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RESEARCH ARTICLE

Identities of the plactic monoid

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Abstract It is shown that the plactic monoid M of rank 3 satisfies the identity $wvvwvw = wvwvvw$ where $v = xyyxxyxyyx$ and $w = xyyxyxxyyx$. This is accomplished by first showing that certain simple monoids related to M satisfy this identity. These simple monoids are natural generalizations of the bicyclic monoid B, which satisfies the identity $w = v$ by a result of Adjan.

Keywords Plactic monoid · Identity · Bicyclic monoid

1 Introduction

For an integer $n \geq 1$ we consider the finitely presented monoid $M_n = \langle a_1, \ldots, a_n \rangle$ defined by the relations

$$\begin{aligned} a_i a_k a_j &= a_k a_i a_j \quad \text{for } i \leq j < k, \\ a_j a_i a_k &= a_j a_k a_i \quad \text{for } i < j \leq k. \end{aligned}$$

It is called the plactic monoid of rank n and was introduced in [14]. It is known that the elements of M_n can be written in a canonical form, which implies that they are in a

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one-to-one correspondence with Young tableaux of certain type. In particular, M_n has polynomial growth of degree $n(n+1)/2$. Because of its strong connections to Young tableaux, the plactic monoid has proved to be a very powerful tool in representation theory and in algebraic combinatorics, see [7,13]. The combinatorics of M_n has been extensively studied, recently including also the aspect of the Gröbner-Shirshov bases [2,6,11]. The algebraic structure and representations of the monoid algebra $K[M_n]$ of M_n over a field K have been investigated, [4,10]. This paper is motivated by the following conjecture.

Conjecture *The plactic monoid of any rank* $n \geq 1$ *satisfies a nontrivial identity.*

On one hand, a motivation comes from general problems concerning existence of identities in classes of finitely generated semigroups of polynomial growth. In particular, it is known (by results of Gromov and Grigorchuk) that a finitely generated cancellative semigroup of polynomial growth satisfies a nontrivial identity, see [15]. However, as shown by Shneerson in [17], this is no longer true without the cancellativity hypothesis. On the other hand, linear semigroups of polynomial growth satisfy a nontrivial identity, [15], Corollary 5.8, while there exist linear semigroups satisfying a nontrivial identity that have subexponential growth, see [15], Example on page 186, and [5] for further results in this direction. Further connections between polynomial growth and the theory of varieties of semigroups were explored in [18], motivated in particular by Sapir's problem on relatively free semigroups of polynomial growth [19]. While simple monoids seem to be of special interest from the point of view of varieties of semigroups, it is also worth mentioning that the structure of M_n heavily depends on certain simple monoids discovered in [10]. These monoids can be considered as generalizations of the bicyclic monoid B. Therefore, a result of Adjan [1], asserting that the bicyclic monoid $B = \langle p, q \mid qp = 1 \rangle$ satisfies the identity

$$xyyxxyxyyx = xyyxyxxyyx. \tag{1}$$

is the starting point for our approach.

On the other hand, the conjecture is motivated by the fact that a class of strongly related monoids, called Chinese monoids C_n, $n \geq 1$, satisfies identity (1). This class was introduced and studied in [3]. Actually, the monoids C_n and M_n are isomorphic if $n \leq 2$. It was shown in [9] that C_n embeds into the product $B^i \times \mathbb{Z}^j$, for some positive integers i, j depending on n, where $\mathbb{Z}$ denotes the additive group of integers. This is a consequence of the description of the minimal prime spectrum of the semigroup algebra $K[C_n]$ over a field K. Therefore, since the bicyclic monoid satisfies identity (1), C_n also satisfies this identity.

Applying some of the results of [10] we prove in this paper that $M = M_3$ satisfies the identity

$$wvvwvw = wvwvvw \tag{2}$$

where $v = v(x, y) = xyyxxyxyyx$ and $w = w(x, y) = xyyxyxxyyx$. Our approach is based on homomorphic images of M arising from natural maps $K[M] \to K[M]/P$, where P are minimal prime ideals of $K[M]$ and on certain infinite dimensional linear representations of M. All such primes P of $K[M]$ and all irreducible representations of M have been determined [10], while in case of plactic monoids M_n with $n \geq 4$ the corresponding problems remain a challenge. In particular, there is no obvious starting point for an attempt to extend our proof to the class of all plactic monoids. One might expect that for every $n \geq 4$ the monoid M_n satisfies some identities, which however would be more complicated than those satisfied by M_{n-1} (unlike the case of Chinese monoids mentioned above) and thus the proofs might be more involved, accordingly.

2 Identities of M_3

Adjan's result was recently reproved in a completely different but quite complicated way in [8], starting from an embedding of B into the semigroup of 2×2 tropical matrices. We start with a simple conceptual proof of this result, and for a more detailed study of identities satisfied by the bicyclic monoid we refer to [16]. It is worth mentioning that one of the key ideas of the proof of our main result is based on a strategy similar to the argument given below.

