Ceng et al. *Journal of Inequalities and Applications* (2015) 2015:280
DOI 10.1186/s13660-015-0805-2

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH | Open Access

Implicit and explicit iterative methods for mixed equilibria with constraints of system of generalized equilibria and hierarchical fixed point problem

Lu-Chuan Ceng[1], Chin-Tzong Pang[2*] and Ching-Feng Wen[3]

*Correspondence: imctpang@saturn.yzu.edu.tw
[2]Department of Information Management, and Innovation Center for Big Data and Digital Convergence, Yuan Ze University, Chung-Li, 32003, Taiwan
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce one composite implicit relaxed extragradient-like scheme and another composite explicit relaxed extragradient-like scheme for finding a common solution of a finite family of generalized mixed equilibrium problems (GMEPs) with the constraints of a system of generalized equilibrium problems (SGEP) and the hierarchical fixed point problem (HFPP) for a strictly pseudocontractive mapping in a real Hilbert space. We establish the strong convergence of these two composite relaxed extragradient-like schemes to the same common solution of finitely many GMEPs and the SGEP, which is the unique solution of the HFPP for a strictly pseudocontractive mapping. In particular, we make use of weaker control conditions than previous ones for the sake of proving strong convergence. Utilizing these results, we first propose the composite implicit and explicit relaxed extragradient-like schemes for finding a common fixed point of a finite family of strictly pseudocontractive mappings, and then we derive their strong convergence to the unique common solution of the SGEP and some HFPP. Our results complement, develop, improve, and extend the corresponding ones given by some authors recently in this area.

MSC: Primary 49J30; 47H09; secondary 47J20; 49M05

Keywords: composite relaxed extragradient-like method; generalized mixed equilibrium problem; system of generalized equilibrium problems; inverse strongly monotone mapping; strictly pseudocontractive mapping; fixed point

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, C be a nonempty, closed, and convex subset of H, and P_C be the metric projection of H onto C. Let $T: C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(T)$ the set of fixed points of T and by $\mathbf{R}$ the set of all real numbers. A mapping $A: H \to H$ is called $\bar{\gamma}$-strongly positive on H if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

A mapping $F : C \to H$ is called L-Lipschitz-continuous if there exists a constant $L \geq 0$ such that

$$\|Fx - Fy\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $L = 1$ then F is called a nonexpansive mapping; if $L \in [0,1)$ then F is called a contraction. A mapping $T : C \to C$ is called k-strictly pseudocontractive (or a k-strict pseudocontraction) if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in C.$$

In particular, if $k = 0$, then T is a nonexpansive mapping. The mapping T is pseudocontractive if and only if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \quad \forall x, y \in C.$$

T is strongly pseudocontractive if and only if there exists a constant $\lambda \in (0,1)$ such that

$$\langle Tx - Ty, x - y\rangle \leq \lambda\|x - y\|^2, \quad \forall x, y \in C.$$

Note that the class of strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is nonexpansive if and only if T is 0-strictly pseudocontractive. The mapping T is also said to be pseudocontractive if $k = 1$ and T is said to be strongly pseudocontractive if there exists a positive constant $\lambda \in (0,1)$ such that $T + (1 - \lambda)I$ is pseudocontractive. Clearly, the class of strictly pseudocontractive mappings falls into the one between the classes of nonexpansive mappings and of pseudocontractive mappings. Also it is clear that the class of strongly pseudocontractive mappings is independent of the class of strictly pseudocontractive mappings (see [1]). The class of pseudocontractive mappings is one of the most important classes of mappings among nonlinear mappings. Recently, many authors have been devoting to the study of the problem of finding fixed points of pseudocontractive mappings; see *e.g.*, [2–9] and the references therein.

Let $\mathcal{A} : C \to H$ be a nonlinear mapping on C. The variational inequality problem (VIP) associated with the set C and the mapping $\mathcal{A}$ is stated as follows: find $x^* \in C$ such that

$$\big\langle \mathcal{A}x^*, x - x^*\big\rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

The solution set of VIP (1.1) is denoted by $\mathrm{VI}(C, \mathcal{A})$.

The VIP (1.1) was first discussed by Lions [10]. There are many applications of VIP (1.1) in various fields; see, *e.g.*, [4, 5, 7, 11]. It is well known that, if $\mathcal{A}$ is a strongly monotone and Lipschitz-continuous mapping on C, then VIP (1.1) has a unique solution. In 1976, Korpelevich [12] proposed an iterative algorithm for solving VIP (1.1) in Euclidean space $\mathbf{R}^n$:

$$\begin{cases} y_n = P_C(x_n - \tau\mathcal{A}x_n), \\ x_{n+1} = P_C(x_n - \tau\mathcal{A}y_n), \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

with $\tau > 0$ a given number, which is known as the extragradient method. The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many authors, who improved it in various ways; see, *e.g.*, [5, 11, 13–29] and references therein, to name but a few.

In 2011, Ceng *et al.* [30] also introduced the following iterative method:

$$x_{n+1} = P_C\big[\alpha_n\gamma Vx_n + (I - \alpha_n\mu F)Tx_n\big], \quad \forall n \geq 0, \tag{1.3}$$

where $T : C \to C$ is a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$, $F : C \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$, $V : C \to H$ is an l-Lipschitzian mapping with constant $l \geq 0$ and $0 < \mu < \frac{2\eta}{\kappa^2}$. They proved that, under mild conditions, the sequence $\{x_n\}$ generated by (1.3) converges strongly to a point $\tilde{x} \in \mathrm{Fix}(T)$ which is the unique solution to the VIP

$$\big\langle(\mu F - \gamma V)\tilde{x}, p - \tilde{x}\big\rangle \geq 0, \quad \forall p \in \mathrm{Fix}(T). \tag{1.4}$$

Their results also improve Tian's results [31] from the contractive mapping f to the Lipschitzian mapping V.

In 2011, Ceng *et al.* [32] introduced one general composite implicit scheme that generates a net $\{x_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma\alpha}\})}$ in an implicit way

$$x_t = (I - \theta_t A)Tx_t + \theta_t\big[Tx_t - t\big(\mu FTx_t - \gamma f(x_t)\big)\big], \tag{1.5}$$

and also proposed another general composite explicit scheme that generates a sequence $\{x_n\}$ in an explicit way

$$\begin{cases} y_n = (I - \alpha_n\mu F)Tx_n + \alpha_n\gamma f(x_n), \\ x_{n+1} = (I - \beta_n A)Tx_n + \beta_n y_n, \quad \forall n \geq 0, \end{cases} \tag{1.6}$$

where $x_0 \in H$ is an arbitrary initial guess, $F : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$, $T : H \to H$ is a nonexpansive mapping, $A : H \to H$ is a $\bar{\gamma}$-strongly positive bounded linear operator, and $f : H \to H$ is an α-contractive mapping with $\alpha \in (0,1)$. They proved that, under appropriate conditions, the net $\{x_t\}$ and the sequence $\{x_n\}$ generated by (1.5) and (1.6), respectively, converge strongly to the same point $\tilde{x} \in \mathrm{Fix}(T)$, which is the unique solution to the VIP

$$\big\langle(A - I)\tilde{x}, p - \tilde{x}\big\rangle \geq 0, \quad \forall p \in \mathrm{Fix}(T). \tag{1.7}$$

Their results supplement and develop the corresponding ones of Marino and Xu [33], Yamada [34] and Tian [31].

Very recently, inspired by Ceng *et al.* [32], Jung [1] introduced one general composite implicit scheme that generates a net $\{x_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})}$ in an implicit way

$$x_t = (I - \theta_t A)T_t x_t + \theta_t\big[t\gamma Vx_t + (I - t\mu F)T_t x_t\big], \tag{1.8}$$

and also proposed another general composite explicit scheme that generates a sequence $\{x_n\}$ in an explicit way,

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_n x_n, \\ x_{n+1} = (I - \beta_n A) T_n x_n + \beta_n y_n, \quad \forall n \geq 0, \end{cases} \tag{1.9}$$

where $x_0 \in H$ is an arbitrary initial guess and the following conditions are satisfied:

$T : H \to H$ is a k-strictly pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$;
A is a $\bar{\gamma}$-strongly positive bounded linear operator on H with $\bar{\gamma} \in (1, 2)$;
$F : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with $0 < \mu < \frac{2\eta}{\kappa^2}$;
$V : H \to H$ is an l-Lipschitzian mapping with $0 \leq \gamma l < \tau$ and $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;
$T_t : H \to H$ is a mapping defined by $T_t x = \lambda_t x + (1 - \lambda_t)Tx$, $t \in (0, 1)$, for $0 \leq k \leq \lambda_t \leq \lambda < 1$ and $\lim_{t\to 0} \lambda_t = \lambda$;
$T_n : H \to H$ is a mapping defined by $T_n x = \lambda_n x + (1 - \lambda_n)Tx$ for $0 \leq k \leq \lambda_n \leq \lambda < 1$ and $\lim_{n\to\infty} \lambda_n = \lambda$;
$\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset (0, 1]$ and $\{\theta_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})} \subset (0, 1)$.

The author of [1] proved that, under weaker control conditions than the previous ones, the net $\{x_t\}$ and the sequence $\{x_n\}$ generated by (1.8) and (1.9), respectively, converge strongly to the same point $\tilde{x} \in \mathrm{Fix}(T)$, which is the unique solution to the VIP

$$\langle (A - I)\tilde{x}, p - \tilde{x} \rangle \geq 0, \quad \forall p \in \mathrm{Fix}(T). \tag{1.10}$$

His results extend and improve Ceng *et al.*'s corresponding ones [32] from the nonexpansive mapping T to the strictly pseudocontractive mapping T and from the contractive mapping f to the Lipschitzian mapping V.

On the other hand, let $\varphi : C \to \mathbf{R}$ be a real-valued function, $\mathcal{A} : C \to H$ be a nonlinear mapping and $\Theta : C \times C \to \mathbf{R}$ be a bifunction. In 2008, Peng and Yao [13] introduced the generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle \mathcal{A}x, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.11}$$

We denote the set of solutions of GMEP (1.11) by $\mathrm{GMEP}(\Theta, \varphi, \mathcal{A})$. The GMEP (1.11) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others. Recently, many authors have been devoting to the study of the GMEP (1.11) and its special cases, *e.g.*, generalized equilibrium problem (GEP), mixed equilibrium problem (MEP),equilibrium problem (EP), *etc.*; see, *e.g.*, [15, 18, 23–29, 31, 35–38] and the references therein.

It was assumed in [13] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)-(A4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (B1) or (B2), where

(A1) $\Theta(x, x) = 0$ for all $x \in C$;
(A2) Θ is monotone, *i.e.*, $\Theta(x, y) + \Theta(y, x) \leq 0$ for any $x, y \in C$;
(A3) Θ is upper-hemicontinuous, *i.e.*, for each $x, y, z \in C$,

$$\limsup_{t\to 0^+} \Theta\bigl(tz + (1 - t)x, y\bigr) \leq \Theta(x, y);$$

(A4) $\Theta(x,\cdot)$ is convex and lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z,y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(B2) C is a bounded set.

