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Some fixed point results for dualistic rational contractions

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Abstract

In this paper, we introduce a new contraction called dualistic contraction of rational type and used it to obtain some fixed point results in ordered dualistic partial metric spaces. These results generalize various comparable results appeared in the literature. We provide an example to show the usefulness of our results among corresponding fixed point results proved in metric spaces.

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1. Introduction

Matthews [3] introduced the concept of partial metric space as a suitable mathematical tool for program verification and proved an analogue of Banach fixed point theorem in complete partial metric spaces. O'Neill [7] introduced the notion of dualistic partial metric, which is more general than partial metric and established a robust relationship between dualistic partial metric and quasi metric. In [9], Oltra and Valero presented a Banach fixed point theorem on complete dualistic partial metric spaces and in this way presented a generalization of famous Banach fixed point theorem. They also showed that the

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contractive condition in Banach fixed point theorem in complete dualistic partial metric spaces cannot be replaced by the contractive condition of Banach fixed point theorem for complete partial metric spaces. Later on, Nazam et al. [4] established a fixed point theorem for Geraghty type contractions in ordered dualistic partial metric spaces and applied this result to show the existence of solution of integral equations .

Harjani et al. [1] extended Banach fixed point principle as follows:

Theorem 1.1 ([1])**.** *Let M be complete ordered metric space and $T : M \to M$ a continuous and non decreasing mapping satisfying,*

$$d(T(j), T(k)) \leq \frac{\alpha d(j, T(j)) \cdot d(k, T(k))}{d(j, k)} + \beta d(j, k),$$

for all comparable $j, k \in M$ with $j \neq k$ and $0 < \alpha + \beta < 1$. Then T has a unique fixed point $m^* \in M$. Moreover, the Picard iterative sequence $\{T^n(j)\}_{n\in\mathbb{N}}$ converges to m^* for every $j \in M$.

Isik and Tukroglu [2] presented an ordered partial metric space version of Theorem 1.1, stated below:

Theorem 1.2 ([2])**.** *Let M be complete ordered partial metric space and $T : M \to M$ a continuous and non decreasing mapping satisfying,*

$$d(T(j), T(k)) \leq \frac{\alpha d(j, T(j)) \cdot d(k, T(k))}{d(j, k)} + \beta d(j, k),$$

for all comparable $j, k \in M$ with $j \neq k$ and $0 < \alpha + \beta < 1$. Then T has a unique fixed point $m^* \in M$. Moreover, the Picard iterative sequence $\{T^n(j)\}_{n\in\mathbb{N}}$ converges to m^* for every $j \in M$.

In this paper, we obtain some fixed point theorems for dualistic contractions of rational type. These results extend the comparable results in [2]. We give examples to show that existing results in partial metric space cannot be applied to obtain fixed points of mappings involved herein.

2. Preliminaries

Following mathematical basics will be needed in the sequel. Throughout this paper, we denote $(0, \infty)$ by $\mathbb{R}^+$, $[0, \infty)$ by $\mathbb{R}_0^+$, $(-\infty, +\infty)$ by $\mathbb{R}$ and set of natural numbers by $\mathbb{N}$.

Let T be a self mapping on a nonempty set M. An element $m^* \in M$ is called a fixed point of T if it remains invariant under the action of T. If j_0 is a given point in M, then a sequence $\{j_n\}$ in M by $j_n = T(j_{n-1}) = T^n(j_0)$, $n \in \mathbb{N}$ is called sequence is called Picard iterative sequence with initial guess j_0.

O'Neill [7] introduced the notion of a dualistic partial metric space as a generalization of partial metric space in order to expand the connections between partial metrics and semantics via valuation spaces. According to O'Neill, a dualistic partial metric can be defined as follow.

