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Research Article

Time Remotely Almost Periodic Viscosity Solutions of Hamilton-Jacobi Equations

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We study some properties of the remotely almost periodic functions. This paper studies viscosity solutions of general Hamilton-Jacobi equations in the time remotely almost periodic case. Existence and uniqueness results are presented under usual hypotheses.

1. Introduction

In this paper we consider the viscosity solutions of first-order Hamilton-Jacobi equations of the form

$$\partial_t u + H(x,u,Du) = f(t), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{1.1}$$

This problem was studied in [1] in the time periodic and almost periodic cases. And papers by Crandall and Lions (see [2–5]) proved the uniqueness and stability of viscosity solutions for a large class of equations, in particular for the initial value problem

$$\begin{aligned} \partial_t u + H(x,t,u,Du) &= 0, \qquad (x,t) \in \mathbb{R}^N \times \,]0,T[, \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}^N \end{aligned} \tag{1.2}$$

and also for the stationary problem

$$H(x,u,Du) = 0, \quad x \in \mathbb{R}^N. \tag{1.3}$$

These results were extended by several papers, for example [6, 7].

Now in this paper we study this problem in a more regular condition, that is, in the time remotely almost periodic case. That is, we will look for such viscosity solutions when the Hamiltonian H and f are continuous functions f is remotely almost periodic in t. The definition of remotely almost periodic was introduced by Sarason in 1984 in [8]. And Zhang and Yang in [9] and Zhang and Jiang in [10] gave such functions' applications.

This paper is structured as follows. In Section 2, we study a new type of almost periodic function—remotely almost periodic function. We present the definitions and prove some properties of such functions. Section 3 proves the uniqueness and existence of time remotely almost periodic viscosity solutions. In Section 3.1, we list some usual hypotheses used for the existence and uniqueness results and present two properties of viscosity solutions. In Section 3.2, we get some theorems for the uniqueness and existence of time remotely almost periodic viscosity solutions. And for the proof of the theorem we give two lemmas which play an important part. In Section 3.3, we concentrate on the asymptotic behaviour of time remotely almost periodic solutions for large frequencies.

In this paper, there are some abbreviations, like *BUC*, *u.s.c*, *l.s.c*, they stand for bounded uniformly continuous, upper semicontinuous, and lower semicontinuous, respectively. For the definition of viscosity subsolution and supersolution the reader can refer to [11].

2. Remotely Almost Periodic Function

It is well known that Bohr almost periodic function space is a Banach space, in which the distance is the supremum of the function. In [8], the author uses the superior limit as the distance in the space and defines a new type of almost periodic function, that is, remotely almost periodic function.

Definition 2.1. Let f be a bounded uniformly continuous function on $\mathbb{R} = (-\infty, +\infty)$. We say that f is remotely almost periodic if and only if for all $\varepsilon > 0$

$$T(f,\varepsilon) = \left\{ \tau \in \mathbb{R} : \limsup_{|t| \to \infty} \left| f(t+\tau) - f(t) \right| < \varepsilon \right\} \tag{2.1}$$

is relatively dense on $\mathbb{R}$. The number $\tau \in T(f,\varepsilon)$ is called ε remotely almost period.

And $\mathrm{RAP}(\mathbb{R})$ denotes all these functions.

Definition 2.2. Let f be a bounded uniformly continuous function on $\mathbb{R}$. We say that f oscillates slowly if and only if for every $\tau \in \mathbb{R}$

$$\lim_{|t| \to \infty} \left| f(t+\tau) - f(t) \right| = 0. \tag{2.2}$$

And $\mathrm{SO}(\mathbb{R})$ denotes all these functions.

Next we will prove two propositions.

Proposition 2.3. *Assume that $f(t)$ is remotely almost periodic and denote by $F(t) = \int_0^t f(s)ds$ a primitive of $f(t)$. Then $F(t)$ is remotely almost periodic if and only if $F(t)$ is bounded.*

Proof. When $F(t)$ is remotely almost periodic, $F(t)$ is certainly bounded. For the converse, let $F(t)$ be bounded, without losing general, and assume that $F(t)$ is a real function. For any $\varepsilon > 0$, there exists $t_0 > 0$ large enough; we have

$$G = \sup_{|t|>t_0} F(t) > g = \inf_{|t|>t_0} F(t); \tag{2.3}$$

take fixed t_1 and t_2, $|t_1| > t_0, |t_2| > t_0$, and assume that $t_1 < t_2$, satisfying

$$F(t_1) < g + \frac{\varepsilon}{6}, \qquad F(t_2) > G - \frac{\varepsilon}{6}. \tag{2.4}$$

Assume that $l = l(\varepsilon_1)$ is an interval length of $T(f, \varepsilon_1)$, where $\varepsilon_1 = \varepsilon/6d,\ d = |t_1 - t_2|$. For every $\alpha \in R$, take $\tau \in T(f, \varepsilon_1) \cap [\alpha - t_1, \alpha - t_1 + l]$.

