exhibiting consistent damping in the region $u > u_0$.

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