Definition 2.1 ([7]). Let M be a nonempty set. A function $D : M \times M \to \mathbb{R}$ is called a dualistic partial metric if for any $j, k, l \in M$, the following conditions hold:

(D_1) $j = k \Leftrightarrow D(j,j) = D(k,k) = D(j,k)$.
(D_2) $D(j,j) \leq D(j,k)$.
(D_3) $D(j,k) = D(k,j)$.
(D_4) $D(j,l) \leq D(j,k) + D(k,l) - D(k,k)$.

We observe that, as in the metric case, if D is a dualistic partial metric then $D(j,k) = 0$ implies $j = k$. In case $D(j,k) \in \mathbb{R}_0^+$ for all $j,k \in M$, then D is a partial metric on M. If (M,D) is a dualistic partial metric space, then the function $d_D : M \times M \to \mathbb{R}_0^+$ defined by

$$d_D(j,k) = D(j,k) - D(j,j),$$

is a quasi metric on M such that $\tau(D) = \tau(d_D)$. In this case, $d_D^s(j,k) = \max\{d_D(j,k), d_D(k,j)\}$ defines a metric on M.

Remark 2.2. *It is obvious that every partial metric is dualistic partial metric but converse is not true. To support this comment, define* $D_\vee : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *by*

$$D_\vee(j,k) = j \vee k = \sup\{j,k\} \ \ \forall\ j,k \in \mathbb{R}.$$

It is easy to check that $D_\vee$ *is a dualistic partial metric. Note that* $D_\vee$ *is not a partial metric, because* $D_\vee(-1,-2) = -1 \notin \mathbb{R}_0^+$. *Nevertheless, the restriction of* $D_\vee$ *to* $\mathbb{R}_0^+$, $D_\vee|_{\mathbb{R}_0^+}$, *is a partial metric.*

Example 2.3. *If* (M,d) *is a metric space and* $c \in \mathbb{R}$ *is an arbitrary constant, then* $D : M \times M \to \mathbb{R}$ *given by*

$$D(j,k) = d(j,k) + c.$$

defines a dualistic partial metric on M.

Following [7], each dualistic partial metric D on M generates a T_0 topology $\tau(D)$ on M. The elements of the topology $\tau(D)$ are open balls of the form $\{B_D(j,\epsilon) : j \in M, \epsilon > 0\}$ where $B_D(j,\epsilon) = \{k \in M : D(j,k) < \epsilon + D(j,j)\}$. A sequence $\{j_n\}_{n\in\mathbb{N}}$ in (M,D) converges to a point $j \in M$ if and only if $D(j,j) = \lim_{n\to\infty} D(j,j_n)$.

Definition 2.4 ([7]). Let (M,D) be a dualistic partial metric space, then

(1) A sequence $\{j_n\}_{n\in\mathbb{N}}$ in (M,D) is called a Cauchy sequence if $\lim_{n,m\to\infty} D(j_n,j_m)$ exists and is finite.
(2) A dualistic partial metric space (M,D) is said to be complete if every Cauchy sequence $\{j_n\}_{n\in\mathbb{N}}$ in M converges, with respect to $\tau(D)$, to a point $j \in M$ such that $D(j,j) = \lim_{n,m\to\infty} D(j_n,j_m)$.

Following lemma will be helpful in the sequel.

Lemma 2.5 ([7, 9]).

(1) A dualistic partial metric (M, D) is complete if and only if the metric space (M, d_D^s) is complete.
(2) A sequence $\{j_n\}_{n\in\mathbb{N}}$ in M converges to a point $j \in M$, with respect to $\tau(d_D^s)$ if and only if $\lim_{n\to\infty} D(j, j_n) = D(j, j) = \lim_{n\to\infty} D(j_n, j_m)$.
(3) If $\lim_{n\to\infty} j_n = \upsilon$ such that $D(\upsilon, \upsilon) = 0$ then $\lim_{n\to\infty} D(j_n, k) = D(\upsilon, k)$ for every $k \in M$.

Oltra and Valero ([6]) extended partial metric space version of the Banach contraction principle to dualistic partial metric spaces.