Given a positive number $r > 0$. Let $T_r^{(\Theta,\varphi)} : H \to C$ be the solution set of the auxiliary mixed equilibrium problem, that is, for each $x \in H$,

$$T_r^{(\Theta,\varphi)}(x) := \left\{ y \in C : \Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y\rangle \geq 0, \forall z \in C \right\}.$$

In particular, if $\varphi \equiv 0$ then $T_r^{(\Theta,\varphi)}$ is rewritten as $T_r^{\Theta} : H \to C$, *i.e.*,

$$T_r^{\Theta}(x) := \left\{ y \in C : \Theta(y,z) + \frac{1}{r}\langle y - x, z - y\rangle \geq 0, \forall z \in C \right\}.$$

Let $\Phi_1, \Phi_2 : C \times C \to \mathbf{R}$ be two bifunctions and $F_1, F_2 : C \to H$ be two mappings. Consider the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \Phi_1(x^*,x) + \langle F_1 y^*, x - x^*\rangle + \frac{1}{\nu_1}\langle x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \Phi_2(y^*,y) + \langle F_2 x^*, y - y^*\rangle + \frac{1}{\nu_2}\langle y^* - x^*, y - y^*\rangle \geq 0, & \forall y \in C, \end{cases} \tag{1.12}$$

which is called a system of generalized equilibrium problems (SGEP) where $\nu_1 > 0$ and $\nu_2 > 0$ are two constants. In 2010, Ceng and Yao [23] transformed the SGEP (1.12) into the fixed point problem of the mapping $G = T_{\nu_1}^{\Phi_1}(I - \nu_1 F_1) T_{\nu_2}^{\Phi_2}(I - \nu_2 F_2)$, that is, $Gx^* = x^*$, where $y^* = T_{\nu_2}^{\Phi_2}(I - \nu_2 F_2)x^*$. Throughout this paper, the fixed point set of the mapping G is denoted by Ξ.

In particular, if $\Phi_1 \equiv \Phi_2 \equiv 0$, then problem (1.12) reduces to the system of variational inequalities (SVI) of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \nu_1 F_1 y^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \nu_2 F_2 x^* + y^* - x^*, y - y^*\rangle \geq 0, & \forall y \in C, \end{cases} \tag{1.13}$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are two constants. Recently, many authors have addressed the study of the SVI (1.13); see, *e.g.*, [11, 14, 15, 17–20, 39–41] and the references therein.

Let $T : C \to C$ be a k-strictly pseudocontractive mapping. In 2010, Ceng and Yao [23] proposed and analyzed the following relaxed extragradient-like iterative scheme for finding a common solution $x^* \in \Omega := \mathrm{Fix}(T) \cap \mathrm{GMEP}(\Theta, \varphi, \mathcal{A}) \cap \Xi$ of the GMEP (1.11), the SGEP (1.12), and the fixed point problem of T:

$$\begin{cases} z_n = T_{\lambda_n}^{(\Theta,\varphi)}(I - \lambda_n \mathcal{A})x_n, \\ y_n = T_{\nu_1}^{\Phi_1}(I - \nu_1 F_1) T_{\nu_2}^{\Phi_2}(I - \nu_2 F_2) z_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n T y_n, & \forall n \geq 0, \end{cases} \tag{1.14}$$

where $0 < \nu_j < 2\zeta_j$ for $j = 1, 2$, and $\{\lambda_n\} \subset [0, 2\eta]$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and $(\gamma_n + \delta_n)k \leq \gamma_n$, $\forall n \geq 0$. Under some mild assumptions, the authors [23] proved that $\{x_n\}$ converges strongly to $x^* = P_\Omega u$ and (x^*, y^*) is a solution of the SGEP (1.12), where $y^* = T_{\nu_2}^{\Phi_2}(I - \nu_2 F_2)x^*$.

In this paper, we introduce one composite implicit relaxed extragradient-like scheme and another composite explicit relaxed extragradient-like scheme for finding a common solution of a finite family of generalized mixed equilibrium problems (GMEP) with the constraints of the SGEP (1.12) and the hierarchical fixed point problem (HFPP) for a strictly pseudocontractive mapping in a real Hilbert space. We establish the strong convergence of these two composite relaxed extragradient-like schemes to the same common solution of finitely many GMEPs and the SGEP (1.12), which is the unique solution of the HFPP for a strictly pseudocontractive mapping. In particular, we make use of weaker control conditions than the previous ones for the sake of proving strong convergence. Utilizing these results, we first propose the composite implicit and explicit relaxed extragradient-like schemes for finding a common fixed point of a finite family of strictly pseudocontractive mappings, and then derive their strong convergence to the unique common solution of the SGEP (1.12) and some HFPP. Our results complement, develop, improve, and extend the corresponding ones given by some authors recently in this area. See, *e.g.*, Ceng *et al.* [32], Jung [1], and Ceng and Yao [23].

2 Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty, closed, and convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, *i.e.*,

$$\omega_w(x_n) := \bigl\{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\bigr\}.$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

The following properties of projections are useful and pertinent to our purpose.

Proposition 2.1 *Given any $x \in H$ and $z \in C$. One has*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z \rangle \leq 0$, $\forall y \in C$;

(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, $\forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2$, $\forall y \in H$, *which hence implies that P_C is nonexpansive and monotone.*

Definition 2.1 A mapping $T : H \to H$ is said to be firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse strongly monotone (1-ism),

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in H;$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

Definition 2.2 A mapping $F : C \to H$ is said to be

(i) monotone if

$$\langle Fx - Fy, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C;$$

(iii) α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Fx - Fy, x - y\rangle \geq \alpha\|Fx - Fy\|^2, \quad \forall x, y \in C.$$

It can easily be seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that the projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

On the other hand, it is obvious that if $F : C \to H$ is α-inverse strongly monotone, then F is monotone and $\frac{1}{\alpha}$-Lipschitz-continuous. Moreover, we also have, for all $u, v \in C$ and $\lambda > 0$,

$$\big\|(I - \lambda F)u - (I - \lambda F)v\big\|^2 \leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Fu - Fv\|^2. \tag{2.1}$$

Consequently, if $\lambda \leq 2\alpha$, then $I - \lambda F$ is a nonexpansive mapping from C to H.

Next we list some elementary conclusions for the MEP.

Proposition 2.2 (see [35]) *Assume that $\Theta : C \times C \to \mathbf{R}$ satisfies* (A1)-(A4) *and let $\varphi : C \to \mathbf{R}$ be a proper lower semicontinuous and convex function. Assume that either* (B1) *or* (B2) *holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows*:

$$T_r^{(\Theta,\varphi)}(x) = \left\{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for all $x \in H$. Then the following hold:

(i) *for each $x \in H$, $T_r^{(\Theta,\varphi)}(x)$ is nonempty and single-valued*;

(ii) *$T_r^{(\Theta,\varphi)}$ is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\big\|T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\big\|^2 \leq \big\langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\big\rangle;$$

(iii) $\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$;

(iv) $\mathrm{MEP}(\Theta, \varphi)$ *is closed and convex*;

(v) $\|T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x\|^2 \leq \frac{s-t}{s}\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x\rangle$ *for all $s, t > 0$ and $x \in H$.*

In 2010, Ceng and Yao [23] transformed the SGEP (1.12) into a fixed point problem in the following way:

Proposition 2.3 (see [23]) *Let $\Phi_1, \Phi_2 : C \times C \to \mathbf{R}$ be two bifunctions satisfying conditions* (A1)-(A4). *Then $(x^*, y^*) \in C \times C$ is a solution of the SGEP* (1.12) *if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$Gx = T_{\nu_1}^{\Phi_1}(I - \nu_1 F_1)T_{\nu_2}^{\Phi_2}(I - \nu_2 F_2)x, \quad \forall x \in C,$$

where $y^* = T_{\nu_2}^{\Phi_2}(I - \nu_2 F_2)x^*$.

In particular, if the mapping $F_j : C \to H$ is ζ_j-inverse strongly monotone for $j = 1, 2$, then the mapping G is nonexpansive provided $\nu_j \in (0, 2\zeta_j]$ for $j = 1, 2$. We denote by Ξ the fixed point set of the mapping G.

In Proposition 2.3, putting $\Phi_1 \equiv \Phi_2 \equiv 0$, we get the following.

Corollary 2.1 (see [15], Lemma 2.1) *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the SVI* (1.13) *if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by $Gx = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x$ for all $x \in C$, where $y^* = P_C(I - \nu_2 F_2)x^*$.*

In particular, if the mapping $F_j : C \to H$ is ζ_j-inverse strongly monotone for $j = 1, 2$, then the mapping G is nonexpansive provided $\nu_j \in (0, 2\zeta_j]$ for $j = 1, 2$. We denote by Ξ the fixed point set of the mapping G.

We need some facts and tools in a real Hilbert space H; these are listed as lemmas below.

Lemma 2.1 *Let X be a real inner product space. Then we have the following inequality*:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X.$$

Lemma 2.2 *Let H be a real Hilbert space. Then the following hold*:

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all $x, y \in H$;*

(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$ *for all $x, y \in H$ and $\lambda, \mu \in [0, 1]$ with* $\lambda + \mu = 1$;

(c) *if $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H.$$

It is clear that, in a real Hilbert space H, $T : C \to C$ is k-strictly pseudocontractive if and only if the following inequality holds:

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2 - \frac{1-k}{2}\big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in C.$$

This immediately implies that if T is a k-strictly pseudocontractive mapping, then $I - T$ is $\frac{1-k}{2}$-inverse strongly monotone; for further detail, we refer to [42] and the references therein. It is well known that the class of strict pseudocontractions strictly includes the class of nonexpansive mappings and that the class of pseudocontractions strictly includes the class of strict pseudocontractions.

Lemma 2.3 (see [42], Proposition 2.1) *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and $T : C \to C$ be a mapping.*

(i) *If T is a k-strictly pseudocontractive mapping, then T satisfies the Lipschitzian condition*

$$\|Tx - Ty\| \leq \frac{1+k}{1-k}\|x - y\|, \quad \forall x, y \in C.$$

(ii) *If T is a k-strictly pseudocontractive mapping, then the mapping $I - T$ is semiclosed at 0, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I - T)x_n \to 0$, then $(I - T)\tilde{x} = 0$.*

(iii) *If T is k-(quasi-)strict pseudocontraction, then the fixed point set $\operatorname{Fix}(T)$ of T is closed and convex so that the projection $P_{\operatorname{Fix}(T)}$ is well defined.*

Lemma 2.4 (see [15]) *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a k-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma + \delta)k \leq \gamma$. Then*

$$\big\|\gamma(x - y) + \delta(Tx - Ty)\big\| \leq (\gamma + \delta)\|x - y\|, \quad \forall x, y \in C.$$

Lemma 2.5 (see [43], Demiclosedness principle) *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let S be a nonexpansive self-mapping on C. Then $I - S$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - S)x_n\}$ strongly converges to some y, it follows that $(I - S)x = y$. Here I is the identity operator of H.*

Lemma 2.6 *Let $F : C \to H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection (see Proposition 2.1(i)) implies*

$$u \in \operatorname{VI}(C, F) \quad \Leftrightarrow \quad u = P_C(u - \lambda Fu), \quad \lambda > 0.$$

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0,1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to C$, we define the mapping $T^\lambda : C \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C,$$

where $F : C \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on C; that is, F satisfies the conditions:

$$\|Fx - Fy\| \leq \kappa\|x - y\| \quad \text{and} \quad \langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2$$

for all $x, y \in C$.

Lemma 2.7 (see [37], Lemma 3.1) *T^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$; that is,*

$$\big\|T^\lambda x - T^\lambda y\big\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0,1]$.