Proposition 2.1 *The bicyclic monoid B satisfies the identity*

$$w(x, y) = v(x, y). \tag{3}$$

Proof We apply the faithful representation $\phi \colon B \to \mathrm{End}(V)$, where V is a vector space over a field K with a basis $\{e_s : s \geq 0\}$ and

$$\phi(p)(e_s) = e_{s+1}, \quad \phi(q)(e_s) = \begin{cases} 0 & \text{if } s = 0, \\ e_{s-1} & \text{if } s > 0, \end{cases}$$

see Exercise 11.9 in [12]. For simplicity we write $be_s = \phi(b)(e_s)$ for $b \in B$. Notice that $p^i q^j(e_s) = e_{s-j+i}$ if it is nonzero, which holds exactly when $j \leq s$. In particular, if $be_s = 0$ then $be_t = 0$ for every $t \leq s$.

Let $x = p^i q^j, y = p^k q^m \in B$. In order to show that $w(x, y) = v(x, y)$ we may assume that $i + k - j - m \geq 0$, because otherwise we apply the involution on B defined by $p \mapsto q, q \mapsto p$, which reduces the problem to this case. It is now enough to show that for every $s \geq 0$ we have $xyxyyxe_s = yxxyyxe_s$. We may assume that one of these elements is nonzero. Then $xyyxe_s \neq 0$. Write $yxe_s = e_{s'}, xyyxe_s = e_{s''}$. Then

$$s'' = s' + (i + k - j - m) \geq s' = s + (i + k - j - m) \geq s$$

by the assumption.

Since $s'' \geq s$ and $yxe_s \neq 0$, we must have $yxxyyxe_s = yxe_{s''} \neq 0$. Second, since $s'' \geq s'$ and $xye_{s'} = e_{s''} \neq 0$, we also have $xyxyyxe_s = xye_{s''} \neq 0$.

Hence, both elements we_s, ve_s are nonzero, and therefore they are both equal to $e_{s+3(i+k-j-m)}$. The result follows. □

It is known that $a_n a_{n-1} \cdots a_2 a_1$ is a central and regular element of $K[M_n]$. In particular $M_2 = \langle a, b \rangle \subseteq M_2 \langle ba \rangle^{-1} \cong B \times \mathbb{Z}$ and M_2 satisfies identity (1) by Proposition 2.1, as was already noticed in [9].

We recall from [10] basic information on certain homomorphic images N_1 and N_2 of $M = M_3$ that were used in the context of the classification of minimal prime ideals of the monoid algebra $K[M]$. Namely, M is described by the presentation

$$M = \langle a, b, c \rangle,$$

where

$$\begin{array}{llll} aba = baa, & bab = bba, & aca = caa, & cac = cca, \\ cbb = bcb, & cbc = ccb, & bac = bca, & acb = cab \end{array}$$

and we define $N_1 = M/(ac = ca)$ and $N_2 = M/(bacb = cbba)$. Here, for any elements $u_1, u_2 \in M$ by $M/(u_1 = u_2)$ we mean the factor semigroup M/τ, where τ is the congruence on M generated by the pair (u_1, u_2). We will denote the generators of M and their natural images in the considered homomorphic images of M in the same way, if unambiguous.

The element $z = cba$ is a central and regular element of $K[M]$ and also of $K[N_i]$ for $i = 1, 2$, and one can consider the central localizations $M\langle z \rangle^{-1}$and $N_i \langle z \rangle^{-1}$. Moreover we have

$$M\langle z \rangle^{-1} \cong M/(z = 1) \times \langle z, z^{-1} \rangle \cong M/(z = 1) \times \mathbb{Z} \tag{4}$$

$$N_i \langle z \rangle^{-1} \cong N_i/(z = 1) \times \langle z, z^{-1} \rangle \cong N_i/(z = 1) \times \mathbb{Z} \tag{5}$$

and

$$N_i \langle z \rangle^{-1}/(z = 1) \cong N_i/(z = 1).$$

The map $f : M \to M$ defined by $a \mapsto cb, b \mapsto ca, c \mapsto ba$ is an antimonomorphism, which induces an involution $M/(z = 1) \to M/(z = 1)$. Moreover, M is a subdirect product of N_1 and N_2 and $M/(z = 1)$ is a subdirect product of $N_1/(z = 1)$ and $N_2/(z = 1)$. Notice that f induces an antiisomorphim of the latter two monoids. It follows that in order to decide whether M satisfies nontrivial identities it is sufficient to verify this for the monoid $N_1/(z = 1)$. Namely, let us say that a pair of words $s = s(x, y)$, $t = t(x, y)$ in x, y is reversive if one obtains the same two words (as a set) when reading $s(x, y)$, $t(x, y)$ backwards. Then we get

Lemma 2.2 *The monoids M, $M/(cba = 1)$ and $N_1/(cba = 1)$ satisfy the same identities of the form $s(x, y) = t(x, y)$, where s, t is a reversive pair of words.*

