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Compound Poisson approximation in systems reliability

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Abstract

The compound Poisson 'local' formulation of the Stein-Chen method is applied to problems in reliability theory. Bounds for the accuracy of the approximation of the reliability by an appropriate compound Poisson distribution are derived under fairly general conditions, and are applied to consecutive-2 and connected-s systems, and the 2-dimensional consecutive-k-out-of-n system, together with a pipeline model. The approximations are usually better than the Poisson 'local' approach would give.

Keywords: reliability bounds, CP-Stein-Chen method, local approach, consecutive-2 systems, connected-s systems, 2-dimensional consecutive-k-out-of-n system.

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1. Introduction

Reliability evaluation is an important and integral feature of planning, design and operation of engineering systems. A review of recent work in reliability is given in Chao, Fu and Koutras [3]. Since it is not always possible to compute the reliability of a system exactly, it is reasonable to look instead for good approximations.

Many reliability systems can be represented as graphs $G(V, E)$ with V a set of vertices (machines) and E a set of edges (connections) between them. One assumes that there are *items* which can be up (work) or can be down (be failed), and in different applications vertices, edges or both may be considered to be items [5]. With each item is associated a Bernoulli random variable which indicates if the item is up or down and takes the value 1 (up) and 0 (down) with probabilities p and $q = 1 - p$ respectively. A common assumption is that the items work and fail independently of each other, though the probabilities of being up need not all be the same. Thus, if there were n items in the system, one would have independent Bernoulli random variables $\mathrm{Be}(p_i)$, $i = 1, \ldots n$ in the model.

In the definition of the system, there are special 'SF (system failure) – subsets' of items: if all items in an SF–subset are failed, the whole system fails. Let Γ denote the set of all SF–subsets of items. For $\alpha \in \Gamma$, let $I_\alpha = I[\text{all items in the SF–subset } \alpha \text{ are failed}]$, and write $\pi_\alpha = \mathbb{P}[I_\alpha = 1] = \prod_{i \in \alpha} q_i$. Then $W = \sum_{\alpha \in \Gamma} I_\alpha$ is the random variable which counts the number of SF–subsets in the system in which all items are failed, and the reliability of the system is just $\mathbb{P}[W = 0]$. Because SF–subsets may overlap, the I_α can be dependent in spite of the independence of individual items.

In such problems, the Stein–Chen approach to Poisson approximation has proved useful in estimating $\mathbb{P}[W = 0]$ by the Poisson probability $e^{-\lambda}$, where $\lambda = \sum_{\alpha \in \Gamma} \pi_\alpha = \mathbb{E}W$ [2,6,7,10,12,13]. However, if the dependence between the I_α is such that they tend to occur in clusters, W may be better approximated by a compound Poisson distribution, and the reliability $\mathbb{P}[W = 0]$ by $e^{-\mu}$, where μ, the intensity of clusters, is smaller than λ. The

purpose of this paper is to show how to exploit this idea, using the 'local' approach to Stein's method for compound Poisson distributions [1,14].

Let $\mathrm{CP}(\boldsymbol{\lambda})$ denote the (compound Poisson) distribution of $\sum_{i\geq 1} iN_i$, where $N_i \sim \mathrm{Po}(\lambda_i)$ are independent for $i = 1, 2, \ldots$, $\sum_{i\geq 1}\lambda_i < \infty$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$. In the case when $\lambda_1 > 0$ and $\lambda_i = 0$ for $i \geq 2$, the usual Poisson distribution results. Then, to state the CP-local approximation, divide Γ into $\{\alpha\}$ and any three subsets Γ_α^{vs}, Γ_α^{b} and Γ_α^{vw}, where it is nonetheless intended that $\{I_\beta,\ \beta \in \Gamma_\alpha^{vs}\}$ should be 'very strongly' dependent on I_α, $\{I_\beta,\ \beta \in \Gamma_\alpha^{vw}\}$ should be only 'very weakly' dependent on $\{I_\gamma,\ \gamma \in \{\alpha\} \cup \Gamma_\alpha^{vs}\}$ and $\Gamma_\alpha^{b} = \Gamma\backslash\{\{\alpha\} \cup \Gamma_\alpha^{vs} \cup \Gamma_\alpha^{vw}\}$ denotes the remaining set of indicators. In the reliability applications sketched above, one could for example take

$$\begin{aligned}\Gamma_\alpha^{vs} =& \{\beta \in \Gamma\backslash\{\alpha\} : \beta \quad \text{overlaps} \quad \alpha\} \qquad \text{and}\\ \Gamma_\alpha^{vw} =& \big\{\beta \in \Gamma\backslash\{\alpha\} : \beta \quad \text{overlaps no element of} \quad \{\{\alpha\} \cup \Gamma_\alpha^{vs}\}\big\}.\end{aligned}$$

Set

$$U_\alpha = \sum_{\beta\in\Gamma_\alpha^{vs}} I_\beta, \qquad Z_\alpha = I_\alpha + U_\alpha \quad \text{and} \quad X_\alpha = \sum_{\beta\in\Gamma_\alpha^{b}} I_\beta, \tag{1.1}$$

so that Z_α can be thought of as the size of the α-clump. Define $\lambda_i = \frac{1}{i}\sum_{\alpha\in\Gamma} \mathbb{E}\{I_\alpha I[Z_\alpha = i]\}$, $\mu = \sum_{i>1}\lambda_i$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$, and set $\phi = \sum_{\alpha\in\Gamma}\sum_{i=1}^{|\Gamma_\alpha^{vs}|+1}\phi_{\alpha i}$, with

$$\phi_{\alpha i} = \mathbb{E}|\mathbb{E}\{I_\alpha I[Z_\alpha = i]|(I_\beta : \beta \in \Gamma_\alpha^{vw})\} - \mathbb{E}\{I_\alpha I[Z_\alpha = i]\}|.$$