Lemma 2.8 (see [44], Lemma 2.1) *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1-\omega_n)a_n + \omega_n\delta_n + r_n, \quad \forall n \geq 0,$$

where $\{\omega_n\}$, $\{\delta_n\}$, *and* $\{r_n\}$ *satisfy the following conditions*:
(i) $\{\omega_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \omega_n = \infty$;
(ii) *either* $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \omega_n|\delta_n| < \infty$;
(iii) $r_n \geq 0$ *for all* $n \geq 0$, *and* $\sum_{n=1}^{\infty} r_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9 (see [33]) *Assume that* A *is a* $\bar{\gamma}$*-strongly positive bounded linear operator on* H *with* $0 < \rho \leq \|A\|^{-1}$. *Then* $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Let LIM be a Banach limit. According to time and circumstances, we use $\mathrm{LIM}_n\, a_n$ instead of $\mathrm{LIM}\, a$ for every $a = \{a_n\} \in l^\infty$. The following properties are well known:
(i) for all $n \geq 1$, $a_n \leq c_n$ implies $\mathrm{LIM}_n\, a_n \leq \mathrm{LIM}_n\, c_n$;
(ii) $\mathrm{LIM}_n\, a_{n+N} = \mathrm{LIM}_n\, a_n$ for any fixed positive integer N;
(iii) $\liminf_{n\to\infty} a_n \leq \mathrm{LIM}_n\, a_n \leq \limsup_{n\to\infty} a_n$ for all $\{a_n\} \in l^\infty$.
The following lemma was given in [39], Proposition 2.

Lemma 2.10 *Let* $a \in \mathbf{R}$ *be a real number and let a sequence* $\{a_n\} \in l^\infty$ *satisfy the condition* $\mathrm{LIM}_n\, a_n \leq a$ *for all Banach limit* LIM. *If* $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, *then* $\limsup_{n\to\infty} a_n \leq a$.

Recall that a set-valued mapping $\widetilde{T} : D(\widetilde{T}) \subset H \to 2^H$ is called monotone if for all $x, y \in D(\widetilde{T})$, $f \in \widetilde{T}x$, and $g \in \widetilde{T}y$ imply

$$\langle f - g, x - y\rangle \geq 0.$$

A set-valued mapping $\widetilde{T}$ is called maximal monotone if $\widetilde{T}$ is monotone and $(I + \lambda\widetilde{T})D(\widetilde{T}) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(\widetilde{T})$ the graph of $\widetilde{T}$. It is well known that a monotone mapping $\widetilde{T}$ is maximal if and only if, for $(x,f) \in H \times H$, $\langle f - g, x - y\rangle \geq 0$ for every $(y,g) \in G(\widetilde{T})$ implies $f \in \widetilde{T}x$. Next we provide an example to illustrate the concept of a maximal monotone mapping.

Let $\Gamma : C \to H$ be a monotone and Lipschitz-continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, *i.e.*,

$$N_C v = \{u \in H : \langle v - p, u\rangle \geq 0, \forall p \in C\}.$$

Define

$$\widetilde{T}v = \begin{cases} \Gamma v + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then it is well known [27] that $\widetilde{T}$ is maximal monotone and $0 \in \widetilde{T}v$ if and only if $v \in \mathrm{VI}(C, \Gamma)$.

3 Main results

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Throughout this section, we always assume the following:

$F : C \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$, and $F_j : C \to H$ is ζ_j-inverse strongly monotone for $j = 1, 2$;
$T : C \to C$ is a k-strictly pseudocontractive mapping and $\mathcal{A}_i : C \to H$ is η_i-inverse strongly monotone for each $i = 1, \dots, N$;
A is a $\bar{\gamma}$-strongly positive bounded linear operator on H with $\bar{\gamma} \in (1, 2)$ and $V : C \to H$ is an l-Lipschitzian mapping with $l \geq 0$;
$\Theta_i, \Phi_j : C \times C \to \mathbf{R}$ are the bifunctions satisfying conditions (A1)-(A4) and $\varphi_i : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with restrictions (B1) or (B2) for each $i = 1, \dots, N$ and $j = 1, 2$;
$0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \gamma l < \tau$ with $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;
$S : C \to C$ is a mapping defined by $Sx = \lambda x + (1 - \lambda)Tx$ for $0 \leq k \leq \lambda < 1$;
$G : C \to C$ is a mapping defined by $Gx = T^{\Phi_1}_{\nu_1}(I - \nu_1 F_1)T^{\Phi_2}_{\nu_2}(I - \nu_2 F_2)x$ with $0 < \nu_j < 2\zeta_j$ for $j = 1, 2$;
$\Delta^N_t : C \to C$ is a mapping defined by $\Delta^N_t x = T^{(\Theta_N,\varphi_N)}_{r_{N,t}}(I - r_{N,t}\mathcal{A}_N) \cdots T^{(\Theta_1,\varphi_1)}_{r_{1,t}}(I - r_{1,t}\mathcal{A}_1)x$, $t \in (0, 1)$, for $\{r_{i,t}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $i = 1, \dots, N$;
$\Delta^N_n : C \to C$ is a mapping defined by $\Delta^N_n x = T^{(\Theta_N,\varphi_N)}_{r_{N,n}}(I - r_{N,n}\mathcal{A}_N) \cdots T^{(\Theta_1,\varphi_1)}_{r_{1,n}}(I - r_{1,n}\mathcal{A}_1)x$ with $\{r_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\lim_{n\to\infty} r_{i,n} = r_i$, for each $i = 1, \dots, N$;
$\Omega := (\bigcap_{i=1}^N \mathrm{GMEP}(\Theta_i, \varphi_i, \mathcal{A}_i)) \cap \mathrm{Fix}(T) \cap \Xi \neq \emptyset$ and P_Ω is the metric projection of H onto Ω;
$\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset (0, 1]$ and $\{\theta_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})} \subset (0, 1)$.

Next, put

$$\Delta^i_t = T^{(\Theta_i,\varphi_i)}_{r_{i,t}}(I - r_{i,t}\mathcal{A}_i)T^{(\Theta_{i-1},\varphi_{i-1})}_{r_{i-1,t}}(I - r_{i-1,t}\mathcal{A}_{i-1}) \cdots T^{(\Theta_1,\varphi_1)}_{r_{1,t}}(I - r_{1,t}\mathcal{A}_1), \quad \forall t \in (0, 1),$$

and

$$\Delta^i_n = T^{(\Theta_i,\varphi_i)}_{r_{i,n}}(I - r_{i,n}\mathcal{A}_i)T^{(\Theta_{i-1},\varphi_{i-1})}_{r_{i-1,n}}(I - r_{i-1,n}\mathcal{A}_{i-1}) \cdots T^{(\Theta_1,\varphi_1)}_{r_{1,n}}(I - r_{1,n}\mathcal{A}_1), \quad \forall n \geq 0,$$

for all $i \in \{1, \dots, N\}$, and $\Delta^0_t = \Delta^0_n = I$, where I is the identity mapping on H.

By Lemma 2.4, we know that S is nonexpansive. It is clear that $\mathrm{Fix}(S) = \mathrm{Fix}(T)$. Since $\{\lambda_{i,t}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, utilizing (2.1) and Proposition 2.2(ii) we have for all $x, y \in C$

$$\begin{aligned}
\|\Delta^N_t x - \Delta^N_t y\| &= \|T^{(\Theta_N,\varphi_N)}_{r_{N,t}}(I - r_{N,t}\mathcal{A}_N)\Delta^{N-1}_t x - T^{(\Theta_N,\varphi_N)}_{r_{N,t}}(I - r_{N,t}\mathcal{A}_N)\Delta^{N-1}_t y\| \\
&\leq \|(I - r_{N,t}\mathcal{A}_N)\Delta^{N-1}_t x - (I - r_{N,t}\mathcal{A}_N)\Delta^{N-1}_t y\| \\
&\leq \|\Delta^{N-1}_t x - \Delta^{N-1}_t y\| \\
&\leq \cdots \\
&\leq \|\Delta^i_t x - \Delta^i_t y\| \\
&\leq \cdots \\
&\leq \|\Delta^0_t x - \Delta^0_t y\| \\
&= \|x - y\|,
\end{aligned}$$

which implies that $\Delta_t^i : C \to C$ is a nonexpansive mapping for all $t \in (0,1)$. Also, since $\{r_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, utilizing (2.1) and Proposition 2.2(ii) we have for all $x, y \in C$

$$\begin{aligned}
\|\Delta_n^N x - \Delta_n^N y\| &= \|T_{r_{N,n}}^{(\Theta_N,\varphi_N)}(I - r_{N,n}\mathcal{A}_N)\Delta_n^{N-1}x - T_{r_{N,n}}^{(\Theta_N,\varphi_N)}(I - r_{N,n}\mathcal{A}_N)\Delta_n^{N-1}y\| \\
&\leq \|(I - r_{N,n}\mathcal{A}_N)\Delta_n^{N-1}x - (I - r_{N,n}\mathcal{A}_N)\Delta_n^{N-1}y\| \\
&\leq \|\Delta_n^{N-1}x - \Delta_n^{N-1}y\| \\
&\leq \cdots \\
&\leq \|\Delta_n^i x - \Delta_n^i y\| \\
&\leq \cdots \\
&\leq \|\Delta_n^0 x - \Delta_n^0 y\| \\
&= \|x - y\|,
\end{aligned}$$

which implies that $\Delta_n^i : C \to C$ is a nonexpansive mapping for all $n \geq 0$.

In this section, we introduce the first composite relaxed extragradient-like scheme that generates a net $\{x_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})}$ in an implicit manner:

$$x_t = P_C\big[(I - \theta_t A)S\Delta_t^N Gx_t + \theta_t\big(t\gamma Vx_t + (I - t\mu F)S\Delta_t^N Gx_t\big)\big]. \tag{3.1}$$

We prove the strong convergence of $\{x_t\}$ as $t \to 0$ to a point $\tilde{x} \in \Omega$ which is a unique solution to the VIP

$$\big\langle (A - I)\tilde{x}, p - \tilde{x}\big\rangle \geq 0, \quad \forall p \in \Omega. \tag{3.2}$$

For arbitrarily given $x_0 \in C$, we also propose the second composite relaxed extragradient-like scheme, which generates a sequence $\{x_n\}$ in an explicit way:

$$\begin{cases} y_n = \alpha_n\gamma Vx_n + (I - \alpha_n\mu F)S\Delta_n^N Gx_n, \\ x_{n+1} = P_C[(I - \beta_n A)S\Delta_n^N Gx_n + \beta_n y_n], \quad \forall n \geq 0, \end{cases} \tag{3.3}$$

and establish the strong convergence of $\{x_n\}$ as $n \to \infty$ to the same point $\tilde{x} \in \Omega$, which is also the unique solution to VIP (3.2).