Every element $u \in N_1$ can be uniquely written in the form

$$u = (cba)^{k_1} (ba)^{k_2} a^{k_3} (cb)^{k_4} b^{k_5} c^{k_6}, \tag{6}$$

where $k_i \geq 0$, see [10], Lemma 2.3. Hence, in view of (5), every element $\overline{u}$ of the monoid $N_1/(z = 1)$ can be uniquely written in the form

$$\overline{u} = (ba)^{k_2} a^{k_3} (cb)^{k_4} b^{k_5} c^{k_6}, \tag{7}$$

where $k_i \geq 0$. And it follows that $N_1/(z = 1)$ is a simple monoid, and hence so is $N_2/(z = 1)$. Since $cb \cdot a = c \cdot ba$ is the unity of $N_1/(z = 1)$, these monoids carry some flavor of the monoid B, see [10], Proposition 2.6.

Consider the natural homomorphisms

$$\phi_1 \colon N_1 \to N_1/(ab = ba)$$

and

$$\phi_2 \colon N_1 \to N_1/(bc = cb).$$

So, the former makes the image of a central. It is easy to see that

$$N_1/(ab = ba) \cong \langle a \rangle \times \langle b, c \rangle \cong \mathbb{N} \times M_2 \tag{8}$$

and similarly

$$N_1/(bc = cb) \cong \langle c \rangle \times \langle a, b \rangle \cong \mathbb{N} \times M_2, \tag{9}$$

with $\mathbb{N}$ denoting the additive monoid of non-negative integers.

Assume that two elements

$$\alpha = (cba)^{k_1} (ba)^{k_2} a^{k_3} (cb)^{k_4} b^{k_5} c^{k_6},$$
$$\alpha' = (cba)^{k'_1} (ba)^{k'_2} a^{k'_3} (cb)^{k'_4} b^{k'_5} c^{k'_6}$$

of N_1 satisfy $\phi_i(\alpha) = \phi_i(\alpha')$ for $i = 1, 2$. Using the canonical form of elements of $\mathbb{N} \times M_2$ it is easy to see that

$$\phi_1(\alpha) = a^{k_1+k_2+k_3} (cb)^{k_1+k_4} b^{k_2+k_5} c^{k_6}, \quad \phi_2(\alpha) = c^{k_1+k_4+k_6} (ba)^{k_1+k_2} a^{k_3} b^{k_4+k_5}$$

and similar presentations hold for the elements $\phi_1(\alpha')$ and $\phi_2(\alpha')$. Consequently, we get

$$k_1 + k_2 + k_3 = k'_1 + k'_2 + k'_3,\ k_1 + k_4 = k'_1 + k'_4,\ k_2 + k_5 = k'_2 + k'_5,\ k_6 = k'_6$$

and

$$k_1 + k_4 + k_6 = k'_1 + k'_4 + k'_6,\ k_1 + k_2 = k'_1 + k'_2,\ k_4 + k_5 = k'_4 + k'_5,\ k_3 = k'_3.$$

This is equivalent to the conditions

$$k_1 + k_2 = k'_1 + k'_2,\ k_1 + k_4 = k'_1 + k'_4,\ k_2 + k_5 = k'_2 + k'_5,\ k_3 = k'_3,\ k_6 = k'_6.$$

If $k_1 = k'_1$ then clearly $k_q = k'_q$ for $q = 1, 2, 3, 4, 5, 6$, so that $\alpha = \alpha'$. Hence, assume for example that $k_1 > k'_1$. Let $k'_1 = i$ and $k_1 = i + s$ for some $s \geq 1$. Then also

$k'_2 = k_2 + s, k'_4 = k_4 + s, k_5 = k'_5 + s$ and $k'_2 = k_2 + s$. Put $j = k_2, k = k_3, l = k_4, m = k_5, r = k_6$. Then α, α' are of the form

$$\alpha = (cba)^{i+s}(ba)^j a^k (cb)^l b^{m+s} c^r = [(cba)^{i+s}(ba)^j a^k] b^s [(cb)^l b^m c^r], \tag{10}$$
$$\alpha' = (cba)^i (ba)^{j+s} a^k (cb)^{l+s} b^m c^r = [(cba)^i (ba)^j a^k]((ba)^s (cb)^s)[(cb)^l b^m c^r] \tag{11}$$

and the images of α, α' in $N_1/(z = 1)$ are of the form

$$(ba)^j a^k b^s (cb)^l b^m c^r, \quad (ba)^j a^k ((ba)^s (cb)^s)(cb)^l b^m c^r, \tag{12}$$

respectively. So α, α' are in the congruence ρ on N_1 generated by all pairs $((cba)^s b^s, (ba)^s (cb)^s)$, $s \geq 1$. It follows that $\rho = \operatorname{Ker}\phi_1 \cap \operatorname{Ker}\phi_2$, the intersection of the congruences determined by ϕ_1 and ϕ_2 and N_1/ρ embeds into the monoid $N_1/(ab = ba) \times N_1/(bc = cb)$.