Theorem 2.6 ([6]). *Let (M, D) be a complete dualistic partial metric space and $T : M \to M$. If there exists $\alpha \in [0, 1[$ such that*

$$|D(T(j), T(k))| \leq \alpha |D(j, k)|,$$

for any $j, k \in M$. Then T has a unique fixed point $m^* \in M$. Moreover, $D(m^*, m^*) = 0$ and for every $j \in M$, the Picard iterative sequence $\{T^n(j)\}_{n\in\mathbb{N}}$ converges with respect to $\tau(d_D^s)$ to m^*.

3. The results

In this section, we shall show that, the dualistic contractions of rational type along with certain conditions have unique fixed point. We will support obtained results by some concrete examples. We introduce the following,

Definition 3.1. Let $(M, \preceq, D)$ be an ordered dualistic partial metric space. A self-mapping T defined on M is called dualistic contraction of rational type if for any $j, k \in M$, we have

$$|D(T(j), T(k))| \leq \alpha \left| \frac{D(j, T(j)) \cdot D(k, T(k))}{D(j, k)} \right| + \beta |D(j, k)|, \tag{3.1}$$

for all comparable $j, k \in M$ and $0 < \alpha + \beta < 1$.

We start with the following result.

Theorem 3.2. *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space and $T : M \to M$ be a continuous and non decreasing dualistic contraction of rational type. Then T has a fixed point m^* in M provided there exists $j_0 \in M$ such that $j_0 \preceq T(j_0)$. Moreover, $D(m^*, m^*) = 0$.*

Proof. Let j_0 be a given point $\in M$ and $j_n = T(j_{n-1})$, $n \geq 1$ an iterative sequence starting with j_0. If there exists a positive integer r such that $j_{r+1} = j_r$, then j_r is a fixed point of T and $D(j_r, j_r) = 0$. Suppose that $j_n \neq j_{n+1}$ for any $n \in \mathbb{N}$. As $j_0 \preceq T(j_0) = j_1$, that is, $j_0 \preceq j_1$ which further implies that $j_1 = T(j_0) \preceq T(j_1) =$ j_2. Continuing this way, we obtain that

$$j_0 \preceq j_1 \preceq j_2 \preceq j_3 \preceq \cdots \preceq j_n \preceq j_{n+1} \cdots.$$

since $j_n \preceq j_{n+1}$ by (3.1), we have

$$\begin{array}{rcl} |D(j_n, j_{n+1})| & = & |D(T(j_{n-1}), T(j_n))| \\ & \leq & \alpha \left| \dfrac{D(j_{n-1}, j_n) \cdot D(j_n, j_{n+1})}{D(j_{n-1}, j_n)} \right| + \beta |D(j_{n-1}, j_n)|, \\ & \leq & \alpha |D(j_n, j_{n+1})| + \beta |D(j_{n-1}, j_n)|, \\ |D(j_n, j_{n+1})| - \alpha |D(j_n, j_{n+1})| & \leq & \beta |D(j_{n-1}, j_n)|, \\ (1-\alpha)|D(j_n, j_{n+1})| & \leq & \beta |D(j_{n-1}, j_n)|, \\ |D(j_n, j_{n+1})| & \leq & (\dfrac{\beta}{1-\alpha})|D(j_{n-1}, j_n)|. \end{array}$$

If $\gamma = \dfrac{\beta}{1-\alpha}$, then $0 < \gamma < 1$ and we have

$$|D(j_n, j_{n+1})| \leq \gamma |D(j_{n-1}, j_n)|. \tag{3.2}$$

Thus

$$|D(j_n, j_{n+1})| \leq \gamma |D(j_{n-1}, j_n)| \leq \gamma^2 |D(j_{n-2}, j_{n-1})| \leq \cdots \leq \gamma^n |D(j_0, j_1)|. \tag{3.3}$$

