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Research Article

On the Conharmonic Curvature Tensor of Generalized Sasakian-Space-Forms

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The object of the present paper is to characterize generalized Sasakian-space-forms satisfying certain curvature conditions on conharmonic curvature tensor. In this paper we study conharmonically semisymmetric, conharmonically flat, ξ-conharmonically flat, and conharmonically recurrent generalized Sasakian-space-forms. Also generalized Sasakian-space-forms satisfying $\tilde{\tilde{C}} \cdot S = 0$ and $\tilde{\tilde{C}} \cdot R = 0$ have been studied.

1. Introduction

Conformal transformations of a Riemannian structures are an important object of study in differential geometry. Of considerable interest in a special type of conformal transformations, conharmonic transformations, which are conformal transformations are preserving the harmonicity property of smooth functions. This type of transformation was introduced by Ishii [1] in 1957 and is now studied from various points of view. It is well known that such transformations have a tensor invariant, the so-called conharmonic curvature tensor. It is easy to verify that this tensor is an algebraic curvature tensor; that is, it possesses the classical symmetry properties of the Riemannian curvature tensor.

Let M and $\overline{M}$ be two Riemannian manifolds with g and $\overline{g}$ being their respective metric tensors related through

$$\overline{g}(X, Y) = e^{2\sigma} g(X, Y), \tag{1.1}$$

where σ is a real function. Then M and $\overline{M}$ are called conformally related manifolds, and the correspondence between M and $\overline{M}$ is known as conformal transformation [2].

It is known that a harmonic function is defined as a function whose Laplacian vanishes. A harmonic function is not invariant, in general. The conditions under which a harmonic function remains invariant have been studied by Ishii [1] who introduced the conharmonic transformation as a subgroup of the conformal transformation (1.1) satisfying the condition

$$\sigma_{,i}^{\ i} + \sigma_{,i}\sigma_{,}^{\ i} = 0, \tag{1.2}$$

where comma denotes the covariant differentiation with respect to metric g. A rank-four tensor $\tilde{\tilde{C}}$ that remains invariant under conharmonic transformation for $(2n+1)$-dimensional Riemannian manifold is given by

$$\begin{aligned}{}'\tilde{\tilde{C}}(X,Y,Z,U) = {}'R(X,Y,Z,Z) - \frac{1}{(2n-1)} \\ \times \left[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) + S(Y,Z)g(X,U) - S(X,Z)g(Y,U)\right],\end{aligned} \tag{1.3}$$

where $'R$ and S denote the Riemannian curvature tensor of type $(0,4)$ defined by $'R(X,Y,Z,U) = g(R(X,Y)Z,U)$ and the Ricci tensor of type $(0,2)$, respectively.

The curvature tensor defined by (1.3) is known as conharmonic curvature tensor. A manifold whose conharmonic curvature tensor vanishes at every point of the manifold is called conharmonically flat manifold. Thus this tensor represents the deviation of the manifold from canharmonic flatness. Conharmonic curvature tensor has been studied by Abdussattar [3], Siddiqui and Ahsan [2], Özgür [4], and many others.

Let M be an almost contact metric manifold equipped with an almost contact metric structure (ϕ,ξ,η,g). At each point $p \in M$, decompose the tangent space T_pM into the direct sum $T_pM = \phi(T_pM) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the 1-dimensional linear subspace of T_pM generated by ξ_p. Thus the conformal curvature tensor C is a map

$$C : T_pM \times T_pM \times T_pM \longrightarrow \phi(T_pM) \oplus \{\xi_p\}, \quad p \in M. \tag{1.4}$$

An almost contact metric manifold M is said to be

(1) conformally symmetric [5] if the projection of the image of C in $\phi(T_pM)$ is zero,

(2) ξ-conformally flat [6] if the projection of the image of C in $\{\xi_p\}$ is zero,

(3) ϕ-conformally flat [7] if the projection of the image of $C \mid T_pM \times T_pM \times T_pM$ in $\phi(T_pM)$ is zero.