Theorem 1 (CP–local approach). *With the above definitions, for any choice of the index sets Γ_α^{vs}, Γ_α^{b} and Γ_α^{vw},*

$$|\mathbb{P}[W = 0] - e^{-\mu}| \leq c_2'(\boldsymbol{\lambda})\sum_{\alpha\in\Gamma}\big((\mathbb{E}I_\alpha)^2 + \mathbb{E}I_\alpha\mathbb{E}\{U_\alpha + X_\alpha\} + \mathbb{E}\{I_\alpha X_\alpha\}\big) + c_1'(\boldsymbol{\lambda})\phi. \tag{1.2}$$

Remark 1.1. The constants $c_1'(\boldsymbol{\lambda})$ and $c_2'(\boldsymbol{\lambda})$ can in general be bounded as follows:

$$c_1'(\boldsymbol{\lambda}) \leq e^{\mu} \quad \text{and} \quad c_2'(\boldsymbol{\lambda}) \leq \min\{1, (C/\lambda_1)\}e^{\mu},$$

where C is a constant. If also $i\lambda_i \geq (i+1)\lambda_{i+1}$ for all i, there are better bounds

$$c_1'(\boldsymbol{\lambda}) \leq \min\Big\{1, \frac{1}{\sqrt{\lambda_1 - 2\lambda_2}}\Big[2 - \frac{1}{\sqrt{\lambda_1 - 2\lambda_2}}\Big]\Big\}$$

and

$$c_2'(\boldsymbol{\lambda}) \leq \min\Big\{1, \frac{1}{2(\lambda_1 - 2\lambda_2)}[1 + 2\log^+\{2(\lambda_1 - 2\lambda_2)\}]\Big\},$$

as follows from [1], where $\log^+ x$ denotes $\max\{0, \log x\}$. When $\lambda_1 - 2\lambda_2$ is large, these bounds make for much tighter estimates than are obtainable from the results of Chapter 10.4 of [6]. In the reliability setting, this is usually less important than the fact that Theorem 1 can be applied in a rather routine fashion, whereas to translate the results of Chapter 10.4 of [6] so as to apply to complicated reliability models would require considerable effort.

Remark 1.2. Theorem 1 is very general, and allows one to make arbitrary choices of Γ_α^{vs}, Γ_α^b and Γ_α^{vw}. Taking $\Gamma_\alpha^{vs} = \emptyset$ gives the Poisson 'local' approach of [4], though, for $\mathrm{Po}(\lambda)$, one has the better bound $c_2(\lambda) \leq \min\{1, 1/\lambda\}$. Other choices of Γ_α^{vs} are usually better if the I_α tend to cluster, because in the estimate (1.2) there is no term of the form $\mathbb{E}(I_\alpha U_\alpha)$. It often helps to choose Γ_α^{vw} in such a way that $\phi = 0$. The examples which follow show that this can frequently be achieved in the reliability context.

Remark 1.3. Note that Theorem 1 can also be applied in many more complicated graph-based reliability models, as, for instance, when the failure of one vertex or edge increases the likelihood of its immediate neighbours failing. It can also be applied in graph-based reliability models with different spatial structures, for example on the line, circle, plane, torus, cylinder or sphere.

To determine the approximating compound Poisson distribution in Theorem 1, one has to compute λ_i for $i \geq 1$. This is not always a simple matter. However, it is sometimes possible to approximate the random variable W by a compound Poisson distribution determined by a smaller number of λ_i's, for which the computations are more tractable. The next theorem is of help in this respect.

Theorem 2. *With the notation of Theorem 1, let* $\boldsymbol{\lambda}^*$ *be defined by* $\lambda_1^* = \lambda_1 + \sum_{i \geq l+1} i\lambda_i$, $\lambda_j^* = \lambda_j$ *for* $j = 2, \ldots, l$ *and* $\lambda_j^* = 0$ *for* $j \geq l+1$ *and set* $\mu^* = \sum_{i=1}^{l} \lambda_i^*$. *Then*

$$|\mathbb{P}[W = 0] - e^{-\mu^*}|$$
$$\leq c_2'(\boldsymbol{\lambda}^*)\Big(\sum_{\alpha \in \Gamma} \big((\mathbb{E}I_\alpha)^2 + \mathbb{E}I_\alpha \mathbb{E}\{U_\alpha + X_\alpha\} + \mathbb{E}\{I_\alpha X_\alpha\}\big) + \sum_{i \geq l+1} i(i-1)\lambda_i \Big) + c_1'(\boldsymbol{\lambda}^*)\phi.$$

□

The proofs of Theorems 1 and 2 can be found in [14]: in fact, the same bounds are shown to be valid also for $|\mathbb{P}[W \in A] - \mathbb{P}[\mathrm{CP}(\boldsymbol{\lambda}) \in A]|$, for any subset $A \in \mathbb{Z}^+$.

The paper is organized as follows. In Section 2, we consider consecutive-2 systems. Our results extend those of [8,16]. In Section 3 we define and consider connected-s systems. This problem seems to be new in the literature. In Section 4 we consider the 2-dimensional consecutive-k-out-of-n: F system and continue the work of [9,11,15]. In Section 5 a double pipeline model is considered.

2. Consecutive-2 systems

A consecutive-2 system is a graph $G(V, E)$ with n vertices and N edges, where vertices are subject to failure. If instead edges were subject to failure, the problem could be transformed into this form by taking the dual graph. The graph is considered failed if any pair of adjacent vertices fail. The SF–subsets in this system are pairs of items connected by an edge in the graph, so we take $\Gamma = E$. With each vertex v_l is associated a Bernoulli random variable Y_l, where $\mathbb{P}[Y_l = 0] = \mathbb{P}[\text{the item in } v_l \text{ fails (is down)}] = q_l$ for $l = 1, \ldots, n$. Set $I_{kl} = I[Y_k = Y_l = 0]$ for all k and l such that the edge $\{k, l\} \in E$, and let $W = \sum_{\{k,l\}} \sum_{\in E} I_{kl}$.