As $j_n \preceq j_n$, for each $n \in \mathbb{N}$, by (3.1) we have

$$\begin{array}{rcl} |D(j_n, j_n)| = |D(T(j_{n-1}), T(j_{n-1}))| & \leq & \dfrac{\alpha |D(j_{n-1}, j_n)|^2}{|D(j_{n-1}, j_{n-1})|} + \beta |D(j_{n-1}, j_{n-1})| \\ & \leq & |D(j_{n-1}, j_{n-1})| \left\{ \alpha \left| \dfrac{D(j_{n-1}, j_n)}{D(j_{n-1}, j_{n-1})} \right|^2 + \beta \right\} \\ & \leq & (\alpha + \beta)|D(j_{n-1}, j_{n-1})|. \text{ Indeed} \left| \dfrac{D(j_{n-1}, j_n)}{D(j_{n-1}, j_{n-1})} \right|^2 = 1. \end{array}$$

Thus we obtain that

$$|D(j_n, j_n)| \leq (\alpha + \beta)^n |D(j_0, j_0)|. \tag{3.4}$$

Now we show that $\{j_n\}$ is a Cauchy sequence in (M, d_D^s). Note that, $d_D(j_n, j_{n+1}) = D(j_n, j_{n+1}) - D(j_n, j_n)$, that is, $d_D(j_n, j_{n+1}) + D(j_n, j_n) = D(j_n, j_{n+1}) \leq |D(j_n, j_{n+1})|$. Thus, we have

$$\begin{array}{rcl} d_D(j_n, j_{n+1}) + D(j_n, j_n) & \leq & \gamma^n |D(j_0, j_1)|. \\ d_D(j_n, j_{n+1}) & \leq & \gamma^n |D(j_0, j_1)| + |D(j_n, j_n)| \\ & \leq & \gamma^n |D(j_0, j_1)| + (\alpha + \beta)^n |D(j_0, j_0)| \end{array}$$

Continuing this way, we obtain that

$$d_D(j_{n+k-1}, j_{n+k}) \leq \gamma^{n+k-1} |D(j_0, j_1)| + (\alpha + \beta)^{n+k-1} |D(j_0, j_0)|$$

Now

$$\begin{aligned} d_D(j_n, j_{n+k}) &\leq d_D(j_n, j_{n+1}) + d_D(j_{n+1}, j_{n+2}) + \cdots + d_D(j_{n+k-1}, j_{n+k}). \\ &\leq \{\gamma^n + \gamma^{n+1} + \cdots + \gamma^{n+k-1}\}|D(j_0, j_1)| \\ &+ \{(\alpha+\beta)^n + (\alpha+\beta)^{n+1} + \cdots + (\alpha+\beta)^{n+k-1}\}|D(j_0, j_0)|. \end{aligned}$$

Thus for $n + k = m > n$

$$d_D(j_n, j_m) \leq \frac{\gamma^n}{1-\gamma}|D(j_0, j_1)| + \frac{(\alpha+\beta)^n}{1-(\alpha+\beta)}|D(j_0, j_0)|. \tag{3.5}$$

Similarly, we have

$$d_D(j_m, j_n) \leq \frac{\gamma^n}{1-\gamma}|D(j_1, j_0)| + \frac{(\alpha+\beta)^n}{1-(\alpha+\beta)}|D(j_0, j_0)|. \tag{3.6}$$

On taking limit as $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} d_D(j_m, j_n) = 0 = \lim_{n,m\to\infty} d_D(j_n, j_m) \text{ and hence } \lim_{n,m\to\infty} d_D^s(j_m, j_n) = 0,$$

we get that $\{j_n\}$ is a Cauchy sequence in (M, d_D^s). Since (M, D) is a complete dualistic partial metric space, so by Lemma 2.5 (M, d_D^s) is also a complete metric space. Thus, there exists m^* in M such that $\lim_{n\to\infty} d_D^s(j_n, m^*) = 0$, again from Lemma 2.5, we get

$$\lim_{n\to\infty} d_D^s(j_n, m^*) = 0 \iff D(m^*, m^*) = \lim_{n\to\infty} D(j_n, m^*) = \lim_{n,m\to\infty} D(j_m, j_n). \tag{3.7}$$