Here cases (1), (2), and (3) are synonymous to conformally symmetric, ξ-conformally flat and ϕ-conformally flat. In [5], it is proved that a conformally symmetric K-contact manifold is locally isometric to the unit sphere. In [6], it is proved that a K-contact manifold is ξ-conformally flat if and only if it is an η-Einstein Sasakian manifold. In [7], some necessary conditions for K-contact manifold to be ϕ-conformally flat are proved. Moreover, in [8] some conditions on conharmonic curvature tensor $\tilde{\tilde{C}}$ are studied which has many applications in physics and mathematics on a hypersurfaces in the semi-Euclidean space E_S^{n+1}. Also, it is shown that every conharmonically Ricci-semisymmetric hypersurface M satisfies the the condition $\tilde{\tilde{C}} \cdot R = 0$ is pseudosymmetric.

On the other hand a generalized Sasakian-space-form was defined by Alegre et al. [9] as the almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ whose curvature tensor R is given by

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3, \tag{1.5}$$

where f_1, f_2, f_3 are some differential functions on M and

$$\begin{aligned} R_1(X,Y)Z &= g(Y,Z)X - g(X,Z)Y, \\ R_2(X,Y)Z &= g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z, \\ R_3(X,Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \end{aligned} \tag{1.6}$$

for any vector fields X, Y, Z on M^{2n+1}. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. This kind of manifold appears as a generalization of the well-known Sasakian-space-forms by taking $f_1 = (c+3)/4$, $f_2 = f_3 = (c-1)/4$. It is known that any three-dimensional (α, β)-trans-Sasakian manifold with α, β depending on ξ is a generalized Sasakian-space-form [10]. Alegre et al. give results in [11] about B. Y Chen's inequality on submanifolds of generalized complex space-forms and generalized Sasakian-space-forms. Al-Ghefari et al. analyse the CR submanifolds of generalized Sasakian-space-forms [12, 13]. In [14], Kim studied conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms. De and Sarkar [15] have studied generalized Sasakian-space-forms regarding projective curvature tensor. Motivated by the above studies, in the present paper, we study flatness and symmetry property of generalized Sasakian-space-forms regarding conharmonic curvature tensor. The present paper is organized as follows.

In this paper, we study the conharmonic curvature tensor of generalized Sasakian-space-forms. In Section 2, some preliminary results are recalled. In Section 3, we study conharmonically semisymmetric generalized Sasakian-space-forms. Section 4 deals with conharmonically flat generalized Sasakian-space-forms. ξ-conharmonically flat generalized Sasakian-space-forms are studied in Section 5 and obtain necessary and sufficient condition for a generalized Sasakian-space-form to be ξ-conharmonically flat. In Section 6, conharmonically recurrent generalized Sasakian-space-forms are studied. Section 7 is devoted to study generalized Sasakian-space-forms satisfying $\tilde{\tilde{C}} \cdot S = 0$. The last section contains generalized Sasakian-space-forms satisfying $\tilde{\tilde{C}} \cdot R = 0$.

2. Preliminaries

If, on an odd-dimensional differentiable manifold M^{2n+1} of differentiability class C^{r+1}, there exists a vector valued real linear function ϕ, a 1-form η, the associated vector field ξ, and the Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \qquad \phi(\xi) = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \qquad g(X,\xi) = \eta(X), \qquad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

for arbitrary vector fields X and Y, then (M^{2n+1}, g) is said to be an almost contact metric manifold [16], and the structure (ϕ,ξ,η,g) is called an almost contact metric structure to M^{2n+1}. In view of (2.1), (2.2) and (2.3), we have

$$\begin{gathered} g(\phi X, Y) = -g(X, \phi Y), \qquad g(\phi X, X) = 0, \\ (\nabla_X \eta)(Y) = g(\nabla_X \xi, Y). \end{gathered} \tag{2.4}$$

Again we know [9] that in a $(2n+1)$-dimensional generalized Sasakian-space-form

$$\begin{aligned} R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned} \tag{2.5}$$

for all vector fields X, Y, Z on M^{2n+1}, where R denotes the curvature tensor of M^{2n+1}:

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.6}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.7}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3. \tag{2.8}$$

We also have for a generalized Sasakian-space-forms

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.9}$$

$$R(\xi,X)Y = -R(X,\xi)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{2.10}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)], \tag{2.11}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X), \tag{2.12}$$

$$Q\xi = 2n(f_1 - f_3)\xi, \tag{2.13}$$

where Q is the Ricci operator, that is, $g(QX,Y) = S(X,Y)$.