As we already know that $f(t)$ is remotely almost periodic, then we have

$$\limsup_{|t|\to\infty} \left| f(t+\tau) - f(t) \right| < \varepsilon_1; \tag{2.5}$$

that is, for $\varepsilon_1 > 0$, there exists $t_0 > 0$, and when $|t| > t_0$, there is

$$\left| f(t+\tau) - f(t) \right| < \varepsilon_1, \qquad -\varepsilon_1 < f(t+\tau) - f(t) < \varepsilon_1. \tag{2.6}$$

Now take $s_i = t_i + \tau\ (i = 1, 2),\ L = l + d$. So $s_1, s_2 \in [\alpha, \alpha + L]$, and

$$\begin{aligned} F(s_2) - F(s_1) &= F(t_2) - F(t_1) - \int_{t_1}^{t_2} f(t)dt + \int_{t_1+\tau}^{t_2+\tau} f(t)dt \\ &= F(t_2) - F(t_1) + \int_{t_1}^{t_2} \left[f(t+\tau) - f(t) \right] dt \\ &> G - g - \frac{\varepsilon}{3} - \varepsilon_1 d = G - g - \frac{\varepsilon}{2}, \end{aligned} \tag{2.7}$$

that is,

$$\left(F(s_1) - g \right) + (G - F(s_2)) < \frac{\varepsilon}{2}; \tag{2.8}$$

as the formulas in two brackets of previous inequality are both nonnegative, so there are two numbers s_1 and s_2 in any interval of length L satisfying simultaneously

$$F(s_1) < g + \frac{\varepsilon}{2}, \qquad F(s_2) > G - \frac{\varepsilon}{2}. \tag{2.9}$$

Now take $\varepsilon_2 = \varepsilon/2L$, and we will prove that when $\tau \in T(f, \varepsilon_2)$, there is $\tau \in T(f, \varepsilon)$. In fact for every $t \in R$, we can choose s_1 and s_2 in the interval $[t, t+L]$ satisfying $F(s_1) < g + (\varepsilon/2)$ and $F(s_2) > G - (\varepsilon/2)$. Hence for $\tau \in T(f, \varepsilon_2)$, there are, respectively,

$$\begin{aligned}\limsup_{|t|\to\infty}(F(t+\tau) - F(t)) &= \limsup_{|t|\to\infty}\left[F(s_1+\tau) - F(s_1) + \int_t^{s_1} f(t)dt - \int_{t+\tau}^{s_1+\tau} f(t)dt\right]\\ &> g - \left(g + \frac{\varepsilon}{2}\right) - \varepsilon_2 L = -\varepsilon,\\ \limsup_{|t|\to\infty}(F(t+\tau) - F(t)) &= \limsup_{|t|\to\infty}\left[F(s_2+\tau) - F(s_2) + \int_t^{s_2} f(t)dt - \int_{t+\tau}^{s_2+\tau} f(t)dt\right]\\ &< G - \left(G - \frac{\varepsilon}{2}\right) + \varepsilon_2 L = \varepsilon.\end{aligned} \tag{2.10}$$

So for $\tau \in T(f, \varepsilon_2)$, we have $\tau \in T(f, \varepsilon)$; hence $F(t)$ is remotely almost periodic. □

Proposition 2.4. *Assume that $f(t)$ is remotely almost periodic. Then $(1/T)\int_a^{a+T} f(t)dt$ converges as $T \to +\infty$ uniformly with respect to $a \in R$. Moreover the limit does not depend on a, and it is called the average of f*

$$\exists\langle f\rangle := \lim_{T\to+\infty}\frac{1}{T}\int_a^{a+T} f(t)dt, \quad \textit{uniformly with respect to } a \in R. \tag{2.11}$$

Proof. As $f(t) \in \mathrm{RAP}(R)$, then $f(t)$ is bounded, and for all $\varepsilon > 0$, for all $\tau \in T(f, \varepsilon)$, there exists $s_0 > 0$, when $|t| > s_0$, $|f(t+\tau) - f(t)| < \varepsilon$. Let $G = \sup_{t\in R}|f(t)|$, take $\varepsilon > 0$, and assume that $l = l(\varepsilon/4)$ is an interval length of $T(f, \varepsilon/4)$. Take $\tau \in T(f, \varepsilon/4) \cap [a, a+l]$; then for any $a, s \in R$