Now, for $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$, and $\theta_t \in (0, \|A\|^{-1}]$, consider a mapping $Q_t : C \to C$ defined by

$$Q_t x = P_C\big[(I - \theta_t A)S\Delta_t^N Gx + \theta_t\big(t\gamma Vx + (I - t\mu F)S\Delta_t^N Gx\big)\big], \quad \forall x \in C.$$

It is easy to see that Q_t is a contractive mapping with constant $1 - \theta_t(\bar{\gamma} - 1 + t(\tau - \gamma l))$. Indeed, by Proposition 2.3 and Lemmas 2.7 and 2.9, we have

$$\begin{aligned}
\|Q_t x - Q_t y\| &\leq \|(I - \theta_t A)S\Delta_t^N Gx + \theta_t\big(t\gamma Vx + (I - t\mu F)S\Delta_t^N Gx\big) \\
&\quad - (I - \theta_t A)S\Delta_t^N Gy - \theta_t\big(t\gamma Vx + (I - t\mu F)S\Delta_t^N Gy\big)\| \\
&\leq \|(I - \theta_t A)S\Delta_t^N Gx - (I - \theta_t A)S\Delta_t^N Gy\|
\end{aligned}$$

$$\begin{aligned}
&+ \theta_t \big\| \big(t\gamma Vx + (I - t\mu F)S\Delta_t^N Gx\big) - \big(t\gamma Vy + (I - t\mu F)S\Delta_t^N Gy\big) \big\| \\
&\le (1 - \theta_t\bar{\gamma}) \big\| S\Delta_t^N Gx - S\Delta_t^N Gy \big\| + \theta_t \big[t\gamma \|Vx - Vy\| \\
&\quad + \big\| (I - t\mu F)S\Delta_t^N Gx - (I - t\mu F)S\Delta_t^N Gy \big\| \big] \\
&\le (1 - \theta_t\bar{\gamma})\|x - y\| + \theta_t \big[t\gamma l \|x - y\| + (1 - t\tau)\|x - y\| \big] \\
&= \big[1 - \theta_t \big(\bar{\gamma} - 1 + t(\tau - \gamma l) \big) \big] \|x - y\|.
\end{aligned}$$

Since $\bar{\gamma} \in (1, 2)$, $\tau - \gamma l > 0$, and

$$0 < t < \min\left\{1, \frac{2 - \bar{\gamma}}{\tau - \gamma l}\right\} \le \frac{2 - \bar{\gamma}}{\tau - \gamma l},$$

it follows that

$$0 < \bar{\gamma} - 1 + t(\tau - \gamma l) < 1,$$

which together with $0 < \theta_t \le \|A\|^{-1} < 1$ yields

$$0 < 1 - \theta_t \big(\bar{\gamma} - 1 + t(\tau - \gamma l) \big) < 1.$$

Hence $Q_t : C \to C$ is a contractive mapping. By the Banach contraction principle, Q_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation (3.1).

We summarize the basic properties of $\{x_t\}$. The argument techniques in [1, 22, 32] extend to developing the new argument ones for these basic properties. We include the argument process for the sake of completeness.

Proposition 3.1 *Let* $\{x_t\}$ *be defined via* (3.1). *Then*

(i) $\{x_t\}$ *is bounded for* $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$;

(ii) $\lim_{t\to 0} \|x_t - Sx_t\| = 0$, $\lim_{t\to 0} \|x_t - Gx_t\| = 0$ *and* $\lim_{t\to 0} \|x_t - \Delta_t^N x_t\| = 0$ *provided* $\lim_{t\to 0} \theta_t = 0$;

(iii) $x_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to H$ *is locally Lipschitzian provided* $\theta_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to (0, \|A\|^{-1}]$ *is locally Lipschitzian, and* $\lambda_{i,t} : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to [a_i, b_i]$ *is locally Lipschitzian for each* $i = 1, \ldots, N$;

(iv) x_t *defines a continuous path from* $(0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$ *into* H *provided* $\theta_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to (0, \|A\|^{-1}]$ *is continuous, and* $\lambda_{i,t} : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to [a_i, b_i]$ *is continuous for each* $i = 1, \ldots, N$.

Proof (i) Let $p \in \Omega$. Noting that $\mathrm{Fix}(S) = \mathrm{Fix}(T)$, $Sp = p$, $Gp = p$, and $\Delta_t^i p = p$ for each $i = 1, \ldots, N$, by the nonexpansivity of S, G, and Δ_t^i, and Lemmas 2.7 and 2.9 we get

$$\begin{aligned}
&\|x_t - p\| \\
&\quad \le \big\| (I - \theta_t A)S\Delta_t^N Gx_t + \theta_t \big(t\gamma Vx_t + (I - t\mu F)S\Delta_t^N Gx_t \big) - p \big\| \\
&\quad = \big\| (I - \theta_t A)S\Delta_t^N Gx_t - (I - \theta_t A)S\Delta_t^N Gp \\
&\qquad + \theta_t \big(t\gamma Vx_t + (I - t\mu F)S\Delta_t^N Gx_t - p \big) + \theta_t (I - A)p \big\| \\
&\quad \le \big\| (I - \theta_t A)S\Delta_t^N Gx_t - (I - \theta_t A)S\Delta_t^N Gp \big\|
\end{aligned}$$

$$
\begin{aligned}
&+\theta_t\big\|t\gamma Vx_t+(I-t\mu F)S\Delta_t^N Gx_t-p\big\|+\theta_t\big\|(I-A)p\big\| \\
&=\big\|(I-\theta_t A)S\Delta_t^N Gx_t-(I-\theta_t A)S\Delta_t^N Gp\big\| \\
&\quad+\theta_t\big\|(I-t\mu F)S\Delta_t^N Gx_t-(I-t\mu F)S\Delta_t^N Gp \\
&\quad+t(\gamma Vx_t-\mu Fp)\big\|+\theta_t\big\|(I-A)p\big\| \\
&\le(1-\theta_t\bar{\gamma})\big\|S\Delta_t^N Gx_t-S\Delta_t^N Gp\big\| \\
&\quad+\theta_t\big[\big\|(I-t\mu F)S\Delta_t^N Gx_t-(I-t\mu F)S\Delta_t^N Gp\big\| \\
&\quad+t\big(\gamma\|Vx_t-Vp\|+\|\gamma Vp-\mu Fp\|\big)\big]+\theta_t\big\|(I-A)p\big\| \\
&\le(1-\theta_t\bar{\gamma})\|x_t-p\|+\theta_t\big[(1-t\tau)\|x_t-p\| \\
&\quad+t\big(\gamma l\|x_t-p\|+\big\|(\gamma V-\mu F)p\big\|\big)\big]+\theta_t\|I-A\|\|p\| \\
&=\big[1-\theta_t\big(\bar{\gamma}-1+t(\tau-\gamma l)\big)\big]\|x_t-p\|+\theta_t\big[\|I-A\|\|p\|+t\big\|(\gamma V-\mu F)p\big\|\big].
\end{aligned}
$$

So, it follows that

$$
\begin{aligned}
\|x_t-p\| &\le \frac{\|I-A\|\|p\|+t\|(\gamma V-\mu F)p\|}{\bar{\gamma}-1+t(\tau-\gamma l)} \\
&\le \frac{\|I-A\|\|p\|+t\|(\gamma V-\mu F)p\|}{\bar{\gamma}-1} \\
&\le \frac{\|I-A\|\|p\|+\|(\gamma V-\mu F)p\|}{\bar{\gamma}-1}.
\end{aligned}
$$

Hence $\{x_t\}$ is bounded and so are $\{Vx_t\}$, $\{\Delta_t^N x_t\}$, $\{S\Delta_t^N Gx_t\}$, and $\{FS\Delta_t^N Gx_t\}$.

(ii) By the definition of $\{x_t\}$, we have

$$
\begin{aligned}
&\big\|x_t-S\Delta_t^N Gx_t\big\| \\
&\quad=\big\|P_C\big[(I-\theta_t A)S\Delta_t^N Gx_t+\theta_t\big(t\gamma Vx_t+(I-t\mu F)S\Delta_t^N Gx_t\big)\big]-P_C S\Delta_t^N Gx_t\big\| \\
&\quad\le\big\|(I-\theta_t A)S\Lambda_t^N Gx_t+\theta_t\big(t\gamma Vx_t+(I-t\mu F)S\Lambda_t^N Gx_t\big)-S\Lambda_t^N Gx_t\big\| \\
&\quad=\big\|\theta_t\big[(I-A)S\Delta_t^N Gx_t+t\big(\gamma Vx_t-\mu FS\Delta_t^N Gx_t\big)\big]\big\| \\
&\quad=\theta_t\big\|(I-A)S\Delta_t^N Gx_t+t\big(\gamma Vx_t-\mu FS\Delta_t^N Gx_t\big)\big\| \\
&\quad\le\theta_t\|I-A\|\big\|S\Delta_t^N Gx_t\big\|+t\big\|\gamma Vx_t-\mu FS\Delta_t^N Gx_t\big\|\to 0\quad \text{as } t\to 0,
\end{aligned}
$$

by the boundedness of $\{Vx_t\}$, $\{S\Delta_t^N Gx_t\}$, and $\{FS\Delta_t^N Gx_t\}$ in the assertion (i). That is,

$$
\lim_{t\to 0}\big\|x_t-S\Delta_t^N Gx_t\big\|=0. \tag{3.4}
$$

Since $p=Gp=T_{\nu_1}^{\Phi_1}(I-\nu_1 F_1)T_{\nu_2}^{\Phi_2}(I-\nu_2 F_2)p$ and F_j is ζ_j-inverse strongly monotone with $0<\nu_j<2\zeta_j$ for $j=1,2$, by Proposition 2.2(ii) we deduce that

$$
\begin{aligned}
&\|Gx_t-p\|^2 \\
&\quad=\big\|T_{\nu_1}^{\Phi_1}(I-\nu_1 F_1)T_{\nu_2}^{\Phi_2}(I-\nu_2 F_2)x_t-T_{\nu_1}^{\Phi_1}(I-\nu_1 F_1)T_{\nu_2}^{\Phi_2}(I-\nu_2 F_2)p\big\|^2 \\
&\quad\le\big\|(I-\nu_1 F_1)T_{\nu_2}^{\Phi_2}(I-\nu_2 F_2)x_t-(I-\nu_1 F_1)T_{\nu_2}^{\Phi_2}(I-\nu_2 F_2)p\big\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \big\| \big[T_{\nu_2}^{\Phi_2}(I-\nu_2F_2)x_t - T_{\nu_2}^{\Phi_2}(I-\nu_2F_2)p\big] \\
&\quad - \nu_1\big[F_1T_{\nu_2}^{\Phi_2}(I-\nu_2F_2)x_t - F_1T_{\nu_2}^{\Phi_2}(I-\nu_2F_2)p\big]\big\|^2 \\
&\le \big\|T_{\nu_2}^{\Phi_2}(I-\nu_2F_2)x_t - T_{\nu_2}^{\Phi_2}(I-\nu_2F_2)p\big\|^2 \\
&\quad + \nu_1(\nu_1-2\zeta_1)\big\|F_1T_{\nu_2}^{\Phi_2}(I-\nu_2F_2)x_t - F_1T_{\nu_2}^{\Phi_2}(I-\nu_2F_2)p\big\|^2 \\
&\le \big\|T_{\nu_2}^{\Phi_2}(I-\nu_2F_2)x_t - T_{\nu_2}^{\Phi_2}(I-\nu_2F_2)p\big\|^2 \\
&\le \big\|(I-\nu_2F_2)x_t-(I-\nu_2F_2)p\big\|^2 \\
&= \big\|(x_t-p)-\nu_2(F_2x_t-F_2p)\big\|^2 \\
&\le \|x_t-p\|^2+\nu_2(\nu_2-2\zeta_2)\|F_2x_t-F_2p\|^2 \\
&\le \|x_t-p\|^2.
\end{aligned}
\tag{3.5}
$$