A generalized Sasakian space-form is said to be η-Einstein if its Ricci tensor S is of the form:

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.14}$$

for arbitrary vector fields X and Y, where a and b are smooth functions on M^{2n+1}. For a $(2n+1)$-dimensional $(n > 1)$ almost contact metric manifold the conharmonic curvature tensor $\tilde{\tilde{C}}$ is given by [17]:

$$\begin{aligned} \tilde{\tilde{C}}(X,Y)Z = {} & R(X,Y)Z - \frac{1}{(2n-1)} \\ & \times \left[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\right]. \end{aligned} \tag{2.15}$$

The conharmonic curvature tensor $\tilde{\tilde{C}}$ in a generalized Sasakian-space-form satisfies

$$\tilde{\tilde{C}}(X,Y)\xi = -\frac{(f_1 - f_3)}{(2n-1)}[\eta(Y)X - \eta(X)Y] - \frac{1}{(2n-1)}[\eta(Y)QX - \eta(X)QY], \tag{2.16}$$

$$\eta\left(\tilde{\tilde{C}}(X,Y)\xi\right) = 0, \tag{2.17}$$

$$\tilde{\tilde{C}}(\xi,Y)Z = -\frac{(f_1 - f_3)}{(2n-1)}[g(Y,Z)\xi - \eta(Z)Y] - \frac{1}{(2n-1)}[S(Y,Z)\xi - \eta(Z)QY], \tag{2.18}$$

$$\begin{aligned} \eta\left(\tilde{\tilde{C}}(\xi,Y)Z\right) = {} & -\frac{(f_1 - f_3)}{(2n-1)} \\ & \times [g(Y,Z) - \eta(Y)\eta(Z)] - \frac{1}{(2n-1)}[S(Y,Z) - 2n(f_1 - f_3)\eta(Y)\eta(Z)], \end{aligned} \tag{2.19}$$

$$\begin{aligned} \eta\left(\tilde{\tilde{C}}(X,Y)Z\right) = {} & -\frac{(f_1 - f_3)}{(2n-1)}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ & - \frac{1}{(2n-1)}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)]. \end{aligned} \tag{2.20}$$

3. Conharmonically Semisymmetric Generalized Sasakian-Space-Forms

Definition 3.1. A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form is said to be conharmonically semisymmetric [15] if it satisfies $R \cdot \tilde{\tilde{C}} = 0$, where R is the Riemannian curvature tensor, and $\tilde{\tilde{C}}$ is the conharmonic curvature tensor of the space-forms.

Theorem 3.2. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form is conharmonically semisymmetric if and only if $f_1 = f_3$.*

Proof. Let us suppose that the generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is conharmonically semisymmetric. Then we can write

$$R(\xi,U)\cdot\tilde{\tilde{C}}(X,Y)\xi = 0. \tag{3.1}$$

The above equation can be written as

$$R(\xi,U)\tilde{\tilde{C}}(X,Y)\xi - \tilde{\tilde{C}}(R(\xi,U)X,Y)\xi - \tilde{\tilde{C}}(X,R(\xi,U)Y)\xi - \tilde{\tilde{C}}(X,Y)R(\xi,U)\ \xi = 0. \tag{3.2}$$

In view of (2.10) the above equation reduces to

$$\begin{aligned}(f_1 - f_3)\Big[g\Big(U,\tilde{\tilde{C}}(X,Y)\xi\Big)\xi - \eta\Big(\tilde{\tilde{C}}(X,Y)\xi\Big)U - g(X,U)\tilde{\tilde{C}}(\xi,Y)\xi \\ + \eta(X)\tilde{\tilde{C}}(U,Y)\xi - g(U,Y)\tilde{\tilde{C}}(X,\xi)\xi + \eta(Y)\tilde{\tilde{C}}(X,U)\xi \\ -\eta(U)\tilde{\tilde{C}}(X,Y)\xi + \tilde{\tilde{C}}(X,Y)U\Big] = 0.\end{aligned} \tag{3.3}$$

Now, taking the inner product of above equation with ξ and using (2.2) and (2.17), we get

$$(f_1 - f_3)\Big[g\Big(U,\tilde{\tilde{C}}(X,Y)\xi\Big) + \eta\Big(\tilde{\tilde{C}}(X,Y)\ U\Big)\Big] = 0. \tag{3.4}$$