We know that $\mathbb{N} \times M_2$ satisfies Adjan's identity (1) because the plactic monoid of rank 2 satisfies every identity holding in the bicyclic monoid. Hence, in view of (8) and (9), N_1/ρ satisfies this identity.

In particular, for every $x, y \in N_1$ the elements $xy^2xxyxy^2x, xy^2xyxxy^2x$ must be of the form described in (10) and (11), while their images in $N_1/(z = 1)$ are of the form (12). Moreover, if $k_1 = k'_1$ (equivalently $s = 0$) then $\alpha = \alpha'$, with the notation as above.

Hence we obtain the following consequence.

Lemma 2.3 *Let w_1, w_2 be two distinct words in the free monoid of rank two of the same total degree. Assume also that every evaluation of the words $w_1(v, w)$ and $w_2(v, w)$ in N_1 (with $v = v(x, y)$, $w = w(x, y)$ defined in (2)) leads to two elements of N_1 with equal exponents of cba in the canonical form. Then N_1 satisfies a nontrivial identity, namely*

$$w_1(xy^2xxyxy^2x, xy^2xyxxy^2x) = w_2(xy^2xxyxy^2x, xy^2xyxxy^2x).$$

For our main result we will use an idea analogous to that used in the proof of Proposition 2.1. Consider the representation ϕ of M coming from Proposition 3.6 in [10] (with $\beta = \gamma = 1$). Namely, let V be a vector space over a field K with basis $\{e_{pq} : p, q \geq 0\}$. Let the action of $a, b, c \in M$ on V be given by

$$ae_{pq} = e_{p,q+1}, \quad be_{pq} = \begin{cases} e_{pq} & \text{if } q = 0, \\ e_{p+1,q-1} & \text{if } q > 0, \end{cases} \quad ce_{pq} = \begin{cases} 0 & \text{if } p = 0, \\ e_{p-1,q} & \text{if } p > 0. \end{cases}$$

Then this action makes V a (simple) left $K[M]$-module. Actually, this is also a $K[N_1]$-module and a $K[N_1/(z = 1)]$-module.

We will use the fact that

$$b^s e_{pq} = e_{p+\min\{s,q\},q-\min\{s,q\}}$$

and another easy consequence of the definition

$$(ba)^s(cb)^s e_{pq} = \begin{cases} 0 & \text{if } p+q<s, \\ e_{p+\min\{s,q\},q-\min\{s,q\}} & \text{if} p+q \geq s. \end{cases}$$

In particular, $b^s e_{pq} = (ba)^s(cb)^s e_{pq}$ if $(ba)^s(cb)^s e_{pq} \neq 0$. Moreover, the latter is 0 only if $p+q<s$ and then $b^s e_{pq} = e_{p+q,0}$.

Lemma 2.4 *The representation ϕ is faithful on $N_1/(z=1)$.*

Proof We use the canonical form (7) of elements in $N_1/(z=1)$. Suppose that $\phi(t) = \phi(t')$ for some

$$t = (ba)^{k_2}a^{k_3}(cb)^{k_4}b^{k_5}c^{k_6}, \quad t' = (ba)^{k'_2}a^{k'_3}(cb)^{k'_4}b^{k'_5}c^{k'_6} \in N_1/(z=1),$$

where $k_i \geq 0$. If $k'_6 > k_6$ then $t'e_{k_6,k_4} = 0$ and $te_{k_6,k_4} \neq 0$. So we may assume $k'_6 = k_6$. If $k'_4 > k_4$ then $t'e_{k_4+k_6,0} = 0$ and $te_{k_4+k_6,0} \neq 0$. So $k_4 = k'_4$. Now $te_{pq} = e_{p-k_6+k_5+k_2,q-k_5-k_4+k_3}$ if $p \geq k_6$ and $q \geq k_4+k_5$. Hence $k_5+k_2 = k'_5+k'_2$ and $-k_5+k_3 = -k'_5+k'_3$. If $k'_5 \geq k_5$ then $t'e_{k_6,k_4+k'_5} = e_{k'_5+k_2,k_3}$ and $te_{k_6,k_4+k'_5} = e_{k_5+k_2,k_3}$. Hence $k'_5 = k_5$ and the assertion follows easily. □

We will often use the following standard fact [10,14].

Lemma 2.5 *If $t \in M$ then $t = cubu'a$ for some $u, u' \in M$ if and only if $t = cbat'$ for some $t' \in M$.*

We will also use the involution $\epsilon\colon M \to M$ determined by $\epsilon(a) = c$, $\epsilon(b) = b$, $\epsilon(c) = a$. It also leads to an involution of N_1, also denoted by ϵ for simplicity.