Now $\lim_{n,m\to\infty} d_D(j_m, j_n) = 0$ implies that $\lim_{n,m\to\infty}[D(j_m, j_n) - D(j_n, j_n)] = 0$ and hence $\lim_{n,m\to\infty} D(j_n, j_m) = \lim_{n\to\infty} D(j_n, j_n)$. By (3.4), we have $\lim_{n\to\infty} D(j_n, j_n) = 0$. Consequently, $\lim_{n,m\to\infty} D(j_n, j_m) = 0$. Thus

$$D(m^*, m^*) = \lim_{n\to\infty} D(j_n, m^*) = 0. \tag{3.8}$$

Now

$$d_D(m^*, T(m^*)) = D(m^*, T(m^*)) - D(m^*, m^*) = D(m^*, T(m^*)).$$

implies that $D(m^*, T(m^*)) \geq 0$. Since T is continuous, for a given $\epsilon > 0$, There exists $\delta > 0$ such that $T(B_D(m^*, \delta)) \subseteq B_D(T(m^*), \epsilon)$. Since $\lim_{n\to\infty} D(j_{n+1}, m^*) = D(m^*, m^*) = 0$, so there exists $r \in \mathbb{N}$ such that $D(j_n, m^*) < D(m^*, m^*) + \delta$ $\forall\ n \geq r$, therefore $\{j_n\} \subset B_D(m^*, \delta)$ $\forall\ n \geq r$. This implies that $T(j_n) \in T(B_D(m^*, \delta) \subseteq B_D(T(m^*), \epsilon)$ and so $D(T(j_n), T(m^*)) < D(T(m^*), T(m^*)) + \epsilon$ $\forall\ n \geq r$. Now for any $\epsilon > 0$, we know that

$$-\epsilon + D(T(m^*), T(m^*)) < D(T(m^*), T(m^*)) \leq D(j_{n+1}, T(m^*))$$

Which yields that

$$|D(j_{n+1}, T(m^*)) - D(T(m^*), T(m^*))| < \epsilon$$

That is $D(T(m^*), T(m^*)) = \lim_{n\to\infty} D(j_{n+1}, T(m^*))$, finally uniqueness of limit in $\mathbb{R}$ implies

$$\lim_{n\to\infty} D(j_{n+1}, T(m^*)) = D(T(m^*), T(m^*)) = D(m^*, T(m^*)). \tag{3.9}$$

Finally, we have $D(T(m^*), m^*) = \lim_{n\to\infty} D(T(j_n), j_n) = \lim_{n\to\infty} D(j_{n+1}, j_n) = 0$. This shows that $D(m^*, T(m^*)) = 0$. So from (3.8) and (3.9) we deduce that

$$D(m^*, T(m^*)) = D(T(m^*), T(m^*)) = D(m^*, m^*).$$

This leads us to conclude that $m^* = T(m^*)$ and hence m^* is a fixed point of T □

In order to prove the uniqueness of fixed point of a mapping T in the above theorem, we need an additional assumption.

Theorem 3.3. *Let $(M, D, \preceq)$ be complete ordered dualistic partial metric space and $T : M \to M$ a mapping which satisfy all conditions of theorem (3.2). Then T has a unique fixed point provided that for each fixed point m^*, n^* of T, there exists $\omega \in M$ which is comparable to both m^* and n^*.*

Proof. From theorem (3.2), it follows that the set of fixed points of T is nonempty. To prove the uniqueness: Let n^* be another fixed point of T, that is, $n^* = T(n^*)$ and $D(n^*, n^*) = 0$. If m^* and n^* are comparable $(m^* \preceq n^*)$, then we have,

$$\begin{aligned} |D(m^*, n^*)| &= |D(T(m^*), T(n^*))|, \\ &\leq \alpha \left| \frac{D(m^*, T(m^*)) \cdot D(n^*, T(n^*))}{D(m^*, n^*)} \right| + \beta |D(m^*, n^*)|. \\ &\leq \alpha \left| \frac{D(m^*, m^*) \cdot D(n^*, n^*)}{D(m^*, n^*)} \right| + \beta |D(m^*, n^*)|. \end{aligned}$$