$$\begin{aligned}\left|\int_a^{a+s} f(t)dt - \int_0^s f(t)dt\right| &= \left|\left(\int_\tau^{\tau+s} - \int_0^s + \int_{\tau+s}^{a+s} + \int_a^\tau\right) f(t)dt\right|\\ &\le \int_0^s \left|f(t+\tau) - f(t)\right|dt + \int_{\tau+s}^{a+s}\left|f(t)\right|dt + \int_0^\tau \left|f(t)\right|dt\\ &= \int_0^{s_0}\left|f(t+\tau) - f(t)\right|dt + \int_{s_0}^s \left|f(t+\tau) - f(t)\right|dt\\ &\quad + \int_{\tau+s}^{a+s}\left|f(t)\right|dt + \int_0^\tau\left|f(t)\right|dt\\ &\le \sup_{[s_0,s]}\left|f(t+\tau) - f(t)\right| \cdot (s - s_0) + 2G(l + s_0)\\ &< \frac{\varepsilon}{4}(s - s_0) + 2G(l + s_0),\end{aligned} \tag{2.12}$$

so

$$\left|\frac{1}{T}\int_a^{a+T} f(t)dt - \frac{1}{T}\int_0^T f(t)dt\right| \le \frac{\varepsilon}{4T}(T - T_0) + \frac{2G(l+T_0)}{T}, \tag{2.13}$$

$$\begin{aligned}\left|\frac{1}{nT}\int_0^{nT} f(t)dt - \frac{1}{T}\int_0^T f(t)dt\right| &= \frac{1}{n}\left|\sum_{k=1}^{n}\frac{1}{T}\left[\int_{(k-1)T}^{kT} f(t)dt - \int_0^T f(t)dt\right]\right| \\ &\le \frac{\varepsilon}{4T}(T - T_0) + \frac{2G(l+T_0)}{T}.\end{aligned} \tag{2.14}$$

By passing $n \to +\infty$ in (2.14), we get

$$\left|\langle f\rangle - \frac{1}{T}\int_0^T f(t)dt\right| \le \frac{\varepsilon}{4T}(T - T_0) + \frac{2G(l+T_0)}{T}. \tag{2.15}$$

Using triangle inequality from (2.13) and (2.15) we deduce

$$\left|\frac{1}{T}\int_a^{a+T} f(t)dt - \langle f\rangle\right| \le \frac{\varepsilon}{2T}(T - T_0) + \frac{4G(l+T_0)}{T} < \varepsilon, \tag{2.16}$$

if only $T > (8G(l+T_0)/\varepsilon) - T_0$. That is, when $T \to \infty$, $(1/T)\int_a^{a+T} f(t)dt$ converges at $\langle f\rangle$ uniformly with respect to $a \in R$. Moreover notice the identical equation

$$\frac{1}{T}\int_a^{a+T} f(t)dt = \frac{1}{T}\int_0^T f(t+a)dt. \tag{2.17}$$

This means that the limit does not depend on a. □

3. Remotely Almost Periodic Viscosity Solutions

In this section we get some results for remotely almost periodic viscosity solutions.

Definition 3.1. One says that $u : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is remotely almost periodic in t uniformly with respect to x if u is bounded and uniformly continuous in t uniformly with respect to x and for all $\varepsilon > 0$, and there exists $l(\varepsilon) > 0$ such that all intervals of length $l(\varepsilon)$ contain a number τ which is ε remotely almost periodic for $u(x,\cdot)$, for all $x \in \mathbb{R}^N$

$$\limsup_{|t|\to\infty} |u(x,t+\tau) - u(x,t)| < \varepsilon, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.1}$$

3.1. Some Hypotheses and Theorems

In this section we list some usual hypotheses used for the uniqueness and existence results and present two properties of viscosity solutions.

First let us list some hypotheses in the stationary case:

$$\begin{gathered} \forall 0 < R < +\infty, \quad \exists \gamma_R > 0 \,:\, H(x,u,p) - H(x,v,p) \geq \gamma_R(u-v), \\ \forall x \in \mathbb{R}^N, \quad -R \leq v \leq u \leq R, \quad p \in \mathbb{R}^N, \end{gathered} \tag{3.2}$$

$$\begin{aligned} \forall R > 0, \quad \exists m_R, \lim_{z\to 0} m_R(z) = 0 \,:\, \left|H(x,u,p) - H(y,u,p)\right| \leq m_R(|x-y| \cdot (1+|p|)), \\ \forall x,y \in \mathbb{R}^N, \quad -R \leq u \leq R, \; p \in \mathbb{R}^N, \end{aligned} \tag{3.3}$$

$$\forall 0 < R < +\infty, \qquad \lim_{|p|\to+\infty} H(x,u,p) = +\infty, \quad \text{uniformly for } (x,u) \in \mathbb{R}^N \times [-R,R], \tag{3.4}$$

$$\forall 0 < R < +\infty, \quad H \text{ is uniformly continuous on } \mathbb{R}^N \times [-R,R] \times \overline{B}_R, \tag{3.5}$$

$$\exists M > 0 : H(x,-M,0) \leq 0 \leq H(x,M,0), \quad \forall x \in \mathbb{R}^N. \tag{3.6}$$

From [1] we know that hypotheses (3.2), (3.3) or (3.4), (3.5), (3.6) ensure the existence of a unique solution for the stationary equation (1.3). And more regularly (3.2) can be replaced by

$$H(x,u,p) - H(x,v,p) \geq 0, \quad \forall x \in \mathbb{R}^N, \; v \leq u, \; p \in \mathbb{R}^N \tag{3.7}$$

(which comes to taking $\gamma_R = 0$ in (3.2)).