From the above equation, we have either $f_1 = f_3$ or

$$g\Big(U,\tilde{\tilde{C}}(X,Y)\xi\Big)\xi + \tilde{\tilde{C}}(X,Y)U = 0, \tag{3.5}$$

which by using (2.15) and (2.16) gives

$$g(Y,U)\eta(X) - g(X,U)\eta(Y) = 0, \tag{3.6}$$

which is not possible in generalized Sasakian-space-form. Conversely, if $f_1 = f_3$, then from (2.10), we have $R(\xi,U) = 0$. Then obviously $R\cdot\tilde{\tilde{C}} = 0$ is satisfied. This completes the proof. □

4. Conharmonically Flat Generalized Sasakian-Space-Forms

Theorem 4.1. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form is conharmonically flat if and only if $f_1 = 3f_2/(1-2n) = f_3$.*

Proof. For a $(2n+1)$-dimensional $(n>1)$ conharmonically flat generalized Sasakian-space-form, we have from (2.15)

$$R(X,Y)Z = \frac{1}{(2n-1)}\left[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\right]. \tag{4.1}$$

In view of (2.6) and (2.7) the above equation takes the form

$$\begin{aligned} R(X,Y)Z = \frac{1}{(2n-1)}&\left[2(2nf_1+3f_2-f_3)\{g(Y,Z)X-g(X,Z)Y\}\right.\\ &-(3f_2+(2n-1)f_3)\{g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi\}\\ &\left.-(3f_2+(2n-1)f_3)\{\eta(Y)X-\eta(X)Y\}\eta(Z)\right]. \end{aligned} \tag{4.2}$$

By virtue of (2.5) the above equation reduces to

$$\begin{aligned} &f_1\{g(Y,Z)X-g(X,Z)Y\}\\ &\quad+f_2\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\}\\ &\quad+f_3\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\}\\ &=\frac{1}{(2n-1)}\left[2(2nf_1+3f_2-f_3)\{g(Y,Z)X-g(X,Z)Y\}\right.\\ &\qquad-(3f_2+(2n-1)f_3)\{g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi\}\\ &\qquad\left.-(3f_2+(2n-1)f_3)\{\eta(Y)X-\eta(X)Y\}\eta(Z)\right]. \end{aligned} \tag{4.3}$$

Now, replacing Z by ϕZ in the above equation, we obtain

$$\left[(2n+1)f_1+3f_2-2f_3\right]\left[g(Y,\phi Z)\eta(X)-g(X,\phi Z)\eta(Y)\right]=0. \tag{4.4}$$

Putting $X=\xi$ in the above equation, we get

$$\left[(2n+1)f_1+3f_2-2f_3\right]g(Y,\phi Z)=0. \tag{4.5}$$

Since $g(Y,\phi Z)\neq 0$, in general, we obtain

$$(2n+1)f_1+3f_2-2f_3=0. \tag{4.6}$$

Again replacing X by ϕX in (4.3), we get

$$\left[(1-2n)f_1-3f_2\right]g(\phi X,Z)\eta(Y)=0, \tag{4.7}$$

which, by putting $Y = \xi$, gives

$$[(1-2n)f_1 - 3f_2]\, g(\phi X, Z) = 0. \tag{4.8}$$

Since $g(\phi X, Z) \neq 0$, in general, we obtain

$$f_1 = \frac{3f_2}{(1-2n)}. \tag{4.9}$$

From (4.6) and (4.9), we have

$$\frac{3f_2}{(1-2n)} = f_3. \tag{4.10}$$

Thus in view of (4.9) and (4.10), we have

$$f_1 = \frac{3f_2}{(1-2n)} = f_3. \tag{4.11}$$

Conversely, suppose that $f_1 = 3f_2/(1-2n) = f_3$ satisfies in generalized Sasakian-space-form, and then we have

$$S(X,Y) = 0, \tag{4.12}$$

$$QX = 0. \tag{4.13}$$

Also, in view of (2.15), we have

$$'\tilde{\tilde{C}}(X,Y,Z,U) = {}'R(X,Y,Z,U), \tag{4.14}$$

where $'\tilde{\tilde{C}}(X,Y,Z,U) = g(\tilde{\tilde{C}}(X,Y)Z,U)$ and $'R(X,Y,Z,U) = g(R(X,Y)Z,U)$. Putting $Y = Z = e_i$ in (4.14) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$\sum_{i=1}^{2n+1} {}'\tilde{\tilde{C}}(X,e_i,e_i,U) = \sum_{i=1}^{2n+1} {}'R(X,e_i,e_i,U) = S(X,U). \tag{4.15}$$