In what follows, as in [8,16], we consider the case where $Y_1, \ldots, Y_n$ are independent. Let $D(l)$ denote the degree of the vertex v_l in the graph and let $D = \max_{1 \leq l \leq n} D(l)$; set $N_j(\alpha) = \{l : d(l, \alpha) = j\}$, where $d(l, \alpha)$ is the distance in G from l to the nearest point of α. Let $S_2(\alpha) \subset N_1(\alpha)$ be defined by $S_2(\alpha) = \big\{s : \{s, k\} \in E \text{ and } \{s, l\} \in E\big\}$, and set

$S_1(\alpha) = N_1(\alpha) \setminus S_2(\alpha)$. Take $\Gamma_\alpha^{vs} = \big\{\{i,l\} \in E : i \in \alpha,\ l \in N_1(\alpha)\big\}$, consisting of edges with one vertex of α and one other vertex; $\Gamma_\alpha^b = \big\{\{i,l\} \in E; i \in N_1(\alpha),\ l \in N_1(\alpha) \cup N_2(\alpha)\big\}$ and $\Gamma_\alpha^{vw} = \Gamma \setminus \big\{\{\alpha\} \cup \Gamma_\alpha^{vs} \cup \Gamma_\alpha^b\big\}$. Note that $\phi = 0$ with this choice of Γ_α^{vs} and Γ_α^b, and that, for $\alpha = \{k,l\}$,

$$|\Gamma_\alpha^{vs}| \le D(k) + D(l) - 2 \le 2(D-1)$$

and

$$\begin{aligned} |\Gamma_\alpha^b| \le & |S_2(\alpha)| \max_{s \in S_2(\alpha)} \{D(s) - 2\} \\ & + (D(k) + D(l) - 2 - 2|S_2(\alpha)|) \max_{s \in N_1(\alpha) \setminus S_2(\alpha)} \{D(s) - 1\} \le 2(D-1)^2. \end{aligned} \tag{2.1}$$

Finally, let $q_{max} = \max_{1 \le i \le n} q_i$.

Theorem 3. *For the consecutive-2 system,*

$$\begin{aligned} |\mathbb{P}[W=0] - e^{-\mu}| &\le c_2'(\boldsymbol{\lambda}) \sum_{\alpha \in \Gamma} \prod_{i \in \alpha} q_i \Big(\prod_{i \in \alpha} q_i + \sum_{\beta \in \Gamma_\alpha^{vs}} \prod_{j \in \beta} q_j + 2 \sum_{\beta \in \Gamma_\alpha^b} \prod_{j \in \beta} q_j \Big) \\ &\le c_2'(\boldsymbol{\lambda})(4D^2 - 6D + 3) q_{max}^2 \mathbb{E}W, \end{aligned}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$, $\mu = \sum_{i \ge 1} \lambda_i$, $\lambda_i = \frac{1}{i} \sum_{\alpha \in \Gamma} \mathbb{E}\{I_\alpha I[I_\alpha + \sum_{\beta \in \Gamma_\alpha^{vs}} I_\beta = i]\}$ *and* $\mathbb{E}W = \sum_{\alpha \in \Gamma} \prod_{i \in \alpha} q_i = \sum_{\{k,l\} \in \Gamma} \sum q_k q_l = \sum_{i \ge 1} i\lambda_i$. □

Remark 2.1. The Poisson 'local' approach here gives

$$|\mathbb{P}[W=0] - e^{-\lambda}| \le c_2(\lambda)\lambda((2D-1)q_{max}^2 + 2(D-1)q_{max}),$$

where $\lambda = \mathbb{E}W$. This estimate is poorer than that of Theorem 3 by a factor of order $(Dq_{max})^{-1}$, if Dq_{max} is small and $c_2(\lambda)$ and $c_2'(\boldsymbol{\lambda})$ are of the same order, as for instance when $i\lambda_i$ is decreasing and $\lambda_1 - 2\lambda_2 > \lambda/2$.

Remark 2.2. Setting

$$X_1(\alpha) = \sum_{\substack{\beta \in \Gamma_\alpha^{vs} \\ |\beta \cap S_1(\alpha)| = 1}} I_\beta \quad \text{and} \quad X_2(\alpha) = \sum_{\substack{\beta \in \Gamma_\alpha^{vs} \\ |\beta \cap S_2(\alpha)| = 1}} I_\beta,$$

we have

$$\begin{aligned}\lambda_i &= \frac{1}{i}\sum_{\alpha\in\Gamma}\mathbb{E}\{I_\alpha I[X_1(\alpha)+2X_2(\alpha)+I_\alpha=i]\}\\ &= \frac{1}{i}\sum_{\alpha\in\Gamma}\pi_\alpha\mathbb{P}[X_1(\alpha)+2X_2(\alpha)=i-1|I_\alpha=1]\end{aligned} \tag{2.2}$$

for $i \geq 1$. If all the failure probabilities are equal to q, then, conditional on $I_\alpha = 1$, $X_1(\alpha) \sim \mathrm{Bi}(|S_1(\alpha)|, q)$ and $X_2(\alpha) \sim \mathrm{Bi}(|S_2(\alpha)|, q)$ are independent. Hence the λ_i can be simply computed.

For example, take the underlying graph to be a complete d-regular tree: that is, a rooted tree in which every non-leaf vertex has degree $d \geq 2$ and all the leaves are at the same distance from the root. The height of such a tree is the distance from a leaf to the root. Let T_t denote the complete d-regular tree of height t. For t fixed, there are $n = 1 + d(\frac{(d-1)^t-1}{d-2})$ vertices, $n_0 = d(d-1)^{t-1}$ leaves, $N = n-1$ edges and $N_0 = n_0$ terminal edges (in which one of the vertices is a leaf). Assume that all failure probabilities are equal to q, so that $\lambda = \mathbb{E}W = Nq^2$. Then we can apply Theorem 3. To compute λ_i's, observe that a tree has no cycles so that $S_2(\alpha) = \emptyset$ for each α, and that $|S_1(\alpha)| = d-1$ for terminal edges α, $|S_1(\alpha)| = 2(d-1)$ otherwise. Hence, from (2.2) and because $X_2(\alpha) = 0$, we have

$$\lambda_2 = q^3(d-1)\Big\{\frac{1}{2}N_0(1-q)^{d-2} + (N-N_0)(1-q)^{2d-3}\Big\}$$

and

$$\begin{aligned}\sum_{i=3}^{2d-1} i(i-1)\lambda_i &= q^3(d-1)\Big\{N_0(1-(1-q)^{d-1}) + 2(N-N_0)(1-(1-q)^{2d-3})\Big\}\\ &\leq Nq^4(d^2+d-4).\end{aligned}$$