Theorem 2.6 *The monoid $M = M_3$ satisfies the identity $wvvwvw = wvwvvw$, where $v = xyyxxyxyyx$ and $w = xyyxyxxyyx$ represent the left-hand and respectively the right-hand side of identity (1).*

Proof The main difficulty of the proof is to show that this identity is satisfied on N_1. Our approach will be based on Lemma 2.4 and on Lemma 2.3. For $x, y \in N_1$ and $w = w(x,y)$, $v = v(x,y)$ we have

$$\{w,v\} = \{(cba)^{i+s}(ba)^j a^k b^m\ b^s(cb)^l c^r,\ (cba)^i(ba)^j a^k b^m\ ((ba)^s(cb)^s)(cb)^l c^r\}$$

for some s, see (10),(11). Hence, in order to decide whether $\alpha(w(x,y), v(x,y)) = \beta(w(x,y), v(x,y))$ in N_1 for some $x, y \in N_1$ and some words α, β of the same degree in each of the two variables, it is enough to show that $\alpha(w(x,y), v(x,y))$ and $\beta(w(x,y), v(x,y))$ have the same kernel on the basis $\{e_{pq} : p, q \geq 0\}$ of the module V introduced before Lemma 2.4. (Indeed, in this case their images must be equal in $N_1/(z=1)$ by the observation made before Lemma 2.4, so that a degree argument shows that they are equal in N_1.) We will use this observation several times without further comment.

First, we consider the case where

$$\begin{aligned} w &= (cba)^{i+s}(ba)^j a^k b^m\ b^s(cb)^l c^r, \\ v &= (cba)^i (ba)^j a^k b^m\ ((ba)^s(cb)^s)(cb)^l c^r. \end{aligned} \tag{13}$$

We will show that in N_1 either $wvwv = vwwv$ and $vwvw = wvvw$, or $wvwv = wvvw$ and $vwvw = vwwv$. The case where the roles of w and v are switched then also follows because of the symmetry of these equalities.

We write $u = (ba)^j a^k$, $t = (cb)^l b^m c^r$, so that

$$w = (cba)^{i+s} u b^s t, \quad v = (cba)^i u((ba)^s(cb)^s)t \tag{14}$$

and consider the following four cases:

1. $r \leq j$ and $l \leq k$,
2. $r \leq j$ and $l > k$,
3. $r > j$ and $l \leq k$,
4. $r > j$ and $l > k$.

It is easy to see that the product tu is divisible by $(cba)^{r+l}$, $(cba)^{r+k+\min\{l-k,j-r\}}$, $(cba)^{j+l}$, $(cba)^{k+j}$ in these cases, respectively. So, the central regular element cba appears in this way the same number of times in each of the elements $wvwv$, $vwwv$, $vwvw$ and $wvvw$. Moreover, the initial segment u of each of these elements and the terminal segment t do not affect the exponent of cba in the canonical form of these elements in N_1. Therefore, deleting from u and from t the appropriate factors (those that produce the maximal power of cba in the product tu) we may assume that u, t have one of the following forms (corresponding to the cases 1–4 listed above):

1. $u = (ba)^j(a)^k$ and $t = (b)^m$,
2. $u = (ba)^j$ and $t = (cb)^l(b)^m$,
3. $u = (a)^k$ and $t = (b)^m(c)^r$,
4. $u = 1$ and $t = (cb)^l(b)^m(c)^r$.

In the second case, if the exponents of ba and cb are nonzero, the situation is also reduced to case 1. or case 4., due to the possible cancellation of cba coming from the factor $cb\,ba$ arising from tu.

So, cancelling the appropriate exponents of cba in w and v, we may assume that the elements w, v are of one of the following forms (corresponding to cases 1, 4, and 3, respectively):

(i) $w = (ba)^j a^k b^{m+s}$ and $v = a^k(ba)^{j+s}(cb)^s b^m$,
(ii) $w = b^{m+s}(cb)^l c^r$ and $v = (ba)^s(cb)^{l+s} b^m c^r$,
(iii) $w = a^k b^{m+s} c^r$ and $v = a^k(ba)^s(cb)^s b^m c^r$.

In the following computations we often apply Lemma 2.5 and the relations defining N_1 without further comment.

We consider the three cases indicated above.

Case (i). The proof of this case will be based on Lemma 2.4. First, we will show that $vwvw = wvvw$. As observed in the first paragraph of the proof, to establish the latter equality it is enough to check that

$$vwvwe_{pq} \neq 0 \text{ iff } wvvwe_{pq} \neq 0 \text{ for all } p, q \geq 0. \tag{15}$$

Before we proceed to the proof of (15) let us first note that $we_{pq} \neq 0$ for all $p, q \geq 0$. Moreover $ve_{pq} = 0$ if and only if $(cb)^s b^m e_{pq} = 0$. Since we have

$$(cb)^s b^m e_{pq} = \begin{cases} 0, & \text{if } p+q < s, \\ e_{p-s+\min\{q,m+s\},q-\min\{q,m+s\}}, & \text{if } p+q \geq s, \end{cases}$$

it is easy to see that $ve_{pq} = 0$ if and only if $p+q < s$. Next, let us note that elements ba and a acting on e_{pq} increase $p+q$ exactly by 1. Similarly, cb annihilates e_{pq} or decreases $p+q$ exactly by 1, while b does not change $p+q$. Hence we conclude that if $e_{p'q'} = we_{pq}$ or $e_{p'q'} = ve_{pq} \neq 0$, then necessarily $p+q \leq p'+q'$.