When the Hamiltonian is time dependent the corresponding assumptions are

$$\begin{gathered} \forall 0 < R < +\infty, \quad \exists \gamma_R > 0 : H(x,t,u,p) - H(x,t,v,p) \geq \gamma_R(u-v), \\ \forall x \in \mathbb{R}^N, \; 0 \leq t \leq T, \; -R \leq v \leq u \leq R, \; p \in \mathbb{R}^N, \end{gathered} \tag{3.8}$$

$$\begin{gathered} \forall R > 0, \quad \exists m_R \,:\, \left|H(x,t,u,p) - H(y,t,u,p)\right| \leq m_R(|x-y| \cdot (1+|p|)), \\ \forall x,y \in \mathbb{R}^N, \; t \in [0,T], \; -R \leq u \leq R, \; p \in \mathbb{R}^N, \text{ where } \lim_{z\to 0} m_R(z) = 0, \end{gathered} \tag{3.9}$$

$$\forall 0 < R < +\infty, \quad H \text{ is uniformly continuous on } \mathbb{R}^N \times [0,T] \times [-R,R] \times \overline{B}_R, \tag{3.10}$$

$$\exists M > 0 \,:\, H(x,t,-M,0) \leq 0 \leq H(x,t,M,0), \quad \forall x \in \mathbb{R}^N, \; t \in [0,T]. \tag{3.11}$$

Now we present two results of viscosity solutions (see [1, 6, 7]).

Theorem 3.2. *Assume that* (3.8), (3.9), (3.10), *and* (3.11) *hold (with $\gamma_R \in \mathbb{R}$, for all $R > 0$). Then for every $u_0 \in \mathrm{BUC}(\mathbb{R}^N)$ there is a unique viscosity solution $u \in \mathrm{BUC}(\mathbb{R}^N \times [0,T])$ of* (1.2)*, for all $T > 0$.*

Theorem 3.3. *Let u be a bounded time periodic viscosity u.s.c. subsolution of $\partial_t u + H(x,t,u,Du) = f(x,t)$ in $\mathbb{R}^N \times \mathbb{R}$ and v a bounded time periodic viscosity l.s.c. supersolution of $\partial_t v + H(x,t,v,Dv) = g(x,t)$ in $\mathbb{R}^N \times \mathbb{R}$, where $f, g \in \mathrm{BUC}(\mathbb{R}^N \times \mathbb{R})$ and H are T periodic such that* (3.8), (3.9), *and* (3.10) *hold. Then one has*

$$\sup_{x\in\mathbb{R}^N}(u(x,t)-v(x,t)) \le \sup_{s\le t}\int_s^t \sup_{x\in\mathbb{R}^N}(f(x,\sigma)-g(x,\sigma))d\sigma. \tag{3.12}$$

Moreover, the hypothesis (3.9) *can be replaced by* $u \in W^{1,\infty}(\mathbb{R}^N \times \mathbb{R})$ *or* $v \in W^{1,\infty}(\mathbb{R}^N \times \mathbb{R})$.

3.2. Uniqueness and Existence of Time Remotely Almost Periodic Viscosity Solutions

In this section we establish uniqueness and existence results for time remotely almost periodic viscosity solutions. For the uniqueness we have the more general result.

Proposition 3.4. *Let u a bounded u.s.c. viscosity subsolution of $\partial_t u + H(x,t,u,Du) = f(x,t)$, in $\mathbb{R}^N \times \mathbb{R}$ and v a bounded l.s.c. viscosity supersolution of $\partial_t v + H(x,t,v,Dv) = g(x,t)$, in $\mathbb{R}^N \times \mathbb{R}$ where $f,g \in \mathrm{BUC}(\mathbb{R}^N \times \mathbb{R})$ and* (3.8), (3.9), (3.10) *hold uniformly for $t \in \mathbb{R}$. Then one has for all $t \in \mathbb{R}$*

$$\sup_{x\in\mathbb{R}^N}(u(x,t)-v(x,t))_+ \le e^{-\gamma t}\int_{-\infty}^t \sup_{x\in\mathbb{R}^N}(f(x,\sigma)-g(x,\sigma))_+ d\sigma. \tag{3.13}$$

Moreover hypotheses (3.9) *can be replaced by* $u \in W^{1,\infty}(\mathbb{R}^N \times \mathbb{R})$ *or* $v \in W^{1,\infty}(\mathbb{R}^N \times \mathbb{R})$.

The proof of this proposition is similar to Proposition 6.5 in [1]. Hence we do not prove it here.