Thus, instead of computing all the λ_i's, take

$$\lambda_1^* = \lambda - 2\lambda_2,\ \lambda_2^* = \lambda_2 \text{ and } \lambda_i^* = 0, i \geq 3,$$

and use Theorem 2 in conjunction with Theorem 3 to give for $\mu^* = \lambda_1^* + \lambda_2^*$

$$|\mathbb{P}[W=0] - e^{-\mu^*}| \leq c_2'(\boldsymbol{\lambda}^*)Nq^4\Big\{(4d^2-6d+3) + (d^2+d-4)\Big\}.$$

Since

$$\lambda_1^* - 2\lambda_2^* \geq Nq^2(1 - 4q(d-1)) \geq 0$$

whenever $q \leq 1/\{4(d-1)\}$, it follows from Remark 1.1 with $\boldsymbol{\lambda}^*$ instead of $\boldsymbol{\lambda}$ that, for such q,

$$c_2'(\boldsymbol{\lambda}^*) \leq \Big(\frac{1}{2} + \log^+[2Nq^2(1-4q(d-1))]\Big) \Big/ \Big(Nq^2(1-4q(d-1))\Big),$$

and hence, for instance, that

$$|\mathbb{P}[W=0] - \exp\{-(Nq^2 - \lambda_2)\}| \leq (1 + 2\log^+(Nq^2))q^2(5d^2 - 5d - 1),$$

whenever $q \leq 1/\{8(d-1)\}$.

In particular, if $t \to \infty$ and $q \to 0$ in such a way that $\mathbb{E}W = Nq^2$ is constant, then the order of approximation is q^2. This contrasts with the order q obtainable for Poisson approximation.

3. Connected-s systems

In this section we generalize the notion of consecutive–2 systems to connected–s systems. We again take the vertices to be independently subject to failure: for edges, use the dual graph. Assume that the SF–subsets are a collection of s–vertex subsets, connected in the underlying graph, and that the system fails if all items in an SF–subset fail.

Let Γ denote the set of all SF–subsets, and let $\alpha = \{k_1, k_2, \ldots, k_s\}$ denote its typical element, where $k_1, k_2, \ldots, k_s$ are the indices of the s vertices. Suppose first that $\Gamma_\alpha^{vs} = \{\beta \in \Gamma : \ \beta \neq \alpha, \ \beta \cap \alpha \neq \emptyset\}$ and that $\Gamma_\alpha^{vw} = \big\{\gamma \in \Gamma : \ \gamma \cap \beta = \emptyset \text{ for all } \beta \in \Gamma_\alpha^{vs} \cup \{\alpha\}\big\}$. Then $\phi = 0$ in (1.2), and X_α and I_α are independent. Hence the right hand side of (1.2) is bounded by $Qc_2'(\boldsymbol{\lambda})\mathbb{E}W$, where

$$Q = q_{max}^s \Big\{1 + \max_{\alpha\in\Gamma} |\Gamma_\alpha^{vs}| + 2 \max_{\alpha\in\Gamma} |\Gamma_\alpha^b|\Big\},$$

and q_{max} is the largest failure probability of an individual item. Thus, for equal q's, if $|\Gamma|$ is large and q small, but $|\Gamma|q^s$ is of order 1, compound Poisson approximation is reasonable if $\max_{\alpha\in\Gamma}|\Gamma_\alpha^{vs}\cup\Gamma_\alpha^b|$ is much less than $|\Gamma|$. In particular, if s is fixed and the underlying graph is part of a fixed regular lattice having M vertices, M growing larger as q decreases in such a way that $|\Gamma|q^s\to c>0$, then the approximation error is of order q^s. This is typically much better than the error in Poisson approximation.

If, however, the structure of this particular Γ_α^{vs} is complicated, it may be difficult to compute the λ_i's, except either numerically or by using computer algebra. In such cases, a smaller Γ_α^{vs} could be chosen, typically leading to an error estimate of larger order, in exchange for explicitly computable λ_i's. An illustration of this is given in the next section.

4. The two dimensional consecutive-k-out-of-n system

The two dimensional consecutive-k-out-of-n system consists of n^2 components placed on a square grid of size n. It fails if there exists a square subgrid of size k $(1<k<n)$ with all k^2 components failed: see [9,11,15].

The system is a particular example of a connected-k^2 system. For any element $\alpha\in\Gamma=\{(r,s):1\le r,s\le n-k+1\}$, let $A_\alpha=A_{rs}$ denote the $k\times k$ subgrid with left lowermost component (r,s) i.e. $A_{rs}=\{(r+x-1,s+y-1):x,y=1,\dots,k\}$. Define $I_\alpha=I[\text{all items in } A_\alpha \text{ are failed}]$ for each $\alpha\in\Gamma$ and $W=\sum_{\alpha\in\Gamma}I_\alpha$. The random variable W counts the number of possibly overlapping $k\times k$ squares with all items failed in the system, and the reliability of the system is just $\mathbb{P}[W=0]$.