In the meantime, utilizing the η_i-inverse strong monotonicity of $\mathcal{A}_i$, we obtain

$$
\begin{aligned}
\big\|\Delta_t^iGx_t-p\big\|^2 &= \big\|T_{r_{i,t}}^{(\Theta_i,\varphi_i)}(I-r_{i,t}\mathcal{A}_i)\Delta_t^{i-1}Gx_t - T_{r_{i,t}}^{(\Theta_i,\varphi_i)}(I-r_{i,t}\mathcal{A}_i)p\big\|^2 \\
&\le \big\|(I-r_{i,t}\mathcal{A}_i)\Delta_t^{i-1}Gx_t-(I-r_{i,t}\mathcal{A}_i)p\big\|^2 \\
&= \big\|\Delta_t^{i-1}Gx_t-p-r_{i,t}\big(\mathcal{A}_i\Delta_t^{i-1}Gx_t-\mathcal{A}_ip\big)\big\|^2 \\
&\le \big\|\Delta_t^{i-1}Gx_t-p\big\|^2+r_{i,t}(r_{i,t}-2\eta_i)\big\|\mathcal{A}_i\Delta_t^{i-1}Gx_t-\mathcal{A}_ip\big\|^2 \\
&\le \|Gx_t-p\|^2+r_{i,t}(r_{i,t}-2\eta_i)\big\|\mathcal{A}_i\Delta_t^{i-1}Gx_t-\mathcal{A}_ip\big\|^2,
\end{aligned}
\tag{3.6}
$$

for each $i\in\{1,2,\dots,N\}$. Simple calculations show that

$$
\begin{aligned}
&x_t-p \\
&\quad = x_t-w_t+w_t-p \\
&\quad = x_t-w_t+(I-\theta_tA)S\Delta_t^NGx_t+\theta_t\big(t\gamma Vx_t+(I-t\mu F)S\Delta_t^NGx_t\big)-p \\
&\quad = x_t-w_t+(I-\theta_tA)S\Delta_t^NGx_t-(I-\theta_tA)S\Delta_t^NGp+\theta_t\big[t\gamma Vx_t \\
&\qquad + (I-t\mu F)S\Delta_t^NGx_t-p\big]+\theta_t(I-A)p \\
&\quad = x_t-w_t+(I-\theta_tA)\big(S\Delta_t^NGx_t-S\Delta_t^NGp\big)+\theta_t\big[t(\gamma Vx_t-\mu Fp) \\
&\qquad + (I-t\mu F)S\Delta_t^NGx_t-(I-t\mu F)p\big]+\theta_t(I-A)p,
\end{aligned}
\tag{3.7}
$$

where $w_t=(I-\theta_tA)S\Delta_t^NGx_t+\theta_t(t\gamma Vx_t+(I-t\mu F)S\Delta_t^NGx_t)$.

For simplicity, we write $\tilde{x}_t=T_{\nu_2}^{\Phi_2}(I-\nu_2F_2)x_t$, $\tilde{p}=T_{\nu_2}^{\Phi_2}(I-\nu_2F_2)p$ and $y_t=T_{\nu_1}^{\Phi_1}(I-\nu_1F_1)\tilde{x}_t$. Then we have $y_t=T_{\nu_1}^{\Phi_1}(I-\nu_1F_1)T_{\nu_2}^{\Phi_2}(I-\nu_2F_2)x_t$ and $p=Gp=T_{\nu_1}^{\Phi_1}(I-\nu_1F_1)\tilde{p}$. Then, by Propositions 2.1(i) and 2.3, and Lemmas 2.7 and 2.9, from (3.5)-(3.7) we obtain

$$
\begin{aligned}
&\|x_t-p\|^2 \\
&\quad = \langle x_t-w_t,x_t-p\rangle+\big\langle(I-\theta_tA)\big(S\Delta_t^NGx_t-S\Delta_t^NGp\big),x_t-p\big\rangle \\
&\qquad + \theta_t\big[t\langle\gamma Vx_t-\mu Fp,x_t-p\rangle+\big\langle(I-t\mu F)S\Delta_t^NGx_t-(I-t\mu F)p,x_t-p\big\rangle\big] \\
&\qquad + \theta_t\big\langle(I-A)p,x_t-p\big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\le \big\langle (I-\theta_t A)\big(S\Delta_t^N Gx_t - S\Delta_t^N Gp\big), x_t - p\big\rangle + \theta_t\big[t\langle \gamma Vx_t - \mu Fp, x_t - p\rangle \\
&\quad + \big\langle (I-t\mu F)S\Delta_t^N Gx_t - (I-t\mu F)p, x_t - p\big\rangle\big] + \theta_t\big\langle (I-A)p, x_t - p\big\rangle \\
&= \big\langle (I-\theta_t A)\big(S\Delta_t^N Gx_t - S\Delta_t^N Gp\big), x_t - p\big\rangle \\
&\quad + \theta_t\big[\big\langle (I-t\mu F)S\Delta_t^N Gx_t - (I-t\mu F)p, x_t - p\big\rangle \\
&\quad + t\big(\gamma\langle Vx_t - Vp, x_t - p\rangle + \langle \gamma Vp - \mu Fp, x_t - p\rangle\big)\big] \\
&\quad + \theta_t\big\langle (I-A)p, x_t - p\big\rangle \\
&\le \big\|(I-\theta_t A)\big(S\Delta_t^N Gx_t - S\Delta_t^N Gp\big)\big\|\|x_t - p\| \\
&\quad + \theta_t\big[\big\|(I-t\mu F)S\Delta_t^N Gx_t - (I-t\mu F)p\big\|\|x_t - p\| \\
&\quad + t\big(\gamma\|Vx_t - Vp\|\|x_t - p\| + \|\gamma Vp - \mu Fp\|\|x_t - p\|\big)\big] + \theta_t\big\|(I-A)p\big\|\|x_t - p\| \\
&\le (1-\theta_t\bar{\gamma})\big\|S\Delta_t^N Gx_t - S\Delta_t^N Gp\big\|\|x_t - p\| + \theta_t\big[(1-t\tau)\big\|\Delta_t^N Gx_t - p\big\|\|x_t - p\| \\
&\quad + t\big(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|\big)\big] + \theta_t\big\|(I-A)p\big\|\|x_t - p\| \\
&\le (1-\theta_t\bar{\gamma})\big\|\Delta_t^N Gx_t - p\big\|\|x_t - p\| + \theta_t\big[(1-t\tau)\big\|\Delta_t^N Gx_t - p\big\|\|x_t - p\| \\
&\quad + t\big(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|\big)\big] + \theta_t\big\|(I-A)p\big\|\|x_t - p\| \\
&= \big(1-\theta_t(\bar{\gamma}-1+t\tau)\big)\big\|\Delta_t^N Gx_t - p\big\|\|x_t - p\| \\
&\quad + \theta_t t\big(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|\big) + \theta_t\big\|(I-A)p\big\|\|x_t - p\| \\
&\le \big(1-\theta_t(\bar{\gamma}-1+t\tau)\big)\frac{1}{2}\big(\big\|\Delta_t^N Gx_t - p\big\|^2 + \|x_t - p\|^2\big) \\
&\quad + \theta_t t\big(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|\big) + \theta_t\big\|(I-A)p\big\|\|x_t - p\| \\
&\le \big(1-\theta_t(\bar{\gamma}-1+t\tau)\big)\frac{1}{2}\big(\big\|\Delta_t^i Gx_t - p\big\|^2 + \|x_t - p\|^2\big) \\
&\quad + \theta_t t\big(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|\big) + \theta_t\big\|(I-A)p\big\|\|x_t - p\| \\
&\le \big(1-\theta_t(\bar{\gamma}-1+t\tau)\big)\frac{1}{2}\big(\|Gx_t - p\|^2 \\
&\quad + r_{i,t}(r_{i,t}-2\eta_i)\big\|\mathcal{A}_i\Delta_t^{i-1}Gx_t - \mathcal{A}_i p\big\|^2 + \|x_t - p\|^2\big) \\
&\quad + \theta_t t\big(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|\big) + \theta_t\big\|(I-A)p\big\|\|x_t - p\| \\
&\le \big(1-\theta_t(\bar{\gamma}-1+t\tau)\big)\frac{1}{2}\big(\|x_t - p\|^2 + \nu_2(\nu_2-2\zeta_2)\|F_2x_t - F_2p\|^2 \\
&\quad + \nu_1(\nu_1-2\zeta_1)\|F_1\tilde{x}_t - F_1\tilde{p}\|^2 \\
&\quad + r_{i,t}(r_{i,t}-2\eta_i)\big\|\mathcal{A}_i\Delta_t^{i-1}Gx_t - \mathcal{A}_i p\big\|^2 + \|x_t - p\|^2\big) \\
&\quad + \theta_t t\big(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|\big) + \theta_t\big\|(I-A)p\big\|\|x_t - p\| \\
&= \big[1-\theta_t\big(\bar{\gamma}-1+t(\tau-\gamma l)\big)\big]\|x_t - p\|^2 - \frac{1-\theta_t(\bar{\gamma}-1+t\tau)}{2}\big[\nu_2(2\zeta_2-\nu_2)\|F_2x_t - F_2p\|^2 \\
&\quad + \nu_1(2\zeta_1-\nu_1)\|F_1\tilde{x}_t - F_1\tilde{p}\|^2 + r_{i,t}(2\eta_i - r_{i,t})\big\|\mathcal{A}_i\Delta_t^{i-1}Gx_t - \mathcal{A}_i p\big\|^2\big] \\
&\quad + \theta_t\big(t\|\gamma Vp - \mu Fp\|\|x_t - p\| + \big\|(I-A)p\big\|\|x_t - p\|\big) \\
&\le \|x_t - p\|^2 - \frac{1-\theta_t(\bar{\gamma}-1+t\tau)}{2}\big[\nu_2(2\zeta_2-\nu_2)\|F_2x_t - F_2p\|^2
\end{aligned}
$$

$$+ \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{x}_t - F_1\tilde{p}\|^2 + r_{i,t}(2\eta_i - r_{i,t})\big\|\mathcal{A}_i\Delta_t^{i-1}Gx_t - \mathcal{A}_ip\big\|^2\big] \\ + \theta_t\big(t\|\gamma Vp - \mu Fp\|\|x_t - p\| + \big\|(I-A)p\big\|\|x_t - p\|\big), \tag{3.8}$$

which together with $\nu_j \in (0, 2\zeta_j)$, $j = 1, 2$, and $\{r_{i,t}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $i = 1, \ldots, N$, implies that

$$\begin{aligned} &\frac{1-\theta_t(\bar{\gamma} - 1 + t\tau)}{2}\big[\nu_2(2\zeta_2 - \nu_2)\|F_2x_t - F_2p\|^2 \\ &\qquad + \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{x}_t - F_1\tilde{p}\|^2 + a_i(2\eta_i - b_i)\big\|\mathcal{A}_i\Delta_t^{i-1}Gx_t - \mathcal{A}_ip\big\|^2\big] \\ &\quad \le \frac{1-\theta_t(\bar{\gamma} - 1 + t\tau)}{2}\big[\nu_2(2\zeta_2 - \nu_2)\|F_2x_t - F_2p\|^2 \\ &\qquad + \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{x}_t - F_1\tilde{p}\|^2 + r_{i,t}(2\eta_i - r_{i,t})\big\|\mathcal{A}_i\Delta_t^{i-1}Gx_t - \mathcal{A}_ip\big\|^2\big] \\ &\quad \le \theta_t\big(t\|\gamma Vp - \mu Fp\|\|x_t - p\| + \big\|(I-A)p\big\|\|x_t - p\|\big). \end{aligned}$$

Since $\lim_{t\to 0}\theta_t = 0$ and $\{x_t\}$ is bounded, we have

$$\begin{aligned} &\lim_{t\to 0}\|F_2x_t - F_2p\| = 0, \qquad \lim_{t\to 0}\|F_1\tilde{x}_t - F_1\tilde{p}\| = 0 \quad \text{and} \\ &\lim_{t\to 0}\big\|\mathcal{A}_i\Delta_t^{i-1}Gx_t - \mathcal{A}_ip\big\| = 0 \end{aligned} \tag{3.9}$$

for each $i = 1, \ldots, N$.