That is, $(1-\beta)|D(m^*, n^*)| \leq 0$ which implies that $|D(m^*, n^*)| \leq 0$ and hence $D(m^*, n^*) = 0 = D(m^*, m^*) = D(n^*, n^*)$. The result follows. Suppose that m^* and n^* are incomparable, there exists ω which is comparable to both m^*, n^*. Without any loss of generality, we assume that $m^* \preceq \omega$, and $n^* \preceq \omega$. As T is non decreasing, $T(m^*) \preceq T(\omega)$ and $T(n^*) \preceq T(\omega)$ imply that $T^{n-1}(m^*) \preceq T^{n-1}(\omega)$ and $T^{n-1}(n^*) \preceq T^{n-1}(\omega)$. Thus

$$\begin{aligned} |D(T^n(m^*), T^n(\omega))| &\leq \alpha \left| \frac{D(T^{n-1}(m^*), T^n(m^*)) \cdot D(T^{n-1}(\omega), T^n(\omega))}{D(T^{n-1}(m^*), T^{n-1}(\omega))} \right| \\ &+ \beta |D(T^{n-1}(m^*), T^{n-1}(\omega))|. \end{aligned}$$

That is, $|D(m^*, T^n(\omega))| \leq \beta |D(m^*, T^{n-1}(\omega))|$. Thus, $\lim_{n\to\infty} D(m^*, T^n(\omega)) = 0$.

Similarly, we can have $\lim_{n\to\infty} D(n^*, T^n(\omega)) = 0$. Note that

$$\begin{aligned} D(n^*, m^*) &\leq D(n^*, T^n(\omega)) + D(T^n(\omega), m^*) - D(T^n(\omega), T^n(\omega)), \\ &\leq D(n^*, T^n(\omega)) + D(T^n(\omega), m^*) - D(T^n(\omega), m^*) - D(m^*, T^n(\omega)) + D(m^*, m^*). \end{aligned}$$

On taking limit as $n \to \infty$ we obtain that $D(n^*, m^*) \leq 0$. Now $d_D(m^*, m^*) = D(n^*, m^*) - D(n^*, n^*)$ implies that $D(n^*, m^*) \geq 0$. Hence $D(n^*, m^*) = 0$ which gives that $n^* = m^*$ □

Example 3.4. *Let $M = \mathbb{R}^2$. Define $D_\vee : M \times M \to \mathbb{R}$ by $D_\vee(j,k) = j_1 \vee k_1$, where $j = (j_1, j_2)$ and $k = (k_1, k_2)$. Note that $(M, D_\vee)$ is a complete dualistic partial metric space. Let $T : M \to M$ be given by*

$$T(j) = \frac{j}{2} \quad \textit{for all } j \in M.$$

In M, we define the relation $\succeq$ in the following way:

$$j \succeq k \text{ if and only if } j_1 \geq k_1, \text{ where } j = (j_1, j_2) \text{ and } k = (k_1, k_2).$$

Clearly, $\succeq$ is a partial order on M and T is continuous, non decreasing mapping with respect to $\succeq$. Moreover, $T(-1,0) \succeq (-1,0)$. We shall show that for all $j, k \in M$, (3.1) is satisfied. For this, note that

$$\begin{aligned}
|D_\vee(T(j), T(k))| = \left| D_\vee\left(\frac{j}{2}, \frac{k}{2}\right)\right| &= \left|\frac{j_1}{2}\right| \text{ for all } j_1 \geq k_1, \\
|D_\vee(j, T(j))| = \left| D_\vee\left(j, \frac{j}{2}\right)\right| &= \begin{cases} \left|\frac{j_1}{2}\right| & \text{if } j_1 \leq 0 \\ |j_1| & \text{if } j_1 \geq 0, \end{cases} \\
|D_\vee(k, T(k))| = \left| D_\vee\left(k, \frac{k}{2}\right)\right| &= \begin{cases} \left|\frac{k_1}{2}\right| & \text{if } k_1 \leq 0 \\ |k_1| & \text{if } k_1 \geq 0, \end{cases} \\
\text{and } |D_\vee(j,k)| &= |j_1| \text{ for all } j_1 \geq k_1.
\end{aligned}$$