To apply Theorem 1, we need to specify Γ_α^{vs} and Γ_α^{vw}. For the latter, we use the standard choice

$$\Gamma_\alpha^{vw}=\big\{\gamma\in\Gamma:\ \gamma\cap\beta=\emptyset \text{ for all } \beta\in\Gamma_\alpha^{vs}\cup\{\alpha\}\big\},$$

so as to make $\phi=0$ in (1.2). If we take $\Gamma_\alpha^{vs}=\emptyset$, we recover the Poisson 'local' approach, as in [11]. It is shown there that

$$|\mathbb{P}[W=0]-e^{-\mathbb{E}W}|\le\epsilon_1, \tag{4.1}$$

where

$$\epsilon_1 = (1 - e^{-\mathbb{E}W})\Big\{(2k-1)^2 q_{max}^{k^2} + 4(\sum_{i=1}^{k}\sum_{j=1}^{k} q_{max}^{k^2-ij} - 1)\Big\},$$

with q_{max}, as before, the largest of the failure probabilities q_{ij}. Note that ϵ_1 can be improved a little to

$$\epsilon_1' = (1 - e^{-\mathbb{E}W})\Big\{(2k-1)^2 q_{max}^{k^2} + 4(\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} q_{max}^{k^2-ij} + \sum_{j=1}^{k-1} q_{max}^{k^2-kj})\Big\}, \tag{4.2}$$

with $\epsilon_1' \sim \epsilon_1/2 \sim 4q_{max}^k(1 - e^{-\mathbb{E}W})$ as $q_{max} \to 0$.

Now take $\Gamma_\alpha^{vs} = \{\beta \in \Gamma :\ \beta \neq \alpha,\ |\beta \cap \alpha| = k^2 - k\}$. Such squares are shifted by one with respect to α in just one of the coordinates. There are $(n-k+1)^2$ possible positions of the $k \times k$ subgrid A_α in the $n \times n$ grid, of which 4 have $|\Gamma_\alpha^{vs}| = 2$ (corners), $4(n-k-1)$ have $|\Gamma_\alpha^{vs}| = 3$ (border) and $(n-k-1)^2$ have $|\Gamma_\alpha^{vs}| = 4$.

If the failure probabilities are equal to q for all items in the grid, then $\lambda = \mathbb{E}W = (n-k+1)^2 q^{k^2}$. Note that $|\Gamma_\alpha^{vs}| \leq 4$ and $|\Gamma_\alpha^{vs}| + |\Gamma_\alpha^{b}| \leq (2k+1)^2 - 1$. First, we compute the following components of the upper bound on the total variation distance given in Theorem 1:

$$\begin{aligned}
&\sum_{\alpha\in\Gamma} (\mathbb{E}I_\alpha)^2 = (n-k+1)^2 q^{2k^2},\\
&\sum_{\alpha\in\Gamma} \mathbb{E}I_\alpha \mathbb{E}\{U_\alpha + X_\alpha\} = \sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma_\alpha^{vs}\cup\Gamma_\alpha^{b}} \mathbb{E}I_\alpha \mathbb{E}I_\beta \leq (n-k+1)^2((2k+1)^2 - 1) q^{2k^2},\\
&\sum_{\alpha\in\Gamma} \mathbb{E}\{I_\alpha X_\alpha\} = \sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma_\alpha^{b}} \mathbb{E}\{I_\alpha I_\beta\}\\
&\qquad \leq (n-k+1)^2 q^{k^2}\Big((8k-4)q^{k^2} + 4(\sum_{u=1}^{k-1}\sum_{v=1}^{k-1} q^{k^2-uv} + \sum_{v=1}^{k-2} q^{k^2-kv})\Big)
\end{aligned} \tag{4.3}$$

and $\phi = 0$. Then adding the above terms gives the following upper bound

$$\begin{aligned}
&|\mathbb{P}[W=0] - e^{-\mu}|\\
&\leq c_2'(\boldsymbol{\lambda})(n-k+1)^2 q^{k^2}\Big((4k^2+12k-3)q^{k^2} + 4(\sum_{u=1}^{k}\sum_{v=1}^{k} q^{k^2-uv} + \sum_{v=1}^{k-2} q^{k^2-kv})\Big),
\end{aligned} \tag{4.4}$$

with $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$, $\mu = \sum_{i\geq 1} \lambda_i$ and

$$\begin{aligned}
\lambda_i =& \frac{1}{i} \sum_{\alpha\in\Gamma} \mathbb{E}\{I_\alpha I[I_\alpha + \sum_{\beta\in\Gamma_\alpha^{vs}} I_\beta = i]\} \\
=& \frac{1}{i} \sum_{r=1}^{n-k+1} \sum_{s=1}^{n-k+1} \mathbb{E}\{I_{rs} I[I_{rs} + I_{r,s-1} + I_{r-1,s} + I_{r,s+1} + I_{r+1,s} = i]\} \\
=& \frac{1}{i} q^{k^2} \{4\pi_1(i) + 4(n-k-1)\pi_2(i) + (n-k-1)^2 \pi_3(i)\}, \quad \text{for} \quad i = 1, \ldots, 5,
\end{aligned}$$

where

$$\begin{aligned}
\pi_1(i) =& \mathbb{P}[\sum_{\beta\in\Gamma_\alpha^{vs}} I_\beta = i-1 | I_\alpha = 1] = \mathbb{P}[\mathrm{Bi}(2, q^k) = i-1] \text{ for the corner indicators,} \\
\pi_2(i) =& \mathbb{P}[\sum_{\beta\in\Gamma_\alpha^{vs}} I_\beta = i-1 | I_\alpha = 1] = \mathbb{P}[\mathrm{Bi}(3, q^k) = i-1] \text{ for the border indicators,} \\
\pi_3(i) =& \mathbb{P}[\sum_{\beta\in\Gamma_\alpha^{vs}} I_\beta = i-1 | I_\alpha = 1] = \mathbb{P}[\mathrm{Bi}(4, q^k) = i-1] \text{ for the central indicators.}
\end{aligned}$$

A simpler result is obtained by invoking Theorem 3, taking $\lambda_1^* = \lambda - 2\lambda_2$ and

$$\lambda_2^* = \lambda_2 = 2q^{k^2+k}(1-q^k)\{(n-k-1)^2(1-q^k)^2 + 3(n-k-1)(1-q^k) + 2\};$$

then

$$\begin{aligned}
\sum_{i=3}^{5} i(i-1)\lambda_i =& 4q^{k^2+2k}\Big\{(n-k-1)^2\big(3(1-q^k)^2 + 3q^k(1-q^k) + q^{2k}\big) \\
&+ (n-k-1)\big(6(1-q^k) + 3q^k\big) + 2\Big\} \\
\leq& 4\lambda q^{2k}\Big(3 + \frac{2}{49}\Big)
\end{aligned} \tag{4.5}$$

and $\lambda_1^* - 2\lambda_2^* = \lambda - 4\lambda_2 \geq \lambda/2$, if $n \geq k+6$ and $q^k \leq 1/24$. This leads to the following theorem.