In order to prove (15) we may assume that one of the elements $vwvwe_{pq}$, $wvvwe_{pq}$ is nonzero. In this situation, we have $vwe_{pq} \neq 0$. Therefore, $e_{p_1q_1} = we_{pq} \neq 0$, $e_{p_2q_2} = ve_{p_1q_1} \neq 0$ and $e_{p_3q_3} = we_{p_2q_2} \neq 0$, for some natural numbers $p_1, p_2, p_3, q_1, q_2, q_3$. As observed above, in this case $p_1+q_1 \leq p_2+q_2 \leq p_3+q_3$. If $ve_{p_2q_2} = 0$, then using $p_1+q_1 \leq p_2+q_2 < s$ we would get $ve_{p_1q_1} = 0$, a contradiction. Hence $e_{p'_3q'_3} = ve_{p_2q_2} \neq 0$ for some p'_3, q'_3. Finally, from $p_2+q_2 \leq p_3+q_3$ we conclude that $ve_{p_3q_3} \neq 0$ and consequently we get

$$\begin{aligned} vwvwe_{pq} &= vwve_{p_1q_1} = vwe_{p_2q_2} = ve_{p_3q_3} \neq 0, \\ wvvwe_{pq} &= wvve_{p_1q_1} = wve_{p_2q_2} = we_{p'_3q'_3} \neq 0. \end{aligned}$$

Hence, by (15), in Case (i) the elements $vwvw$ and $wvvw$ are equal in N_1. In a similar way one shows that $wvwve_{pq} \neq 0$ if and only if $vwwve_{pq} \neq 0$ for all $p, q \geq 0$ and hence $wvwv = vwwv$.

Case (ii). In this case we apply the involution ϵ of N_1 and we get

$$\begin{aligned} \epsilon(w) &= \epsilon(b^{m+s}(cb)^l c^r) = a^r (ba)^l b^{m+s}, \\ \epsilon(v) &= \epsilon((ba)^s (cb)^{l+s} b^m c^r) = a^r b^m (ba)^{l+s} (cb)^s. \end{aligned}$$

This leads to Case (i). Therefore, we get relations: $vwvw = vwwv$ and $wvwv = wvvw$ in N_1.

Case (iii). First, assume that $r \geq k$. We consider three subcases:

(I) $s \leq k \leq r$,
(II) $k \leq s \leq r$,
(III) $k \leq r \leq s$.

If (I) holds then

$$wv = [a^k b^{m+s} c^r][a^k (ba)^s (cb)^s b^m c^r] = (cba)^s u$$

for some $u \notin cbaM$ and

$$vw = [a^k(ba)^s(cb)^s b^m c^r][a^k b^{m+s} c^r] = (cba)^s u'$$

for some $u' \notin cbaM$. By Lemma 2.3 we get $wv = vw$ in N_1.

If (II) holds then

$$\begin{aligned} wv &= [a^k b^{m+s} c^r][a^k(ba)^s(cb)^s b^m c^r] \\ &= a^k b^{m+s} c^{r-s} a^k (cba)^s (cb)^s b^m c^r \\ &= (cba)^s a^k (ba)^k b^{m+s-k} c^{r-s} (cb)^s b^m c^r \\ &= (cba)^s a^k (ba)^k t \end{aligned}$$

where $t \in \langle b, c\rangle$. Then

$$v(wv) = u(cb)^s c^r (cba)^s a^k (ba)^k t = u(cba)^{s+k+k} t_1$$

where $u \in \langle a, b\rangle$, $t_1 \in \langle b, c\rangle$. So $vwv \notin (cba)^{s+2k+1}M$. Moreover

$$\begin{aligned} vw &= [a^k(ba)^s(cb)^s b^m c^r][a^k b^{m+s} c^r] \\ &= a^k (ba)^s (cba)^k b^m (cb)^{s-k} c^r b^{m+s} c^r \\ &= (cba)^k a^k (ba)^s t' \end{aligned}$$

where $t' \in \langle b, c\rangle$. Hence

$$v(vw) = u(cb)^s c^r (cba)^k a^k (ba)^s t' = u(cba)^{s+k+k} t_2$$

where $t_2 \in \langle b, c\rangle$. So $vvw \notin (cba)^{s+2k+1}M$.

By Lemma 2.3 we get $vwv = vvw$ in N_1.

Similarly

$$w(wv) = [a^k b^{m+s} c^r][(cba)^s a^k (ba)^k t] \in (cba)^{s+k}M \setminus (cba)^{s+k+1}M$$

and

$$w(vw) = [a^k b^{m+s} c^r][(cba)^k a^k (ba)^s t'] \in (cba)^{s+k}M \setminus (cba)^{s+k+1}M.$$

Hence we also have $wwv = wvw$. So in Case (II) we also get $wvwv = wvvw$ and $vwwv = vwvw$ in N_1.