Before we concentrate on the existence part, let us see two important lemmas first. Now take $h(t) = \int_{-\infty}^t e^{\gamma(\sigma-t)} f(\sigma)d\sigma$, where $\gamma > 0$ is a constant, $t \in \mathbb{R}$.

Lemma 3.5. *If $f(t) \in \mathrm{SO}(\mathbb{R})$, then $h(t) \in \mathrm{SO}(\mathbb{R})$.*

Proof. As $f(t) \in \mathrm{SO}(\mathbb{R})$, so for every $\tau \in \mathbb{R}$

$$\lim_{|t|\to\infty}\left|f(t+\tau)-f(t)\right| = 0. \tag{3.14}$$

Now for every $\tau \in \mathbb{R}$

$$\begin{aligned}
|h(t+\tau)-h(t)| &= \left|\int_{-\infty}^{t+\tau} e^{\gamma(\sigma-t-\tau)} f(\sigma)d\sigma - \int_{-\infty}^{t} e^{\gamma(\sigma-t)} f(\sigma)d\sigma\right| \\
&= \left|\int_{-\infty}^{0} e^{\gamma\sigma} f(t+\sigma+\tau)d\sigma - \int_{-\infty}^{0} e^{\gamma\sigma} f(t+\sigma)d\sigma\right| \\
&= \left|\int_{-\infty}^{0} e^{\gamma\sigma} \left[f(t+\sigma+\tau) - f(t+\sigma)\right]d\sigma\right| \\
&\le \int_{-\infty}^{0} e^{\gamma\sigma} \left|f(t+\sigma+\tau) - f(t+\sigma)\right|d\sigma \\
&\le \sup_{\sigma}\left|f(t+\sigma+\tau) - f(t+\sigma)\right| \cdot \frac{1}{\gamma},
\end{aligned} \tag{3.15}$$

hence

$$\lim_{|t|\to\infty}|h(t+\tau)-h(t)| \le \limsup_{|t|\to\infty}\left|f(t+\sigma+\tau)-f(t+\sigma)\right|\cdot\frac{1}{\gamma}=0. \tag{3.16}$$

Since we already know that $f(t)\in \mathrm{BUC}(\mathbb{R})$, we deduce also that $h(t)\in \mathrm{BUC}(\mathbb{R})$. That is, $h(t)\in \mathrm{SO}(\mathbb{R})$. □

Lemma 3.6. *If $f(t)\in \mathrm{RAP}(\mathbb{R})$, then $h(t)\in \mathrm{RAP}(\mathbb{R})$.*

Proof. The main result in [8] proved that $f(t)\in \mathrm{RAP}(\mathbb{R})$ is the closed subalgebra in $C(\mathbb{R})$ created by $\mathrm{AP}(\mathbb{R})$ and $\mathrm{SO}(\mathbb{R})$. Hence, if $f(t)\in\mathrm{RAP}(\mathbb{R})$, for every $\varepsilon>0$, take $\varepsilon_1=\gamma\cdot\varepsilon$, there exists $g_1,g_2\in \mathrm{AP}(\mathbb{R})$ and $\varphi_1,\varphi_2\in\mathrm{SO}(\mathbb{R})$; hence

$$\left\| f-\left[g_1+\varphi_1+g_2\varphi_2\right]\right\| < \frac{\varepsilon_1}{4}. \tag{3.17}$$

If $\varphi_2=0$, consider a number τ which is an $\varepsilon_1/2$ remotely almost period of g_1:

$$\begin{aligned}
|h(t+\tau)-h(t)| &= \left|\int_{-\infty}^{0} e^{\gamma\sigma}\left[f(t+\sigma+\tau)-f(t+\sigma)\right]d\sigma\right| \\
&\le \int_{-\infty}^{0} e^{\gamma\sigma}\left|f(t+\sigma+\tau)-f(t+\sigma)\right|d\sigma \\
&\le \int_{-\infty}^{0} e^{\gamma\sigma}\left|f(t+\sigma+\tau)-\left[g_1(t+\sigma+\tau)+\varphi_1(t+\sigma+\tau)\right]\right|d\sigma \\
&\quad+\int_{-\infty}^{0} e^{\gamma\sigma}\left|g_1(t+\sigma+\tau)-g_1(t+\sigma)\right|d\sigma \\
&\quad+\int_{-\infty}^{0} e^{\gamma\sigma}\left|\varphi_1(t+\sigma+\tau)-\varphi_1(t+\sigma)\right|d\sigma \\
&\quad+\int_{-\infty}^{0} e^{\gamma\sigma}\left|f(t+\sigma)-\left[g_1(t+\sigma)+\varphi_1(t+\sigma)\right]\right|d\sigma \\
&< \frac{\varepsilon_1}{4\gamma}+\frac{\varepsilon_1}{2\gamma}+\int_{-\infty}^{0} e^{\gamma\sigma}\left|\varphi_1(t+\sigma+\tau)-\varphi_1(t+\sigma)\right|d\sigma+\frac{\varepsilon_1}{4\gamma}.
\end{aligned} \tag{3.18}$$

By using Lemma 3.5 we deduce

$$\limsup_{|t|\to\infty}|h(t+\tau)-h(t)| < \frac{\varepsilon_1}{4\gamma}+\frac{\varepsilon_1}{2\gamma}+\frac{\varepsilon_1}{4\gamma}=\varepsilon. \tag{3.19}$$

Thus this proves that any $\varepsilon_1/2$ remotely almost period of g_1 is an ε remotely almost period of h.