Theorem 4. *For the two dimensional consecutive-k-out-of-n system, when the failure probabilities are equal and $n \geq k+6$, $q^k \leq 1/24$, then*

$$\begin{aligned}
&|\mathbb{P}[W=0] - e^{-\lambda+\lambda_2}| \\
&\leq (1 + 2\log^+\lambda)\Big\{(4k^2+12k-3)q^{k^2} + 4\Big(\sum_{u=1}^{k-1}\sum_{v=1}^{k-1} q^{k^2-uv} + \sum_{v=1}^{k-2} q^{k^2-kv}\Big) + 13q^{2k}\Big\}.
\end{aligned} \tag{4.6}$$

□

Remark 4.1. The above bounds are of order $O(q^{2k-1})$. The argument presented above provides an improvement on the order $O(q^k)$, obtained when using the Poisson 'local' approach, by the factor q^{k-1}. Note that the best possible order would be $O(q^{k^2})$.

Remark 4.2. The above result can be extended to the case of unequal failure probabilities. The bound (4.6) is then true if q is replaced by q_{max} throughout, though λ and λ_2 now have to be calculated taking the unequal q_{rs} into account. These bounds may be rather crude, because of the use of q_{max}, if only few items in the system have large values of q_{rs}, and all others are highly reliable.

However, one can obtain a better upper bound for the failure probability by exact computation of the expression

$$\sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma_\alpha^{vs}\cup\Gamma_\alpha^b\cup\{\alpha\}}\mathbb{E}I_\alpha\mathbb{E}I_\beta+\sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma_\alpha^b}\mathbb{E}\{I_\alpha I_\beta\}$$

in the error estimate, without making approximations. This solution was programmed and computed in Mathematica, using the following formulae.

$$\sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma_\alpha^{vs}\cup\Gamma_\alpha^b\cup\{\alpha\}}\mathbb{E}I_\alpha\mathbb{E}I_\beta=\sum_{r=1}^{n-k+1}\sum_{s=1}^{n-k+1}x_1(r,s)x_2(r,s)$$

and (4.7)

$$\sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma_\alpha^b}\mathbb{E}\{I_\alpha I_\beta\}=\sum_{r=1}^{n-k+1}\sum_{s=1}^{n-k+1}x_1(r,s)x_3(r,s),$$

where

$$\begin{aligned}
x_1(r,s)&=\prod_{u=r}^{r+k-1}\prod_{v=s}^{s+k-1}q_{uv},\\
x_2(r,s)&=\sum_{a=r-k}^{r+k}\sum_{b=s-k}^{s+k}\prod_{u=a}^{a+k-1}\prod_{v=s}^{b+k-1}q_{uv},\qquad (4.8)\\
x_3(r,s)&=\sum_{a=r-k}^{r+k}\sum_{b=s-k}^{s+k}\Big(\prod_{u=a}^{a+k-1}\prod_{v=s}^{b+k-1}q^*_{uv}-\alpha_1^*-\alpha_2^*-\alpha_3^*-\alpha_4^*-1\Big)
\end{aligned}$$

and

$$\alpha_1^* = \prod_{u=r}^{r+k-1} q_{u,s-1}^*,\ \alpha_2^* = \prod_{v=s}^{s-k+1} q_{r-1,v}^*,\ \alpha_3^* = \prod_{u=r}^{r+k-1} q_{u,s+k}^* \text{ and } \alpha_4^* = \prod_{v=s}^{s+k-1} q_{r+k,v}^*,$$

with $q_{uv} = q_{uv}^* = 0$ if at least one of the indices $u, v \in \{-k+1, \ldots, 0, n+1, \ldots, n+k\}$ and $q_{uv}^* = 1$ for $u = r, \ldots, r+k-1$, $v = s, \ldots, s+k-1$ for $1 \le r, s \le n-k+1$. Numerically, the $n \times n$ matrix was imbedded in the centre of an $(n+2k) \times (n+2k)$ matrix, whose remaining cells were set equal to 0. The λ_i's were also computed, using the formulae

$$\lambda_i = \frac{1}{i} \sum_{r=1}^{n-k+1} \sum_{s=1}^{n-k+1} \prod_{u=r}^{r+k-1} \prod_{v=s}^{s+k-1} q_{uv}\, Q^{(i)}, \tag{4.9}$$

where

$$Q^{(1)} = \prod_{l=1}^{4}(1-\alpha_l); \quad Q^{(2)} = Q^{(1)} \sum_{j=1}^{4} \frac{\alpha_j}{1-\alpha_j};$$

$$Q^{(3)} = Q^{(1)} \sum_{j=1}^{3} \sum_{k=j+1}^{4} \frac{\alpha_j}{1-\alpha_j} \frac{\alpha_k}{1-\alpha_k}; \quad Q^{(4)} = Q^{(5)} \sum_{j=1}^{4} \frac{1-\alpha_j}{\alpha_j}; \quad Q^{(5)} = \prod_{l=1}^{4} \alpha_l.$$

Theorem 5. *For the two dimensional consecutive-k-out-of-n system, when the failure probabilities q_{rs} are unequal,*

$$|\mathbb{P}[W=0] - e^{-\mu}| \le \epsilon_3, \tag{4.10}$$

where $\mu = \sum_{i=1}^{5} \lambda_i$, $c_2'(\boldsymbol{\lambda})$ is bounded as in Remark 1.1 and the remaining notation is as defined above. □

The adaptation of the results in this section to more general models, such as the consecutive-k_1k_2 system, with rectangular subgrids in the rectangular $n_1 \times n_2$ grid, and its generalizations to higher dimensions, would in principle be rather straightforward: see also [9].