On the other hand, in terms of the firm nonexpansivity of $T_{\nu_j}^{\Phi_j}$ and the ζ_j-inverse strong monotonicity of F_j for $j = 1, 2$, we obtain from $\nu_j \in (0, 2\zeta_j)$, $j = 1, 2$, and (3.5)

$$\begin{aligned} \|\tilde{x}_t - \tilde{p}\|^2 &= \big\|T_{\nu_2}^{\Phi_2}(I - \nu_2F_2)x_t - T_{\nu_2}^{\Phi_2}(I - \nu_2F_2)p\big\|^2 \\ &\le \big\langle (I - \nu_2F_2)x_t - (I - \nu_2F_2)p, \tilde{x}_t - \tilde{p}\big\rangle \\ &= \frac{1}{2}\big[\big\|(I - \nu_2F_2)x_t - (I - \nu_2F_2)p\big\|^2 + \|\tilde{x}_t - \tilde{p}\|^2 \\ &\quad - \big\|(I - \nu_2F_2)x_t - (I - \nu_2F_2)p - (\tilde{x}_t - \tilde{p})\big\|^2\big] \\ &\le \frac{1}{2}\big[\|x_t - p\|^2 + \|\tilde{x}_t - \tilde{p}\|^2 - \big\|(x_t - \tilde{x}_t) - \nu_2(F_2x_t - F_2p) - (p - \tilde{p})\big\|^2\big] \\ &= \frac{1}{2}\big[\|x_t - p\|^2 + \|\tilde{x}_t - \tilde{p}\|^2 - \big\|(x_t - \tilde{x}_t) - (p - \tilde{p})\big\|^2 \\ &\quad + 2\nu_2\big\langle (x_t - \tilde{x}_t) - (p - \tilde{p}), F_2x_t - F_2p\big\rangle - \nu_2^2\|F_2x_t - F_2p\|^2\big] \end{aligned}$$

and

$$\begin{aligned} \|y_t - p\|^2 &= \big\|T_{\nu_1}^{\Phi_1}(I - \nu_1F_1)\tilde{x}_t - T_{\nu_1}^{\Phi_1}(I - \nu_1F_1)(I - \nu_1F_1)\tilde{p}\big\|^2 \\ &\le \big\langle (I - \nu_1F_1)\tilde{x}_t - (I - \nu_1F_1)\tilde{p}, y_t - p\big\rangle \\ &= \frac{1}{2}\big[\big\|(I - \nu_1F_1)\tilde{x}_t - (I - \nu_1F_1)\tilde{p}\big\|^2 + \|y_t - p\|^2 \\ &\quad - \big\|(I - \nu_1F_1)\tilde{x}_t - (I - \nu_1F_1)\tilde{p} - (y_t - p)\big\|^2\big] \\ &\le \frac{1}{2}\big[\|\tilde{x}_t - \tilde{p}\|^2 + \|y_t - p\|^2 - \big\|(\tilde{x}_t - y_t) + (p - \tilde{p})\big\|^2 \end{aligned}$$

$$
\begin{aligned}
&\quad + 2\nu_1\langle F_1\tilde{x}_t - F_1\tilde{p}, (\tilde{x}_t - y_t) + (p - \tilde{p})\rangle - \nu_1^2\|F_1\tilde{x}_t - F_1\tilde{p}\|^2\big] \\
&\leq \frac{1}{2}\big[\|x_t - p\|^2 + \|y_t - p\|^2 - \big\|(\tilde{x}_t - y_t) + (p - \tilde{p})\big\|^2 \\
&\quad + 2\nu_1\langle F_1\tilde{x}_t - F_1\tilde{p}, (\tilde{x}_t - y_t) + (p - \tilde{p})\rangle\big].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\|\tilde{x}_t - \tilde{p}\|^2 \\
&\quad \leq \|x_t - p\|^2 - \big\|(x_t - \tilde{x}_t) - (p - \tilde{p})\big\|^2 + 2\nu_2\langle (x_t - \tilde{x}_t) - (p - \tilde{p}), F_2x_t - F_2p\rangle \\
&\qquad - \nu_2^2\|F_2x_t - F_2p\|^2
\end{aligned}
\tag{3.10}
$$

and

$$
\|y_t - p\|^2 \leq \|x_t - p\|^2 - \big\|(\tilde{x}_t - y_t) + (p - \tilde{p})\big\|^2 + 2\nu_1\|F_1\tilde{x}_t - F_1\tilde{p}\|\big\|(\tilde{x}_t - y_t) + (p - \tilde{p})\big\|. \tag{3.11}
$$

Consequently, from (3.5), (3.8), and (3.10) it follows that

$$
\begin{aligned}
&\|x_t - p\|^2 \\
&\quad \leq \big(1 - \theta_t(\bar{\gamma} - 1 + t\tau)\big)\frac{1}{2}\big(\|Gx_t - p\|^2 \\
&\qquad + r_{i,t}(r_{i,t} - 2\eta_i)\big\|\mathcal{A}_i\Delta_t^{i-1}Gx_t - \mathcal{A}_ip\big\|^2 + \|x_t - p\|^2\big) \\
&\qquad + \theta_t t\big(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|\big) + \theta_t\big\|(I - A)p\big\|\|x_t - p\| \\
&\quad \leq \big(1 - \theta_t(\bar{\gamma} - 1 + t\tau)\big)\frac{1}{2}\big(\|Gx_t - p\|^2 + \|x_t - p\|^2\big) \\
&\qquad + \theta_t t\big(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|\big) + \theta_t\big\|(I - A)p\big\|\|x_t - p\| \\
&\quad \leq \big(1 - \theta_t(\bar{\gamma} - 1 + t\tau)\big)\frac{1}{2}\big[\|\tilde{x}_t - \tilde{p}\|^2 + \nu_1(\nu_1 - 2\zeta_1)\|F_1\tilde{x}_t - F_1\tilde{p}\|^2 + \|x_t - p\|^2\big] \\
&\qquad + \theta_t t\big(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|\big) + \theta_t\big\|(I - A)p\big\|\|x_t - p\| \\
&\quad \leq \big(1 - \theta_t(\bar{\gamma} - 1 + t\tau)\big)\frac{1}{2}\big[\|\tilde{x}_t - \tilde{p}\|^2 + \|x_t - p\|^2\big] \\
&\qquad + \theta_t t\big(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|\big) + \theta_t\big\|(I - A)p\big\|\|x_t - p\| \\
&\quad \leq \big(1 - \theta_t(\bar{\gamma} - 1 + t\tau)\big)\frac{1}{2}\big[\|x_t - p\|^2 - \big\|(x_t - \tilde{x}_t) - (p - \tilde{p})\big\|^2 \\
&\qquad + 2\nu_2\langle (x_t - \tilde{x}_t) - (p - \tilde{p}), F_2x_t - F_2p\rangle - \nu_2^2\|F_2x_t - F_2p\|^2 + \|x_t - p\|^2\big] \\
&\qquad + \theta_t t\big(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|\big) + \theta_t\big\|(I - A)p\big\|\|x_t - p\| \\
&\quad \leq \big[1 - \theta_t\big(\bar{\gamma} - 1 + t(\tau - \gamma l)\big)\big]\|x_t - p\|^2 \\
&\qquad - \big(1 - \theta_t(\bar{\gamma} - 1 + t\tau)\big)\frac{1}{2}\big\|(x_t - \tilde{x}_t) - (p - \tilde{p})\big\|^2 \\
&\qquad + \nu_2\big\|(x_t - \tilde{x}_t) - (p - \tilde{p})\big\|\|F_2x_t - F_2p\| \\
&\qquad + \theta_t\big(t\|\gamma Vp - \mu Fp\|\|x_t - p\| + \big\|(I - A)p\big\|\|x_t - p\|\big)
\end{aligned}
$$

$$\begin{aligned}&\le \|x_t-p\|^2-\big(1-\theta_t(\bar{\gamma}-1+t\tau)\big)\frac{1}{2}\big\|(x_t-\tilde{x}_t)-(p-\tilde{p})\big\|^2\\&\quad+\nu_2\big\|(x_t-\tilde{x}_t)-(p-\tilde{p})\big\|\|F_2x_t-F_2p\|\\&\quad+\theta_t\big(t\|\gamma Vp-\mu Fp\|\|x_t-p\|+\big\|(I-A)p\big\|\|x_t-p\|\big),\end{aligned}$$

which hence leads to

$$\begin{aligned}&\big(1-\theta_t(\bar{\gamma}-1+t\tau)\big)\frac{1}{2}\big\|(x_t-\tilde{x}_t)-(p-\tilde{p})\big\|^2\\&\quad\le \nu_2\big\|(x_t-\tilde{x}_t)-(p-\tilde{p})\big\|\|F_2x_t-F_2p\|\\&\qquad+\theta_t\big(t\|\gamma Vp-\mu Fp\|\|x_t-p\|+\big\|(I-A)p\big\|\|x_t-p\|\big).\end{aligned}$$

Since $\lim_{t\to 0}\theta_t=0$ and $\lim_{t\to 0}\|F_2x_t-F_2p\|=0$ (due to (3.9)), we deduce from the boundedness of $\{x_t\}$ and $\{\tilde{x}_t\}$ that

$$\lim_{t\to 0}\big\|(x_t-\tilde{x}_t)-(p-\tilde{p})\big\|=0. \tag{3.12}$$

Furthermore, from (3.5), (3.8), and (3.11) it follows that

$$\begin{aligned}&\|x_t-p\|^2\\&\quad\le \big(1-\theta_t(\bar{\gamma}-1+t\tau)\big)\frac{1}{2}\big(\|Gx_t-p\|^2\\&\qquad+r_{i,t}(r_{i,t}-2\eta_i)\big\|\mathcal{A}_i\Delta_t^{i-1}Gx_t-\mathcal{A}_ip\big\|^2+\|x_t-p\|^2\big)\\&\qquad+\theta_tt\big(\gamma l\|x_t-p\|^2+\|\gamma Vp-\mu Fp\|\|x_t-p\|\big)+\theta_t\big\|(I-A)p\big\|\|x_t-p\|\\&\quad\le \big(1-\theta_t(\bar{\gamma}-1+t\tau)\big)\frac{1}{2}\big(\|Gx_t-p\|^2+\|x_t-p\|^2\big)\\&\qquad+\theta_tt\big(\gamma l\|x_t-p\|^2+\|\gamma Vp-\mu Fp\|\|x_t-p\|\big)+\theta_t\big\|(I-A)p\big\|\|x_t-p\|\\&\quad= \big(1-\theta_t(\bar{\gamma}-1+t\tau)\big)\frac{1}{2}\big(\|y_t-p\|^2+\|x_t-p\|^2\big)\\&\qquad+\theta_tt\big(\gamma l\|x_t-p\|^2+\|\gamma Vp-\mu Fp\|\|x_t-p\|\big)+\theta_t\big\|(I-A)p\big\|\|x_t-p\|\\&\quad\le \big(1-\theta_t(\bar{\gamma}-1+t\tau)\big)\frac{1}{2}\big[\|x_t-p\|^2-\big\|(\tilde{x}_t-y_t)+(p-\tilde{p})\big\|^2\\&\qquad+2\nu_1\|F_1\tilde{x}_t-F_1\tilde{p}\|\big\|(\tilde{x}_t-y_t)+(p-\tilde{p})\big\|+\|x_t-p\|^2\big]\\&\qquad+\theta_tt\big(\gamma l\|x_t-p\|^2+\|\gamma Vp-\mu Fp\|\|x_t-p\|\big)+\theta_t\big\|(I-A)p\big\|\|x_t-p\|\\&\quad\le \big[1-\theta_t\big(\bar{\gamma}-1+t(\tau-\gamma l)\big)\big]\|x_t-p\|^2\\&\qquad-\big(1-\theta_t(\bar{\gamma}-1+t\tau)\big)\frac{1}{2}\big\|(\tilde{x}_t-y_t)+(p-\tilde{p})\big\|^2+\nu_1\|F_1\tilde{x}_t-F_1\tilde{p}\|\big\|(\tilde{x}_t-y_t)+(p-\tilde{p})\big\|\\&\qquad+\theta_t\big(t\|\gamma Vp-\mu Fp\|\|x_t-p\|+\big\|(I-A)p\big\|\|x_t-p\|\big)\\&\quad\le \|x_t-p\|^2-\big(1-\theta_t(\bar{\gamma}-1+t\tau)\big)\frac{1}{2}\big\|(\tilde{x}_t-y_t)+(p-\tilde{p})\big\|^2\\&\qquad+\nu_1\|F_1\tilde{x}_t-F_1\tilde{p}\|\big\|(\tilde{x}_t-y_t)+(p-\tilde{p})\big\|\\&\qquad+\theta_t\big(t\|\gamma Vp-\mu Fp\|\|x_t-p\|+\big\|(I-A)p\big\|\|x_t-p\|\big),\end{aligned}$$