In view of (2.5) and (4.14), we have

$$\begin{aligned} '\tilde{\tilde{C}}(X,Y,Z,U) = {} & f_1\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} \\ & + f_2\{g(X,\phi Z)g(\phi Y,U) - g(Y,\phi Z)g(\phi X,U) + 2g(X,\phi Y)g(\phi Z,U)\} \\ & + f_3\{\eta(X)\eta(Z)g(Y,U) - \eta(Y)\eta(Z)g(X,U) \\ & \quad + g(X,Z)\eta(Y)\eta(U) - g(Y,Z)\eta(X)\eta(U)\}. \end{aligned} \tag{4.16}$$

Now, putting $Y = Z = e_i$ in above equation and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$\sum_{i=1}^{2n+1} {}'\tilde{\tilde{C}}(X, e_i, e_i, U) = 2nf_1 g(X, U) + 3f_2 g(\phi X, \phi U) - f_3\{(2n-1)\eta(X)\eta(U) + g(X, U)\}. \tag{4.17}$$

In view of (4.12), (4.15) and (4.17), we have

$$2nf_1 g(X, U) + 3f_2 g(\phi X, \phi U) - f_3\{(2n-1)\eta(X)\eta(U) + g(X, U)\} = 0. \tag{4.18}$$

Putting $X = W = e_i$ in above equation and taking summation over i, $1 \leq i \leq 2n+1$, we get $f_1 = 0$. Then in view of (4.11), $f_2 = f_3 = 0$. Therefore, we obtain from (2.5)

$$R(X, Y)Z = 0. \tag{4.19}$$

Hence in view of (4.12), (4.13) and (4.19), we have $\tilde{\tilde{C}}(X, Y)Z = 0$. This completes the proof. □

5. ξ-Conharmonically Flat Generalized Sasakian-Space-Forms

Definition 5.1. A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form is said to be ξ-conharmonically flat [6] if $\tilde{\tilde{C}}(X, Y)\xi = 0$ for all $X, Y \in TM$.

Theorem 5.2. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form is ξ-conharmonically flat if and only if it is η-Einstein manifold.*

Proof. Let us consider that a generalized Sasakian-space-form is ξ-conharmonically flat, that is, $\tilde{\tilde{C}}(X, Y)\xi = 0$. Then in view of (2.15), we have

$$R(X, Y)\xi = \frac{1}{(2n-1)}[\eta(Y)QX - \eta(X)QY + S(Y, \xi)X - S(X, \xi)Y]. \tag{5.1}$$

In virtue of (2.9) and (2.12) the above equation reduces to

$$-(f_1 - f_3)[\eta(Y)X - \eta(X)Y] = [\eta(Y)QX - \eta(X)QY], \tag{5.2}$$

which by putting $Y = \xi$ gives

$$QX = -(f_1 - f_3)X + (2n+1)(f_1 - f_3)\eta(X)\xi. \tag{5.3}$$

Now, taking the inner product of the above equation with U, we get

$$S(X, U) = -(f_1 - f_3)[g(X, U) + (-2n-1)\eta(X)\eta(U)], \tag{5.4}$$

which shows that generalized Sasakian-space-form is an η-Einstein manifold. Conversely, suppose that (5.4) is satisfied. Then by virtue of (5.1) and (5.3), we have $\tilde{\tilde{C}}(X,Y)\xi = 0$. This completes the proof. □

6. Conharmonically Recurrent Generalized Sasakian-Space-Forms

Definition 6.1. A nonflat Riemannian manifold M^{2n+1} is said to be conharmonically recurrent if its conharmonic curvature tensor $\tilde{\tilde{C}}$ satisfies the condition

$$\nabla\tilde{\tilde{C}} = A \otimes \tilde{\tilde{C}}, \tag{6.1}$$

where A is nonzero 1-form.