In Tables 1 and 2, bounds computed according to Theorem 5 are compared with the results of simulations, for two models already considered in the literature:

Model I

$$q_{ij} = \begin{cases} 0.3 & \text{if } (i+j) \text{ odd}, \\ 0.25 & \text{if } (i+j) \text{ even}, \end{cases} \quad \text{for} \quad 1 \le i, j \le n, \tag{4.11}$$

and

Model II

$$q_{ij} = \begin{cases} 0.5 & \text{if } |i-j| \le 1, \\ \frac{1}{|i-j|} & \text{if } |i-j| \ge 2, \end{cases} \quad \text{for} \quad 1 \le i, j \le n. \tag{4.12}$$

We also compute bounds based on Poisson approximation, using the analogue of (4.2) for unequal failure probabilities.

The results computed using compound Poisson (4.10) and Poisson approximation, given in Tables 1 and 2 for Models I and II respectively, indicate that the compound Poisson bounds are a substantial improvement over the Poisson bounds. In the case $k = 2$, neither works well. For the other values of k, the difference between the upper and lower compound Poisson bounds is smaller than that between the bounds in [9].

5. The double pipeline

Consider a double pipeline with a total of n junctions, where n is an even integer, each of the two pipelines having $\frac{n}{2}$ junctions. In the underlying graph there are two lines of $\frac{n}{2}$ vertices. Label the vertices of the first line by the even integers $i = 0, 2, 4, \ldots, n-2$, and the second line with $i = 1, 3, 5, \ldots, n-1$. In each line there are connections between consecutive vertices, so that $(i, i+2) \in E$ for $i = 0, 1, \ldots n-3$. In order to improve the reliability of the system, there are also additional connections between the lines, with $(i, i+1) \in E$ for $i = 1, 2, 3, \ldots, n-3$. Call the first two vertices labelled by 0 and 1 sources, and the last two vertices labelled by $n-2$ and $n-1$ sinks. The edges (pipes) are supposed to be up or down with equal probabilities of failure q. Any combination of the failed edges which does not allow any flow between the source and the sink will be called a cut. The possible cuts have the following form: one edge from each line, $(i, i+2)$ and $(i+k, i+k+2)$ say, with k odd, $1 \le k \le n-3$, and the k consecutive interconnecting

edges $(i+j, i+j+1)$, $1 \le j \le k$. The system fails if W, the number of cuts in the system, is at least one.

To apply the methods of Section 1, we transfer attention to the dual graph G'. Assign label $i+j$ to the vertex representing the edge (i,j), $(i,j) \in E$. In the dual graph the vertices are supposed to be failed with equal probabilities q and the cuts defined in the initial graph have following representation. The first and the last vertices in the cut have even labels $i < j$, one of them divisible by 4 and the other not. The rest of the vertices in the cut have as labels the odd numbers between i and j. Note that the number of vertices in each cut is equal to $\frac{1}{2}(j-i)+2$. Let $\alpha(k,l)$ denote the cut of length k with leftmost vertex l. Then

$$\Gamma = \{\alpha(2u+1, 2s) : 1 \le u \le \frac{n-2}{2}, 1 \le s \le n-2u-1\}.$$

Define $W = \sum_{\alpha\in\Gamma} I_\alpha$, where $I_\alpha = I[\text{all vertices in the cut } \alpha \text{ are failed}]$. Note that $\mathbb{E}W = \sum_{\alpha\in\Gamma} \mathbb{E}I_\alpha = (n-3)q^3 + (n-5)q^5 + O(nq^7)$. The shortest cuts all have probability q^3 of occurring, and the error in the Poisson approximation to the number occuring would be of order q^3, if they were all independent. To obtain an error of order q^3 for our compound Poisson approximation, it is enough to define

$$\Gamma^{vs}_{\alpha(3,2)} = \{\alpha(3,4)\}, \quad \Gamma^{vs}_{\alpha(3,2n-6)} = \{\alpha(3,2n-8)\},$$

$$\Gamma^{vs}_{\alpha(3,2s)} = \{\alpha(3,2s-2), \alpha(3,2s+2)\}, \quad \text{for} \quad 2 \le s \le n-4,$$

$$\Gamma^{vs}_{\alpha(2u+1,2s)} = \emptyset \quad \text{for} \quad u \ge 2,$$

$$\Gamma^{b}_{\alpha(3,2s)} = \{\alpha(2r+1, 2t) : 1 \le r \le \frac{n-2}{2},$$
$$(s-2r) \vee 1 \le t \le (s+2) \wedge (n-2r-1)\}$$
$$\setminus \{\Gamma^{vs}_{\alpha(3,2s)} \cup \{\alpha(3,2s)\}\}, \quad \text{for} \quad 1 \le s \le n-3,$$

$$\Gamma^{b}_{\alpha(2u+1,2s)} = \{\alpha(2r+1, 2t) : 1 \le r \le \frac{n-2}{2},$$
$$(s-2r+1) \vee 1 \le t \le (s+2u-1) \wedge (n-2r-1)\}$$
$$\setminus \{\alpha(2u+1, 2s)\}, \quad \text{for} \quad 1 \le s \le n-2u-1 \quad \text{and} \quad 2 \le u \le \frac{n-2}{2}.$$

Then

$$\lambda_i = \frac{1}{i}\sum_{\alpha\in\Gamma}\mathbb{E}\{I_\alpha I[I_\alpha + \sum_{\beta\in\Gamma_\alpha^{vs}} I_\beta = i]\}, \quad i = 1,2,3$$

because $\max_{1\le s\le n-3}\{|\Gamma^{vs}_{\alpha(3,2s)}|\} = 2$ and $|\Gamma^{vs}_{\alpha(2u+1,2s)}| = 0$ for $u \ge 2$. Thus, by direct calculation,

$$\begin{aligned}
\lambda_1 &= (n-3)q^3(1-q^2) + \frac{(n-7)q^7 - (n-5)q^9 + q^{n-2} + q^{n-4}}{(1-q^2)^2},\\
\lambda_2 &= q^5\{(n-5)(1-q^2)+1\} \quad \text{and} \quad \lambda_3 = \frac{(n-5)}{3}q^7;
\end{aligned} \tag{5.1}$$

for $q \le 1/3$, note that $\lambda_1 - 2\lambda_2 \ge (n-3)q^3(1-q^2)^3$ and that $2\lambda_2 \ge 3\lambda_3$.