Now for $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$. If $j_1 \leq 0$, $k_1 \leq 0$, then

$$|D_\vee(T(j), T(k))| \leq \alpha \left| \frac{D_\vee(j, T(j)) \cdot D_\vee(k, T(k))}{D_\vee(j,k)} \right| + \beta |D_\vee(j,k)| \text{ for all } j \succeq k$$

holds if and only if $6|j_1| \leq |k_1| + 6|j_1|$.

For if $j_1 \geq 0$, $k_1 \geq 0$, then contractive condition

$$|D_\vee(T(j), T(k))| \leq \alpha \left| \frac{D_\vee(j, T(j)) \cdot D_\vee(k, T(k))}{D_\vee(j,k)} \right| + \beta |D_\vee(j,k)| \text{ for all } j \succeq k$$

holds if and only if $j_1 \leq \frac{2}{3}k_1 + j_1$.

Finally, if $j_1 \geq 0$, $k_1 \leq 0$, then

$$|D_\vee(T(j), T(k))| \leq \alpha \left| \frac{D_\vee(j, T(j)) \cdot D_\vee(k, T(k))}{D_\vee(j,k)} \right| + \beta |D_\vee(j,k)| \ \forall\ j \succeq k$$

holds if and only if $3j_1 \leq |k_1| + 3j_1$. Thus, all the conditions of Theorem 3.2 are satisfied. Moreover, $(0,0)$ is a fixed point of T.

Remark 3.5. *As every dualistic partial metric D is an extension of partial metric p, therefore, Theorem 3.2 is an extension of Theorem 1.2.*

There arises the following natural question:

Whether the contractive condition in the statement of Theorem 3.2 can be replaced by the contractive condition in Theorem 1.2.

The following example provides a negative answer to the above question.

Example 3.6. Define the mapping $T_0 : \mathbb{R}^+ \to \mathbb{R}$ by

$$T_0(j) = \begin{cases} 0 & \text{if } j > 1 \\ -5 & \text{if } j = 1 \end{cases}.$$

Clearly, for any $j, k \in \mathbb{R}$, following contractive condition is satisfied

$$D_\vee(T_0(j), T_0(k)) \le \frac{\alpha D_\vee(j, T_0(j)) \cdot D_\vee(k, T_0(k))}{D_\vee(j,k)} + \beta D_\vee(j,k)$$

where $D_\vee$ is a complete dualistic partial metric on $\mathbb{R}$. Here, T has no fixed point. Thus a fixed point free mapping satisfies the contractive condition of Theorem 1.2. On the other hand, for all $0 < \alpha + \beta < 1$, we have

$$5 = |D_\vee(-5,-5)| = |D_\vee(T_0(1), T_0(1))| > \alpha \left| \frac{D_\vee(1, T_0(1)) \cdot D(1, T_0(1))}{D_\vee(1,1)} \right| + \beta |D_\vee(1,1)|.$$

Thus contractive condition of Theorem 3.2 does not hold.

Theorem 3.2 remains true if we replace the continuity hypothesis by the following property:

(H): If $\{j_n\}$ is a non decreasing sequence in M such that $j_n \to \upsilon$, then

$$j_n \preceq \upsilon \quad \text{for all } n \in \mathbb{N}. \tag{3.10}$$

This statement is given as follows:

Theorem 3.7. *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space and if,*

(i) *$T : M \to M$ be a non decreasing dualistic contraction of rational type.*
(ii) *there exists $j_0 \in M$ such that $j_0 \preceq T(j_0)$.*
(ii) *(H) holds.*

Then T has a fixed point m^* in M. Moreover, $D(m^*, m^*) = 0$.