Theorem 6.2. *A* $(2n+1)$*-dimensional* $(n > 1)$ *generalized Sasakian-space-form is conharmonically recurrent if and only if* $f_1 = f_3$*.*

Proof. We define a function $f^2 = g(\tilde{\tilde{C}}, \tilde{\tilde{C}})$ on M^{2n+1}, where the metric g is extended to the inner product between the tensor fields. Then we have

$$f(Yf) = f^2 A(Y). \tag{6.2}$$

This can be written as

$$Yf = f(A(Y)), \quad (f \neq 0). \tag{6.3}$$

From the above equation, we have

$$X(Yf) - Y(Xf) = \{XA(Y) - YA(X) - A([X,Y])\}f. \tag{6.4}$$

Since the left hand side of the above equation is identically zero and $f \neq 0$ on M^{2n+1}, then

$$dA(X,Y) = 0, \tag{6.5}$$

that is, 1-form A is closed.

Now from

$$\left(\nabla_X \tilde{\tilde{C}}\right)(Y,Z)U = A(X)\tilde{\tilde{C}}(Y,Z)U, \tag{6.6}$$

we have

$$\left(\nabla_W \nabla_X \tilde{\tilde{C}}\right)(Y,Z)U = \{WA(X) + A(W)A(X)\}\tilde{\tilde{C}}(Y,Z)U. \tag{6.7}$$

In view of (6.5) and (6.7), we have

$$\left(R(W,X)\cdot \tilde{\tilde{C}}\right)(Y,Z)U = [2dA(W,X)]\tilde{\tilde{C}}(Y,Z)U = 0. \tag{6.8}$$

Thus in view of Theorem 3.2, we have $f_1 = f_3$. Converse follows from retreating the steps. □

Corollary 6.3. *Conharmonically recurrent generalized Sasakian-space-form is conharmonically semisymmetric.*

Proof. Proof follows from the above theorem. □

7. Generalized Sasakian-Space-Forms Satisfying $\tilde{\tilde{C}} \cdot S = 0$

Theorem 7.1. *A* $(2n+1)$*-dimensional* $(n > 1)$ *generalized Sasakian-space-form satisfying* $\tilde{\tilde{C}} \cdot S = 0$ *is an* η*-Einstein manifold.*

Proof. Let us consider generalized Sasakian-space-form M^{2n+1} satisfying $\tilde{\tilde{C}}(\xi,X)\cdot S = 0$. In this case we can write

$$S\left(\tilde{\tilde{C}}(\xi,X)Y,Z\right) + S\left(Y,\tilde{\tilde{C}}(\xi,X)Z\right) = 0. \tag{7.1}$$

In view of (2.18) the above equation reduces to

$$\begin{aligned} &-(f_1-f_3)\left[2n(f_1-f_3)\{g(X,Y)\eta(Z)+g(X,Z)\eta(Y)\}-\eta(Y)S(X,Z)-\eta(Z)S(X,Y)\right] \\ &\quad -\left[2n(f_1-f_3)\{S(X,Y)\eta(Z)+S(X,Z)\eta(Y)\}-\eta(Y)S(QX,Z)-\eta(Z)S(QX,Y)\right]=0. \end{aligned} \tag{7.2}$$

Now, putting $Z = \xi$ in the above equation, we get

$$S(QX,Y) = (f_1-f_3)\left[(2n-1)S(X,Y) + 2n(f_1-f_3)g(X,Y)\right]. \tag{7.3}$$

In virtue of (2.6) the above equation takes the form:

$$S(X,Y) = \frac{2n(f_1 - f_3)}{K}\left[(f_1 - f_3)g(X,Y) + (3f_2 + (2n-1)f_3)\eta(X)\eta(Y)\right], \tag{7.4}$$

where $K = 3f_2 - 2(n-1)f_3$. This completes the proof. □

8. Generalized Sasakian-Space-Forms Satisfying $\tilde{\tilde{C}} \cdot R = 0$

Theorem 8.1. *A* $(2n+1)$*-dimensional* $(n > 1)$ *generalized Sasakian-space-form satisfying* $\tilde{\tilde{C}} \cdot R = 0$ *is an* η*-Einstein manifold.*