If $\varphi_2 \neq 0$, assume that $\delta = \min\{\varepsilon_1/4, \varepsilon_1/(4 \cdot \|\varphi_2\|)\}$, and take number τ which is a common δ remotely almost period of g_1 and g_2. We will prove that τ is an $\varepsilon_1/2$ remotely almost period of $(g_1 + \varphi_1 + g_2\varphi_2)$, and an ε remotely almost period of h:

$$\begin{aligned}
&\left|g_2(t+\sigma+\tau)\varphi_2(t+\sigma+\tau) - g_2(t+\sigma)\varphi_2(t+\sigma)\right| \\
&\quad \leq \left|g_2(t+\sigma+\tau)\varphi_2(t+\sigma+\tau) - g_2(t+\sigma+\tau)\varphi_2(t+\sigma)\right| \\
&\qquad + \left|g_2(t+\sigma+\tau)\varphi_2(t+\sigma) - g_2(t+\sigma)\varphi_2(t+\sigma)\right| \\
&\quad \leq \|g_2\| \cdot \left|\varphi_2(t+\sigma+\tau) - \varphi_2(t+\sigma)\right| + \|\varphi_2\| \cdot \left|g_2(t+\sigma+\tau) - g_2(t+\sigma)\right|.
\end{aligned} \tag{3.20}$$

We have

$$\limsup_{|t|\to\infty}\left|g_2(t+\sigma+\tau)\varphi_2(t+\sigma+\tau) - g_2(t+\sigma)\varphi_2(t+\sigma)\right| < \frac{\varepsilon_1}{4}. \tag{3.21}$$

Hence

$$\begin{aligned}
&\limsup_{|t|\to\infty}\Big|\left[g_1(t+\sigma+\tau) + \varphi_1(t+\sigma+\tau) + g_2(t+\sigma+\tau)\varphi_2(t+\sigma+\tau)\right] \\
&\qquad\qquad - \left[g_1(t+\sigma) + \varphi_1(t+\sigma) + g_2(t+\sigma)\varphi_2(t+\sigma)\right]\Big| \\
&\quad \leq \limsup_{|t|\to\infty}\left|g_1(t+\sigma+\tau) - g_1(t+\sigma)\right| \\
&\qquad + \limsup_{|t|\to\infty}\left|\varphi_1(t+\sigma+\tau) - \varphi_1(t+\sigma)\right| \\
&\qquad + \limsup_{|t|\to\infty}\left|g_2(t+\sigma+\tau)\varphi_2(t+\sigma+\tau) - g_2(t+\sigma)\varphi_2(t+\sigma)\right| \\
&\quad < \frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4} = \frac{\varepsilon_1}{2}, \\
&\left|f(t+\sigma+\tau) - f(t+\sigma)\right| \\
&\quad \leq \left|f(t+\sigma+\tau) - \left[g_1(t+\sigma+\tau) + \varphi_1(t+\sigma+\tau) + g_2(t+\sigma+\tau)\varphi_2(t+\sigma+\tau)\right]\right| \\
&\qquad + \Big|\left[g_1(t+\sigma+\tau) + \varphi_1(t+\sigma+\tau) + g_2(t+\sigma+\tau)\varphi_2(t+\sigma+\tau)\right] \\
&\qquad\quad - \left[g_1(t+\sigma) + \varphi_1(t+\sigma) + g_2(t+\sigma)\varphi_2(t+\sigma)\right]\Big| \\
&\qquad + \left|\left[g_1(t+\sigma) + \varphi_1(t+\sigma) + g_2(t+\sigma)\varphi_2(t+\sigma)\right] - f(t+\sigma)\right|.
\end{aligned} \tag{3.22}$$

So we have

$$\begin{aligned}
\limsup_{|t|\to\infty}|h(t+\tau) - h(t)| &= \limsup_{|t|\to\infty}\left|\int_{-\infty}^{0} e^{\gamma\sigma}\left[f(t+\sigma+\tau) - f(t+\sigma)\right]d\sigma\right| \\
&\leq \limsup_{|t|\to\infty}\left|f(t+\sigma+\tau) - f(t+\sigma)\right| \cdot \frac{1}{\gamma} \\
&< \left(\frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{4}\right) \cdot \frac{1}{\gamma} = \varepsilon.
\end{aligned} \tag{3.23}$$

Since we already know that $f(t) \in \mathrm{BUC}(\mathbb{R})$, we deduce also that $h(t) \in \mathrm{BUC}(\mathbb{R})$. So this proves that $h(t) \in \mathrm{RAP}(\mathbb{R})$. □

Now we concentrate on the existence part.