which hence yields

$$\begin{aligned}
&\big(1-\theta_t(\bar{\gamma}-1+t\tau)\big)\frac{1}{2}\big\|(\tilde{x}_t-y_t)+(p-\tilde{p})\big\|^2\\
&\quad\le \nu_1\|F_1\tilde{x}_t-F_1\tilde{p}\|\big\|(\tilde{x}_t-y_t)+(p-\tilde{p})\big\|\\
&\qquad+\theta_t\big(t\|\gamma Vp-\mu Fp\|\|x_t-p\|+\big\|(I-A)p\big\|\|x_t-p\|\big).
\end{aligned}$$

Since $\lim_{t\to 0}\theta_t=0$ and $\lim_{t\to 0}\|F_1\tilde{x}_t-F_1\tilde{p}\|=0$ (due to (3.9)), we deduce from the boundedness of $\{x_t\}$, $\{y_t\}$, and $\{\tilde{x}_t\}$ that

$$\lim_{t\to 0}\big\|(\tilde{x}_t-y_t)+(p-\tilde{p})\big\|=0. \tag{3.13}$$

Note that

$$\|x_t-y_t\|\le\big\|(x_t-\tilde{x}_t)-(p-\tilde{p})\big\|+\big\|(\tilde{x}_t-y_t)+(p-\tilde{p})\big\|.$$

Hence from (3.12) and (3.13) we get

$$\lim_{t\to 0}\|x_t-Gx_t\|=\lim_{t\to 0}\|x_t-y_t\|=0. \tag{3.14}$$

Utilizing Proposition 2.2(ii) and Lemma 2.2(a), we obtain for each $i\in\{1,\ldots,N\}$

$$\begin{aligned}
&\big\|\Delta_t^iGx_t-p\big\|^2\\
&\quad=\big\|T_{r_{i,t}}^{(\Theta_i,\varphi_i)}(I-r_{i,t}\mathcal{A}_i)\Delta_t^{i-1}Gx_t-T_{r_{i,t}}^{(\Theta_i,\varphi_i)}(I-r_{i,t}\mathcal{A}_i)p\big\|^2\\
&\quad\le\big\langle(I-r_{i,t}\mathcal{A}_i)\Delta_t^{i-1}Gx_t-(I-r_{i,t}\mathcal{A}_i)p,\Delta_t^iGx_t-p\big\rangle\\
&\quad=\frac{1}{2}\big(\big\|(I-r_{i,t}\mathcal{A}_i)\Delta_t^{i-1}Gx_t-(I-r_{i,t}\mathcal{A}_i)p\big\|^2+\big\|\Delta_t^iGx_t-p\big\|^2\\
&\qquad-\big\|(I-r_{i,t}\mathcal{A}_i)\Delta_t^{i-1}Gx_t-(I-r_{i,t}\mathcal{A}_i)p-\big(\Delta_t^iGx_t-p\big)\big\|^2\big)\\
&\quad\le\frac{1}{2}\big(\big\|\Delta_t^{i-1}Gx_t-p\big\|^2+\big\|\Delta_t^iGx_t-p\big\|^2-\big\|\Delta_t^{i-1}Gx_t-\Delta_t^iGx_t\\
&\qquad-r_{i,t}\big(\mathcal{A}_i\Delta_t^{i-1}Gx_t-\mathcal{A}_ip\big)\big\|^2\big)\\
&\quad\le\frac{1}{2}\big(\|x_t-p\|^2+\big\|\Delta_t^iGx_t-p\big\|^2-\big\|\Delta_t^{i-1}Gx_t-\Delta_t^iGx_t-r_{i,t}\big(\mathcal{A}_i\Delta_t^{i-1}Gx_t-\mathcal{A}_ip\big)\big\|^2\big),
\end{aligned}$$

which immediately leads to

$$\begin{aligned}
&\big\|\Delta_t^iGx_t-p\big\|^2\\
&\quad\le\|x_t-p\|^2-\big\|\Delta_t^{i-1}Gx_t-\Delta_t^iGx_t-r_{i,t}\big(\mathcal{A}_i\Delta_t^{i-1}Gx_t-\mathcal{A}_ip\big)\big\|^2\\
&\quad=\|x_t-p\|^2-\big\|\Delta_t^{i-1}Gx_t-\Delta_t^iGx_t\big\|^2-r_{i,t}^2\big\|\mathcal{A}_i\Delta_t^{i-1}Gx_t-\mathcal{A}_ip\big\|^2\\
&\qquad+2r_{i,t}\big\langle\Delta_t^{i-1}Gx_t-\Delta_t^iGx_t,\mathcal{A}_i\Delta_t^{i-1}Gx_t-\mathcal{A}_ip\big\rangle\\
&\quad\le\|x_t-p\|^2-\big\|\Delta_t^{i-1}Gx_t-\Delta_t^iGx_t\big\|^2\\
&\qquad+2r_{i,t}\big\|\Delta_t^{i-1}Gx_t-\Delta_t^iGx_t\big\|\big\|\mathcal{A}_i\Delta_t^{i-1}Gx_t-\mathcal{A}_ip\big\|.
\end{aligned} \tag{3.15}$$

Combining (3.8) and (3.15) we conclude that

$$\begin{aligned}
&\|x_t - p\|^2 \\
&\quad \le \bigl(1 - \theta_t(\bar{\gamma} - 1 + t\tau)\bigr)\frac{1}{2}\bigl(\bigl\|\Delta_t^i Gx_t - p\bigr\|^2 + \|x_t - p\|^2\bigr) \\
&\qquad + \theta_t t\bigl(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|\bigr) + \theta_t\bigl\|(I - A)p\bigr\|\|x_t - p\| \\
&\quad \le \bigl(1 - \theta_t(\bar{\gamma} - 1 + t\tau)\bigr)\frac{1}{2}\bigl[\|x_t - p\|^2 - \bigl\|\Delta_t^{i-1}Gx_t - \Delta_t^i Gx_t\bigr\|^2 \\
&\qquad + 2r_{i,t}\bigl\|\Delta_t^{i-1}Gx_t - \Delta_t^i Gx_t\bigr\|\bigl\|\mathcal{A}_i\Delta_t^{i-1}Gx_t - \mathcal{A}_i p\bigr\| + \|x_t - p\|^2\bigr] \\
&\qquad + \theta_t t\bigl(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|\bigr) + \theta_t\bigl\|(I - A)p\bigr\|\|x_t - p\| \\
&\quad \le \bigl[1 - \theta_t\bigl(\bar{\gamma} - 1 + t(\tau - \gamma l)\bigr)\bigr]\|x_t - p\|^2 \\
&\qquad - \bigl(1 - \theta_t(\bar{\gamma} - 1 + t\tau)\bigr)\frac{1}{2}\bigl\|\Delta_t^{i-1}Gx_t - \Delta_t^i Gx_t\bigr\|^2 \\
&\qquad + r_{i,t}\bigl\|\Delta_t^{i-1}Gx_t - \Delta_t^i Gx_t\bigr\|\bigl\|\mathcal{A}_i\Delta_t^{i-1}Gx_t - \mathcal{A}_i p\bigr\| \\
&\qquad + \theta_t\bigl(t\|\gamma Vp - \mu Fp\|\|x_t - p\| + \bigl\|(I - A)p\bigr\|\|x_t - p\|\bigr) \\
&\quad \le \|x_t - p\|^2 - \frac{1 - \theta_t(\bar{\gamma} - 1 + t\tau)}{2}\bigl\|\Delta_t^{i-1}Gx_t - \Delta_t^i Gx_t\bigr\|^2 \\
&\qquad + r_{i,t}\bigl\|\Delta_t^{i-1}Gx_t - \Delta_t^i Gx_t\bigr\|\bigl\|\mathcal{A}_i\Delta_t^{i-1}Gx_t - \mathcal{A}_i p\bigr\| \\
&\qquad + \theta_t\bigl(t\|\gamma Vp - \mu Fp\|\|x_t - p\| + \bigl\|(I - A)p\bigr\|\|x_t - p\|\bigr),
\end{aligned}$$

which hence yields

$$\begin{aligned}
&\frac{1 - \theta_t(\bar{\gamma} - 1 + t\tau)}{2}\bigl\|\Delta_t^{i-1}Gx_t - \Delta_t^i Gx_t\bigr\|^2 \\
&\quad \le r_{i,t}\bigl\|\Delta_t^{i-1}Gx_t - \Delta_t^i Gx_t\bigr\|\bigl\|\mathcal{A}_i\Delta_t^{i-1}Gx_t - \mathcal{A}_i p\bigr\| \\
&\qquad + \theta_t\bigl(t\|\gamma Vp - \mu Fp\|\|x_t - p\| + \bigl\|(I - A)p\bigr\|\|x_t - p\|\bigr).
\end{aligned}$$

Since $\{r_{i,t}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $\lim_{t\to 0}\theta_t = 0$ and $\lim_{t\to 0}\|\mathcal{A}_i\Delta_t^{i-1}Gx_t - \mathcal{A}_i p\| = 0$ (due to (3.9)), we deduce from the boundedness of $\{x_t\}$ and $\{\Delta_t^i Gx_t\}$ that

$$\lim_{t\to 0}\bigl\|\Delta_t^{i-1}Gx_t - \Delta_t^i Gx_t\bigr\| = 0, \quad \forall i \in \{1, \dots, N\}. \tag{3.16}$$

Note that

$$\begin{aligned}
\bigl\|Gx_t - \Delta_t^N Gx_t\bigr\| &= \bigl\|\Delta_t^0 Gx_t - \Delta_t^N Gx_t\bigr\| \\
&\le \bigl\|\Delta_t^0 Gx_t - \Delta_t^1 Gx_t\bigr\| + \bigl\|\Delta_t^1 Gx_t - \Delta_t^2 Gx_t\bigr\| + \cdots \\
&\quad + \bigl\|\Delta_t^{N-1}Gx_t - \Delta_t^N Gx_t\bigr\|.
\end{aligned}$$