Proof. Let generalized Sasakian-space-form satisfying

$$\tilde{\tilde{C}}(\xi, X) \cdot R(Y,Z)U = 0. \tag{8.1}$$

This can be written as

$$\begin{aligned} &\tilde{\tilde{C}}(\xi, X)R(Y,Z)U - R\left(\tilde{\tilde{C}}(\xi, X)Y, Z\right)U - R\left(Y, \tilde{\tilde{C}}(\xi, X)Z\right)U \\ &\quad - R(Y,Z)\tilde{\tilde{C}}(\xi, X)U = 0, \end{aligned} \tag{8.2}$$

which on using (2.18) takes the following form:

$$\begin{aligned} &\frac{(f_1 - f_3)}{2n-1}\left[-g(X, R(Y,Z)U)\xi + \eta(R(Y,Z)U)X + g(X,Y)R(\xi,Z)U - \eta(Y)R(X,Z)U\right. \\ &\qquad \left. + g(X,Z)R(Y,\xi)U - \eta(Z)R(Y,X)U + g(X,U)R(Y,Z)\xi - \eta(U)R(Y,Z)X\right] \\ &\quad + \frac{1}{2n-1}\left[-S(X, R(Y,Z)U)\xi + \eta(R(Y,Z)U)QX + S(X,Y)R(\xi,Z)U - \eta(Y)R(QX,Z)U\right. \\ &\qquad \left. + S(X,Z)R(Y,\xi)U - \eta(Z)R(Y,QX)U + S(X,U)R(Y,Z)\xi - \eta(U)R(Y,Z)QX\right] = 0 \end{aligned} \tag{8.3}$$

Now taking the inner product of the above equation with ξ, we get

$$\begin{aligned} &(f_1 - f_3)\left[-g(X, R(Y,Z)U) + \eta(R(Y,Z)U)\eta(X) + g(X,Y)\eta(R(\xi,Z)U) - \eta(Y)\eta(R(X,Z)U)\right. \\ &\qquad \left. + g(X,Z)\eta(R(Y,\xi)U) - \eta(Z)\eta(R(Y,X)U) + g(X,U)\eta(R(Y,Z)\xi) - \eta(U)\eta(R(Y,Z)X)\right] \\ &\quad + \left[-S(X, R(Y,Z)U) + \eta(R(Y,Z)U)\eta(QX) + S(X,Y)\eta(R(\xi,Z)U) - \eta(Y)\eta(R(QX,Z)U)\right. \\ &\quad + S(X,Z)\eta(R(Y,\xi)U) - \eta(Z)\eta(R(Y,QX)U) + S(X,U)\eta(R(Y,Z)\xi) \\ &\quad \left. - \eta(U)\eta(R(Y,Z)QX)\right] = 0. \end{aligned} \tag{8.4}$$

In consequence of (2.5), (2.9), (2.10), and (2.11) the above equation takes the form:

$$\begin{aligned}
&(f_1 - f_3)\left[-f_1\{g(Z,U)g(X,Y) - g(X,Z)g(Y,U)\}\right.\\
&\qquad - f_2\{g(Y,\phi U)g(X,\phi Z) - g(Z,\phi U)g(X,\phi Y) + 2g(Y,\phi Z)g(\phi U,X)\}\\
&\qquad - f_3\{\eta(Y)\eta(U)g(Z,X) - \eta(Z)\eta(U)g(X,Y) + g(Y,U)\eta(X)\eta(Z) - g(Z,U)\eta(X)\eta(Y)\}\\
&\qquad \left.+(f_1 - f_3)\{g(X,Y)g(Z,U) - g(X,Z)g(Y,U)\}\right]\\
&\quad - f_1\{g(Z,U)S(X,Y) - g(Y,U)S(X,Z)\}\\
&\quad - f_2\{g(Y,\phi U)S(X,\phi Z) - g(Z,\phi U)S(X,\phi Y) + 2g(Y,\phi Z)S(\phi U,X)\}\\
&\quad - f_3\{\eta(Y)\eta(U)S(Z,X) - \eta(Z)\eta(U)S(X,Y)\\
&\qquad +2n(f_1 - f_3)\{g(Y,U)\eta(X)\eta(Z) - g(Z,U)\eta(X)\eta(Y)\}\}\\
&\quad +(f_1 - f_3)\{S(X,Y)g(Z,U) - S(X,Z)g(Y,U)\}] = 0.
\end{aligned} \tag{8.5}$$

Putting $Z = U = e_i$ in the above equation and taking summation over i, $1 \le i \le 2n+1$, we get

$$S(X,Y) = (f_1 - f_3)\left[-g(X,Y) + (2n+1)\eta(X)\eta(Y)\right], \tag{8.6}$$

which shows that M^{2n+1} is an η-Einstein manifold. This completes the proof. □

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