Proposition 3.7. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is remotely almost periodic and that the Hamiltonian $H = H(x, z, p)$ satisfying the hypotheses* (3.2), (3.3), (3.5), *and there exists $M > 0$ such that $H(x, -M, 0) \le f(t) \le H(x, M, 0)$, for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$. Then there is a time remotely almost periodic viscosity solution in* $\mathrm{BUC}(\mathbb{R}^N \times \mathbb{R})$ *of $\partial_t u + H(x, u, Du) = f(t)$, in $\mathbb{R}^N \times \mathbb{R}$.*

Proof. We consider the unique viscosity solution of the problem

$$\begin{aligned} &\partial_t u_n + H(x, u_n, Du_n) = f(t), \quad (x, t) \in \mathbb{R}^N \times]-n, +\infty[, \\ &u_n(x, -n) = 0, \ \ x \in \mathbb{R}^N \end{aligned} \tag{3.24}$$

for all $n \ge 1$. Such a solution exists by Theorem 3.2. Next we will prove that for all $t \in \mathbb{R}$, $(u_n(t))_{n \ge -t}$ converges to a remotely almost periodic viscosity solution of $\partial_t u + H(x, u, Du) = f(t)$, in $\mathbb{R}^N \times \mathbb{R}$. Similar to the proof of Proposition 6.6 in [1], we obtain by fixing $t \in \mathbb{R}$ and n large enough

$$|u_n(x, t) - u_n(x, t + \tau)| \le 2M \cdot e^{-\gamma(t - t_n)} + e^{-\gamma t} \int_{t_n}^{t} e^{\gamma\sigma} \left| f(\sigma + \tau) - f(\sigma) \right| d\sigma. \tag{3.25}$$

By passing $n \to +\infty$ we have $t_n \to -\infty$, and therefore

$$|u(x, t) - u(x, t + \tau)| \le \int_{-\infty}^{t} e^{-\gamma(t - \sigma)} \left| f(\sigma + \tau) - f(\sigma) \right| d\sigma. \tag{3.26}$$

As f is remotely almost periodic, using Lemma 3.6 we deduce

$$\limsup_{|t| \to \infty} |u(x, t) - u(x, t + \tau)| \le \limsup_{|t| \to \infty} \int_{-\infty}^{t} e^{\gamma(\sigma - t)} \left| f(\sigma + \tau) - f(\sigma) \right| d\sigma < \varepsilon. \tag{3.27}$$

Since we already know that $u \in \mathrm{BUC}(\mathbb{R}^N \times [a, b])$, for all $a, b \in \mathbb{R}, a \le b$, by time remotely almost periodicity we deduce also that $u \in \mathrm{BUC}(\mathbb{R}^N \times \mathbb{R})$. □

Now we will study the time remotely almost periodic viscosity solutions of

$$\partial_t u + H(x, u, Du) = f(t), \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.28}$$

for Hamiltonians satisfying (3.7). We introduce also the stationary equation

$$H(x, u, Du) = \langle f \rangle := \frac{1}{T} \int_0^T f(t) dt, \quad x \in \mathbb{R}^N. \tag{3.29}$$

We have the following theorem for the existence of time remotely almost periodic viscosity solution.

Theorem 3.8. *Assume that Hamiltonian $H = H(x,z,p)$ satisfies hypotheses* (3.7), (3.4), (3.5), $\sup\{|H(x,0,0)| : x \in \mathbb{R}\} = C < +\infty$ *and f is a time remotely almost periodic function such that $F(t) = \int_0^t \{f(\sigma) - \langle f \rangle\} d\sigma$ is bounded on $\mathbb{R}$. Then there is a bounded Lipschitz time remotely almost periodic viscosity solution of* (3.28) *if and only if there is a bounded viscosity solution of* (3.29).