If (III) holds then

$$\begin{aligned} wv &= [a^k b^{m+s} c^r][a^k(ba)^s(cb)^s b^m c^r] \\ &= a^k b^{m+s} (cba)^r (ba)^{s-r} a^k (cb)^s b^m c^r \\ &= (cba)^r a^k b^{2m+s-k} (ba)^{s-r+k} (cb)^s c^r \end{aligned}$$

$$\begin{aligned} vw &= [a^k(ba)^s(cb)^s b^m c^r][a^k b^{m+s} c^r] \\ &= a^k(ba)^s b^m (cba)^k (cb)^{s-k} c^r b^{m+s} c^r \\ &= (cba)^k a^k (ba)^s b^m (cb)^{s-k+r} b^{m+s-r} c^r \\ &= (cba)^k a^k (ba)^s b^{2m+s-r} (cb)^{s-k+r} c^r. \end{aligned}$$

Then

$$\begin{aligned} wvvw &= [(cba)^r a^k b^{2m+s-k}(ba)^{s-r+k}(cb)^s c^r][(cba)^k a^k (ba)^s b^{2m+s-r}(cb)^{s-k+r} c^r] \\ &= (cba)^{k+r} u (cb)^s c^r a^k (ba)^s t \\ &= (cba)^{k+r} u (cba)^{k+r} (cb)^{s-k} (ba)^{s-r} t \\ &= (cba)^{2k+s+r} u' t' \end{aligned}$$

where $u, u' \in \langle a, b\rangle$, $t, t' \in \langle b, c\rangle$ and

$$\begin{aligned} wvwv &= [(cba)^r a^k b^{2m+s-k}(ba)^{s-r+k}(cb)^s c^r][(cba)^r a^k b^{2m+s-k}(ba)^{s-r+k}(cb)^s c^r] \\ &= (cba)^{r+r} u' (cb)^s c^r a^k (ba)^{s-r+k} t'' \\ &= (cba)^{r+r} u' (cba)^k (cb)^{s-k} c^r (ba)^{s-r+k} t'' \\ &= (cba)^{s+r+2k} u' t''' \end{aligned}$$

where $u' \in \langle a, b\rangle$, $t'', t''' \in \langle b, c\rangle$. By Lemma 2.3 we get $wvwv = wvvw$ in N_1.

Moreover

$$\begin{aligned} vwwv &= [(cba)^k a^k (ba)^s b^{2m+s-r}(cb)^{s-k+r} c^r][(cba)^r a^k b^{2m+s-k}(ba)^{s-r+k}(cb)^s c^r] \\ &= u (cba)^{k+r} (cb)^{s-k+r} c^r a^k (ba)^{s-r+k} t \\ &= u (cba)^{k+r+s-r+k} c^r b^{s-r+k} (cb)^{2r-2k} a^k t \\ &= u (cba)^{2k+s+k} t' \end{aligned}$$

where $u \in \langle a, b\rangle$, $t, t' \in \langle b, c\rangle$ (because the degree of b in $(cb)^{2r-2k} b^{s-r+k}$ is $s + r - k \geq k$). Also

$$\begin{aligned} vwvw &= [(cba)^k a^k (ba)^s b^{2m+s-r}(cb)^{s-k+r} c^r][(cba)^k a^k (ba)^s b^{2m+s-r}(cb)^{s-k+r} c^r] \\ &= u' (cba)^{k+k} (cb)^{s-k+r} c^r a^k (ba)^s t'' \\ &= u' (cba)^{k+k} (cb)^{s-k+r} a^k (cba)^r (ba)^{s-r} t'' \\ &= u' (cba)^{2k+s+k} t''' \end{aligned}$$

where $u' \in \langle a, b\rangle$, $t'', t''' \in \langle b, c\rangle$ (because the degree of a in $a^k(ba)^{s-r}$ is $s + k - r \leq s - k + r$).

By Lemma 2.3 we get $vwwv = vwvw$ in N_1.

It remains to consider the situation where $r < k$ in Case (iii). Then applying the involution ϵ on N_1 we get

$$\epsilon(v) = (a)^r b^m (ba)^s (cb)^s c^k, \quad \epsilon(w) = a^r b^{m+s} c^k.$$

Therefore, this reduces the situation to the case where $r > k$, considered above. This means that $vwwv = wvwv$ and $wvvw = vwvw$ in N_1.

Conclusion: if $w = w(x, y)$ and $v = v(x, y)$ for some $x, y \in N_1$ then either $wvwv = vwwv$ and $vwvw = wvvw$ or $wvwv = wvvw$ and $vwvw = vwwv$. Hence in the former case we have

$$wv(vwvw) = wv(wvvw)$$

while in the latter

$$(wvvw)vw = (wvwv)vw.$$

Thus, the identity $wvvwvw = wvwvvw$ is satisfied in N_1, as claimed at the beginning of the proof.