Proof. Following the proof of Theorem 3.2, we know that $\{j_n\}$ is non decreasing sequence in M such that $j_n \to m^*$. By (H), we have $j_n \preceq m^*$. As T is non decreasing, we have $T(j_n) \preceq T(m^*)$, that is, $j_{n+1} \preceq T(m^*)$. Also, $j_0 \preceq j_1 \preceq T(m^*)$ and $j_n \preceq m^*$, $n \ge 1$ imply that

$$m^* \preceq T(m^*). \tag{3.11}$$

From the proof of Theorem 3.2, we deduce that $\{T^n(m^*)\}$ is non decreasing sequence. Suppose that $\lim_{n\to+\infty} T^n(m^*) = \mu$ for some $\mu \in M$. Now $j_0 \preceq m^*$ gives $T^n(j_0) \preceq T^n(m^*)$, that is, $j_n \preceq T^n(m^*)$ for all $n \ge 1$. Thus we have

$$j_n \preceq m^* \preceq T(m^*) \preceq T^n(m^*) \quad n \ge 1.$$

By (3.1), we have

$$\begin{aligned} |D(j_{n+1}, T^{n+1}(m^*))| &= |D(T(j_n), T(T^n(m^*)))|, \\ &\le \alpha \left| \frac{D(j_n, j_{n+1}) \cdot D(T^n(m^*), T^{n+1}(m^*))}{D(j_n, T^n(m^*))} \right| + \beta |D(j_n, T^n(m^*))|. \end{aligned}$$

On taking limit as n approaches to plus infinity, we obtain that

$$|D(m^*, \mu)| \leq \beta |D(m^*, \mu)|.$$

which implies that $m^* = \mu$. Thus $\lim_{n \to +\infty} T^n(m^*) = \mu$ implies that $\lim_{n \to +\infty} T^n(m^*) = m^*$. Hence

$$T(m^*) \preceq m^* \tag{3.12}$$

From (3.11) and (3.12), it follows that $m^* = T(m^*)$. □

Some deductions are given below.

Corollary 3.8. *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space and $T : M \to M$ be a non decreasing mapping such that,*

(1)
$$|D(T(j), T(k))| \leq \alpha \left| \frac{D(j, T(j)) \cdot D(k, T(k))}{D(j, k)} \right|, \quad \text{where } 0 < \alpha < 1.$$

(2) *there exists $j_0 \in M$ such that $j_0 \preceq T(j_0)$.*

(3) *either T is continuous or (H) holds.*

Then T has a fixed point m^* in M. Moreover, $D(m^*, m^*) = 0$.

Proof. Set $\beta = 0$ in Theorem 3.2. □

The next deduction generalizes Theorem 2.6 presented by Valero in [6]

Corollary 3.9. *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space and $T : M \to M$ be a non decreasing mapping such that,*

(1) $|D(T(j), T(k))| \leq \beta |D(j, k)|$, *where $0 < \beta < 1$.*

(2) *there exists $j_0 \in M$ such that $j_0 \preceq T(j_0)$.*

(3) *either T is continuous or (H) holds.*

Then T has a fixed point m^* in M. Moreover, $D(m^*, m^*) = 0$.

Proof. Set $\alpha = 0$ in Theorem 3.2. □

Remark 3.10.

(1) If we set $D(j, k) \in \mathbb{R}_0^+$ for all $j, k \in M$ in Corollary 3.8 and in Corollary 3.9, we obtain results in partial metric spaces.

(2) If we set $D(j, k) \in \mathbb{R}_0^+$ for all $j, k \in M$ and $D(j, j) = 0$ for all $j \in M$ in Corollary 3.8 and in Corollary 3.9, we obtain results in metric spaces.

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