Proof. Assume that V is a bounded viscosity of (3.29). We deduce that V is a Lipschitz function as the Hamiltonian satisfies (3.4). For any $\alpha > 0$, take $M_\alpha = \|V\|_{L^\infty(\mathbb{R}^N)} + (1/\alpha)(C + \|f\|_{L^\infty(\mathbb{R})})$. By Propositions 3.4 and 3.7 we can construct the family of time remotely almost periodic solutions v_α for

$$\alpha(v_\alpha - V(x)) + \partial_t v_\alpha + H(x, v_\alpha, Dv_\alpha) = f(t), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.30}$$

Similar to Theorems 4.1 and 6.1 in [1], we can extract a sequence which converges uniformly on compact sets of $\mathbb{R}^N \times \mathbb{R}$ towards a bounded Lipschitz solution v of (3.28). Next we will prove that v is remotely almost periodic. By the hypotheses and Proposition 2.3 we deduce that F is remotely almost periodic, and thus, for all $\varepsilon > 0$, there is $l(\varepsilon/2)$ such that any interval of length $l(\varepsilon/2)$ contains an $\varepsilon/2$ remotely almost period of F. Take an interval of length $l(\varepsilon/2)$ and τ an $\varepsilon/2$ remotely almost period of F in this interval. We have for all $\alpha > 0$, $(x,t) \in \mathbb{R}^N \times \mathbb{R}$

$$\begin{aligned}
|v_\alpha(x,t+\tau) - v_\alpha(x,t)| &\le \left| \sup_{s \le t} \int_s^t \{f(\sigma+\tau) - f(\sigma)\} d\sigma \right| \\
&= \left| \sup_{s \le t} \left\{ \int_{s+\tau}^{t+\tau} (f(\sigma) - \langle f \rangle) d\sigma - \int_s^t (f(\sigma) - \langle f \rangle) d\sigma \right\} \right| \\
&= \left| \sup_{s \le t} \{ (F(t+\tau) - F(t)) - (F(s+\tau) - F(s)) \} \right| \\
&\le 2 \sup_t |F(t+\tau) - F(t)|.
\end{aligned} \tag{3.31}$$

After passing to the limit for $\alpha \searrow 0$ one gets $|v(x,t+\tau) - v(x,t)| \le 2\sup_t |F(t+\tau) - F(t)|$, and hence

$$\limsup_{|t| \to \infty} |v(x,t+\tau) - v(x,t)| \le 2 \limsup_{|t| \to \infty} |F(t+\tau) - F(t)| \le \varepsilon. \tag{3.32}$$

By using the uniform continuity of F, we can prove exactly in the same manner that v is continuous in t uniformly with respect to x. The converse implication follows similarlyTheorem 4.1 in [1]; here we do not prove it. □

3.3. Asymptotic Behaviour for Large Frequencies

In this section we study the asymptotic behaviour of time remotely almost periodic viscosity solutions of

$$\partial_t u_n + H(x, u_n, Du_n) = f_n(t), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.33}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a remotely almost periodic function. For all $n \geq 1$ notice that $f_n(t) = f(nt)$, for all $t \in \mathbb{R}$ is remotely almost periodic and has the same average as f. Now suppose that such a hypothesis exists

$$\exists M > 0 \text{ such that } H(x, -M, 0) \leq f(t), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.34}$$

Theorem 3.9. *Let $H = H(x, z, p)$ be a Hamiltonian satisfying* (3.7), (3.3), (3.5), (3.34) *where f is remotely almost periodic function. Suppose also that there is a bounded l.s.c viscosity supersolution $\widetilde{V} \geq -M$ of* (3.29)*, that $t \to F(t) = \int_0^t \{f(s) - \langle f \rangle\} ds$ is bounded, and denote by V the minimal stationary l.s.c. viscosity supersolution of* (3.29)*, v_n the time remotely almost periodic l.s.c. viscosity supersolution of* (3.33)*. Then the sequence $(v_n)_n$ converges uniformly on $\mathbb{R}^N \times \mathbb{R}$ towards V and $\|v_n - V\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})} \leq (2/n)\|F\|_{L^\infty(\mathbb{R})}$, for all $n \geq 1$.*

Proof. As $v_n = \sup_{\alpha > 0} v_{n,\alpha}$ is remotely almost periodic, we introduce $w_{n,\alpha}(x,t) = v_{n,\alpha}(x, t/n)$, $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, which is also remotely almost periodic. Similar to Theorem 5.1 in [1] and by using Theorem 3.3 we deduce that

$$w_{n,\alpha}(x,t) - V_\alpha(x) \leq \sup_{s \leq t} \frac{1}{n} \int_s^t (f(\sigma) - \langle f \rangle) d\sigma \leq \frac{2}{n} \|F\|_{L^\infty(\mathbb{R})}, \tag{3.35}$$

and similarly $V_\alpha(x) - w_{n,\alpha}(x,t) \leq (2/n)\|F\|_{L^\infty(\mathbb{R})}$, for all $n \geq 1$. We have for all $n \leq 1$

$$|w_{n,\alpha}(x,t) - V_\alpha(x)| \leq \frac{2}{n} \|F\|_{L^\infty(\mathbb{R})}, \tag{3.36}$$

and after passing to the limit for $\alpha \searrow 0$ one gets for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$

$$|w_n(x,t) - V(x)| \leq \frac{2}{n} \|F\|_{L^\infty(\mathbb{R})}. \tag{3.37}$$

Finally we deduce that $\|v_n - V\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})} \leq (2/n)\|F\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})}$ for all $n \geq 1$. □

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