For the upper bound, we have $\phi = 0$ and, after more computation,

$$\begin{aligned}
&\sum_{\alpha\in\Gamma}\{(\mathbb{E}I_\alpha)^2 + \mathbb{E}I_\alpha\mathbb{E}\{U_\alpha + X_\alpha\} + \mathbb{E}\{I_\alpha X_\alpha\}\}\\
&\qquad \le (n-3)q^6\{\frac{1}{(1-q^4)} + \frac{4+4q}{(1-q^2)^2} + \frac{5+3q}{1-q^2}\}\\
&\qquad + (n-5)q^8\{\frac{4+2q^3}{(1-q^2)^3} + \frac{3+5q+5q^2}{(1-q^2)^2} + \frac{2+2q}{1-q^2}\}\\
&\qquad \le (n-3)q^6\,\frac{263}{16}
\end{aligned} \tag{5.2}$$

when $q \le 1/3$. Using Theorem 1, we now have the following result.

Theorem 6. *For the double pipeline with equal probabilities $q \le \frac{1}{3}$ of failure of a connection, the reliability can be estimated by*

$$|\mathbb{P}[W=0] - e^{-\mu}| \le [1 + 2\log^+ 2nq^3]12q^3,$$

where $\mu = \lambda_1 + \lambda_2 + \lambda_3$ is as given in (5.1). □

Thus the order of the error is essentially $O(q^3)$, as desired. Note that the consecutively connected systems introduced in [17] do not include this model, but that his systems could also be analysed using Stein's method.

Tables. *Lower and upper bounds for the reliability of a two dimensional consecutive-k-out-of-n system.*

The simulations are based on $N = 100'000$ trials.

*** period?? Function: Random[ ], Mathematica 1.2 ***

Stderr= $\sqrt{\frac{p(1-p)}{N}}$.

LPo and UPo: lower and upper Poisson bounds

LCP and UCP: lower and upper compound Poisson bounds

LFK and UFK: from Fu and Koutras [9]

Table 1.

Model I as defined in (4.11)

n	k	LPo	LFK	LCP	Simul	UCP	UFK	UPo	Stderr
3	2	0.97341	0.97768	0.97834	0.97886	0.98038	0.98337	0.98209	0.00045
5	2	0.88750	0.91369	0.90928	0.92273	0.93620	0.94481	0.94036	0.00084
5	3	0.99991	0.99992	0.99992	0.99986	0.99993	0.99995	0.99993	0.00004
10	2	0.49672	0.63323	0.56496	0.66862	0.77732	0.77082	0.77138	0.00149
10	3	0.99940	0.99944	0.99945	0.99937	0.99947	0.99968	0.99949	0.00008
20	2	-0.22092	0.13050	-0.34719	0.17089	0.69066	0.32924	0.48344	0.00119
20	3	0.99694	0.99718	0.99726	0.99759	0.99732	0.99845	0.99743	0.00016
50	2	-0.42230	0.00000	-1.56480	0.00000	1.56482	0.00074	0.42231	0.00000
50	3	0.97838	0.98015	0.98074	0.98100	0.98111	0.98928	0.98192	0.00043
50	4	0.99999	0.99999	0.99999	0.99999	1.	1.	1.	0.00001
100	3	0.91261	0.91983	0.92210	0.92226	0.92367	0.95639	0.92705	0.00085
100	4	0.99999	0.99999	0.99999	0.99998	1.	1.	1.	0.00001

Table 2.

Model II as defined in (4.12)

n	k	LPo	LFK	LCP	Simul	UCP	UFK	UPo	Stderr
3	2	0.58525	0.77247	0.73097	0.81618	0.91848	0.86608	0.97235	0.00122
5	2	-0.29922	0.43016	-0.46071	0.53674	1.58230	0.71558	1.18058	0.00158
5	3	0.98423	0.98812	0.98867	0.98944	0.99044	0.99421	0.99204	0.00032
5	4	0.99997	0.99998	0.99998	1.	0.99998	0.99999	0.99999	0.00000
10	2	-1.23483	0.08723	-3.72406	0.16708	4.11407	0.43418	1.42096	0.00118
10	3	0.94475	0.96105	0.96125	0.96819	0.97171	0.98512	0.97743	0.00055
10	4	0.99989	0.99990	0.99991	0.99993	0.99992	0.99996	0.99992	0.00003
20	2	-1.48958	0.00348	-9.70771	0.01641	9.75359	0.15696	1.49764	0.00040
20	3	0.86893	0.90909	0.90792	0.92271	0.93591	0.96720	0.94939	0.00084
20	4	0.99973	0.99976	0.99977	0.99985	0.99978	0.99991	0.99979	0.00004
50	2	-1.51572	0.00000	-27.1654	0.00001	27.1655	0.00737	1.51572	0.00001
50	3	0.66519	0.76947	0.75990	0.80439	0.84044	0.91537	0.87404	0.00125
50	4	0.99925	0.99932	0.99935	0.99930	0.99937	0.99975	0.99940	0.00008
100	3	0.39270	0.58277	0.54786	0.63615	0.71599	0.83509	0.77329	0.00152
100	4	0.99845	0.99860	0.99865	0.99864	0.99868	0.99948	0.99875	0.00012

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