Recall that v (viewed as a word in x, y) is obtained by reading backwards the word w. Therefore, an argument as that leading to Lemma 2.2 shows that also in N_2 we get that either $wvwv = vwwv$ and $vwvw = wvvw$, or $wvwv = wvvw$ and $vwvw = vwwv$. So N_2 also satisfies the identity $wvvwvw = wvwvvw$. Since M is a subdirect product of N_1 and N_2, this identity is also satisfied in M. $\square$

One can ask whether M satisfies a simpler identity. We conclude by showing that M does not satisfy Adjan's identity $w(x, y) = v(x, y)$. Let $x = a^3c$ and $y = (ba)b(cb)^2$. Then

$$xy = (cba)a^3b(cb)^2, \qquad yx = (cba)^2(ba)^2c$$

and

$$xyyx = (cba)a^3b(cb)^2\ (cba)^2(ba)^2c = (cba)^5a^3b^3c,$$

so that it is easy to see that

$$\begin{aligned}
w(x, y) &= (cba)^5a^3b^3c\ (cba)a^3b(cb)^2\ (cba)^5a^3b^3c \\
&= (cba)^{11}a^3b^3a^3(cb)^3a^3b^3c \\
&= (cba)^{14}a^3(ba)^3b^3c, \\
v(x, y) &= (cba)^5a^3b^3c\ (cba)^2(ba)^2c\ (cba)^5a^3b^3c \\
&= (cba)^{12}a^3b^3c(ba)^2ca^3b^3c \\
&= (cba)^{13}a^3(ba)b^3ca^3b^3c \\
&= (cba)^{13}a^3(ba)^4b^2(cb)c
\end{aligned}$$

are written in the canonical forms in M, whence they are different. Notice that vw and wv again have a different exponent of cba in their canonical forms, namely 29 and 28, respectively; whence they are different. But we get $wvvw = vwvw$ and $vwwv = wvwv$.

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References

1. Adjan, S.I.: Defining relations and algorithmic problems for groups and semigroups. In: Proceedings of Steklov Institute of Mathematics, vol. 85. American Mathematical Society, Providence (1966)
2. Cain, A.J., Gray, R., Malheiro, A.: Finite Gröbner–Shirshov bases for Plactic algebras and biautomatic structures for Plactic monoids. arXiv:1205.4885v2
3. Cassaigne, J., Espie, M., Krob, D., Novelli, J.-C., Hivert, F.: The Chinese monoid. Int. J. Algebra Comput. **11**(3), 301–334 (2001)
4. Cedó, F., Okniński, J.: Plactic algebras. J. Algebra **274**, 97–117 (2004)
5. Cedó, F., Okniński, J.: Semigroups of matrices of intermediate growth. Adv. Math. **212**, 669–691 (2007)
6. Chen, Y., Li, J.: New approach to Schensted–Knuth normal forms. arXiv:1106.4753v1
7. Fulton, W.: Young Tableaux, London Mathematical Society Student Texts, vol. 35. Cambridge University Press, New York (1997)
8. Izhakian, Z., Margolis, S.W.: Semigroup identities in the monoid of two-by-two tropical matrices. Semigr. Forum **80**, 191–218 (2010)
9. Jaszuńska, J., Okniński, J.: Structure of Chinese algebras. J. Algebra **346**, 31–81 (2011)
10. Kubat, Ł., Okniński, J.: Plactic algebra of rank 3. Semigr. Forum **84**, 241–266 (2012)
11. Kubat, Ł, Okniński J.: Gröbner-Shirshov bases for plactic algebras. Algebra Colloquium (to appear)
12. Lam, T.Y.: Exercises in Clasical Ring Theory. Springer, New York (2003)
13. Lascoux, A., Leclerc, B., Thibon, J.-Y.: The plactic monoid. In: Algebraic Combinatorics on Words, pp. 164–196. Cambridge University Press, Cambridge (2002)
14. Lascoux, A., Schützenberger, M.P.: Le monoide plaxique. In: Noncommutative Structures in Algebra and Geometric Combinatorics (Naples, 1978), vol. 109 of Quad. "Ricerca Sci.", pp. 129–156. CNR, Rome (1981)
15. Okniński, J.: Semigroups of Matrices. World Scientific, Singapore (1998)
16. Pastijn, F.: Polyhedral convex cones and the equational theory of the bicyclic semigroup. J. Aust. Math. Soc. **81**, 63–96 (2006)
17. Shneerson, L.M.: Identities in finitely generated semigroups of polynomial growth. J. Algebra **154**, 67–85 (1993)
18. Shneerson, L.M.: Polynomial growth in semigroup varieties. J. Algebra **320**, 2218–2279 (2008)
19. Shevrin, L.N. (ed.): The Sverdlovsk Notebook. Unsolved Problems of the Theory of Semigroups (Russian), 3rd ed. Ural State University, Sverdlovsk (1989)