Hence, from (3.16) we get

$$\lim_{t\to 0}\bigl\|Gx_t - \Delta_t^N Gx_t\bigr\| = 0. \tag{3.17}$$

Also, observe that

$$\begin{aligned}\|x_t - \Delta_t^N x_t\| &\le \|x_t - Gx_t\| + \|Gx_t - \Delta_t^N Gx_t\| + \|\Delta_t^N Gx_t - \Delta_t^N x_t\| \\ &\le 2\|x_t - Gx_t\| + \|Gx_t - \Delta_t^N Gx_t\|.\end{aligned}$$

So, it follows from (3.14) and (3.17) that

$$\lim_{t\to 0}\|x_t - \Delta_t^N x_t\| = 0. \tag{3.18}$$

In addition, it is not hard to find that

$$\begin{aligned}\|x_t - Sx_t\| &\le \|x_t - S\Delta_t^N Gx_t\| + \|S\Delta_t^N Gx_t - S\Delta_t^N x_t\| + \|S\Delta_t^N x_t - Sx_t\| \\ &\le \|x_t - S\Delta_t^N Gx_t\| + \|Gx_t - x_t\| + \|\Delta_t^N x_t - x_t\|.\end{aligned}$$

Consequently, from (3.4), (3.14), and (3.18) we deduce that

$$\lim_{t\to 0}\|x_t - Sx_t\| = 0. \tag{3.19}$$

(iii) Let $t, t_0 \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$. Utilizing Proposition 2.2(ii), (v), we deduce that

$$\begin{aligned}&\|\Delta_t^N Gx_t - \Delta_{t_0}^N Gx_{t_0}\| \\ &= \|T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}\mathcal{A}_N)\Delta_t^{N-1}Gx_t - T_{r_{N,t_0}}^{(\Theta_N,\varphi_N)}(I - r_{N,t_0}\mathcal{A}_N)\Delta_{t_0}^{N-1}Gx_{t_0}\| \\ &\le \|T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}\mathcal{A}_N)\Delta_t^{N-1}Gx_t - T_{r_{N,t_0}}^{(\Theta_N,\varphi_N)}(I - r_{N,t_0}\mathcal{A}_N)\Delta_t^{N-1}Gx_t\| \\ &\quad + \|T_{r_{N,t_0}}^{(\Theta_N,\varphi_N)}(I - r_{N,t_0}\mathcal{A}_N)\Delta_t^{N-1}Gx_t - T_{r_{N,t_0}}^{(\Theta_N,\varphi_N)}(I - r_{N,t_0}\mathcal{A}_N)\Delta_{t_0}^{N-1}Gx_{t_0}\| \\ &\le \|T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}\mathcal{A}_N)\Delta_t^{N-1}Gx_t - T_{r_{N,t_0}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}\mathcal{A}_N)\Delta_t^{N-1}Gx_t\| \\ &\quad + \|T_{r_{N,t_0}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}\mathcal{A}_N)\Delta_t^{N-1}Gx_t - T_{r_{N,t_0}}^{(\Theta_N,\varphi_N)}(I - r_{N,t_0}\mathcal{A}_N)\Delta_t^{N-1}Gx_t\| \\ &\quad + \|(I - r_{N,t_0}\mathcal{A}_N)\Delta_t^{N-1}Gx_t - (I - r_{N,t_0}\mathcal{A}_N)\Delta_{t_0}^{N-1}Gx_{t_0}\| \\ &\le \frac{|r_{N,t} - r_{N,t_0}|}{r_{N,t}}\|T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}\mathcal{A}_N)\Delta_t^{N-1}Gx_t - (I - r_{N,t}\mathcal{A}_N)\Delta_t^{N-1}Gx_t\| \\ &\quad + |r_{N,t} - r_{N,t_0}|\|\mathcal{A}_N\Delta_t^{N-1}Gx_t\| + \|\Delta_t^{N-1}Gx_t - \Delta_{t_0}^{N-1}Gx_{t_0}\| \\ &= |r_{N,t} - r_{N,t_0}|\biggl[\|\mathcal{A}_N\Delta_t^{N-1}Gx_t\| + \frac{1}{r_{N,t}}\|T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}\mathcal{A}_N)\Delta_t^{N-1}Gx_t \\ &\quad - (I - r_{N,t}\mathcal{A}_N)\Delta_t^{N-1}Gx_t\|\biggr] + \|\Delta_t^{N-1}Gx_t - \Delta_{t_0}^{N-1}Gx_{t_0}\| \\ &\le \cdots \\ &\le |r_{N,t} - r_{N,t_0}|\biggl[\|\mathcal{A}_N\Delta_t^{N-1}Gx_t\| + \frac{1}{r_{N,t}}\|T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}\mathcal{A}_N)\Delta_t^{N-1}Gx_t \\ &\quad - (I - r_{N,t}\mathcal{A}_N)\Delta_t^{N-1}Gx_t\|\biggr] + \cdots + |r_{1,t} - r_{1,t_0}|\biggl[\|\mathcal{A}_1\Delta_t^0 Gx_t\|\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{r_{1,t}} \left\| T_{r_{1,t}}^{(\Theta_1,\varphi_1)}(I - r_{1,t}\mathcal{A}_1)\Delta_t^0 Gx_t - (I - r_{1,t}\mathcal{A}_1)\Delta_t^0 Gx_t \right\| \Bigg] \\
&+ \left\| \Delta_t^0 Gx_t - \Delta_{t_0}^0 Gx_{t_0} \right\| \\
&\le \widetilde{M}_0 \sum_{i=1}^N |r_{i,t} - r_{i,t_0}| + \|x_t - x_{t_0}\|,
\end{aligned}
\tag{3.20}
$$

where $\widetilde{M}_0 > 0$ is a constant such that for each $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$

$$
\sum_{i=1}^N \left[\left\| \mathcal{A}_i \Delta_t^{i-1} Gx_t \right\| + \frac{1}{r_{i,t}} \left\| T_{r_{i,t}}^{(\Theta_i,\varphi_i)}(I - r_{i,t}\mathcal{A}_i)\Delta_t^{i-1} Gx_t - (I - r_{i,t}\mathcal{A}_i)\Delta_t^{i-1} Gx_t \right\| \right] \le \widetilde{M}_0.
$$

In terms of (3.20) we calculate

$$
\begin{aligned}
&\|x_t - x_{t_0}\| \\
&\quad \le \left\| (I - \theta_t A) S\Delta_t^N Gx_t + \theta_t \big(t\gamma Vx_t + (I - t\mu F) S\Delta_t^N Gx_t \big) \right. \\
&\qquad \left. - (I - \theta_{t_0} A) S\Delta_{t_0}^N Gx_{t_0} - \theta_{t_0} \big(t_0\gamma Vx_{t_0} + (I - t_0\mu F) S\Delta_{t_0}^N Gx_{t_0} \big) \right\| \\
&\quad \le \left\| (I - \theta_t A) S\Delta_t^N Gx_t - (I - \theta_{t_0} A) S\Delta_t^N Gx_t \right\| \\
&\qquad + \left\| (I - \theta_{t_0} A) S\Delta_t^N Gx_t - (I - \theta_{t_0} A) S\Delta_{t_0}^N Gx_{t_0} \right\| \\
&\qquad + |\theta_t - \theta_{t_0}| \left\| t\gamma Vx_t + (I - t\mu F) S\Delta_t^N Gx_t \right\| \\
&\qquad + \theta_{t_0} \left\| \big[t\gamma Vx_t + (I - t\mu F) S\Delta_t^N Gx_t \big] - \big[t_0\gamma Vx_{t_0} + (I - t_0\mu F) S\Delta_{t_0}^N Gx_{t_0} \big] \right\| \\
&\quad \le |\theta_t - \theta_{t_0}| \|A\| \left\| S\Delta_t^N Gx_t \right\| + (1 - \theta_{t_0}\bar{\gamma}) \left\| S\Delta_t^N Gx_t - S\Delta_{t_0}^N Gx_{t_0} \right\| \\
&\qquad + |\theta_t - \theta_{t_0}| \left\| t\gamma Vx_t + (I - t\mu F) S\Delta_t^N Gx_t \right\| + \theta_{t_0} \left\| (t - t_0)\gamma Vx_t \right. \\
&\qquad + t_0\gamma (Vx_t - Vx_{t_0}) - (t - t_0)\mu FS\Delta_t^N Gx_t + (I - t_0\mu F) S\Delta_t^N Gx_t \\
&\qquad \left. - (I - t_0\mu F) S\Delta_{t_0}^N Gx_{t_0} \right\| \\
&\quad \le |\theta_t - \theta_{t_0}| \|A\| \left\| S\Delta_t^N Gx_t \right\| + (1 - \theta_{t_0}\bar{\gamma}) \left\| \Delta_t^N Gx_t - \Delta_{t_0}^N Gx_{t_0} \right\| \\
&\qquad + |\theta_t - \theta_{t_0}| \left\| t\gamma Vx_t + (I - t\mu F) S\Delta_t^N Gx_t \right\| + \theta_{t_0} \left\| (t - t_0)\gamma Vx_t \right. \\
&\qquad + t_0\gamma (Vx_t - Vx_{t_0}) - (t - t_0)\mu FS\Delta_t^N Gx_t + (I - t_0\mu F) S\Delta_t^N Gx_t \\
&\qquad \left. - (I - t_0\mu F) S\Delta_{t_0}^N Gx_{t_0} \right\| \\
&\quad \le |\theta_t - \theta_{t_0}| \|A\| \left\| S\Delta_t^N Gx_t \right\| + (1 - \theta_{t_0}\bar{\gamma}) \left[\|x_t - x_{t_0}\| + \widetilde{M}_0 \sum_{i=1}^N |r_{i,t} - r_{i,t_0}| \right] \\
&\qquad + |\theta_t - \theta_{t_0}| \big[\left\| S\Delta_t^N Gx_t \right\| + t \big(\gamma \|Vx_t\| + \mu \left\| FS\Delta_t^N Gx_t \right\| \big) \big] \\
&\qquad + \theta_{t_0} \big[\big(\gamma \|Vx_t\| + \mu \left\| FS\Delta_t^N Gx_t \right\| \big) |t - t_0| + t_0\gamma l \|x_t - x_{t_0}\| \\
&\qquad + (1 - t_0\tau) \left\| \Delta_t^N Gx_t - \Delta_{t_0}^N Gx_{t_0} \right\| \big] \\
&\quad \le |\theta_t - \theta_{t_0}| \|A\| \left\| S\Delta_t^N Gx_t \right\| + (1 - \theta_{t_0}\bar{\gamma}) \left(\|x_t - x_{t_0}\| + \widetilde{M}_0 \sum_{i=1}^N |r_{i,t} - r_{i,t_0}| \right) \\
&\qquad + |\theta_t - \theta_{t_0}| \big(\left\| S\Delta_t^N Gx_t \right\| + \gamma \|Vx_t\| + \mu \left\| FS\Delta_t^N Gx_t \right\| \big)
\end{aligned}
$$

$$+ \theta_{t_0}\big(\gamma\|Vx_t\| + \mu\big\|FS\Delta_t^N Gx_t\big\|\big)|t-t_0| + \theta_{t_0}t_0\gamma l\|x_t - x_{t_0}\| + \theta_{t_0}(1-t_0\tau)\left(\|x_t - x_{t_0}\| + \widetilde{M}_0\sum_{i=1}^N |r_{i,t} - r_{i,t_0}|\right).$$

This immediately implies that

$$\begin{aligned}\|x_t - x_{t_0}\| \le{}& \frac{\|A\|\|S\Delta_t^N Gx_t\| + \|S\Delta_t^N Gx_t\| + \gamma\|Vx_t\| + \mu\|FS\Delta_t^N Gx_t\|}{\theta_{t_0}(\bar{\gamma} - 1 + t_0(\tau - \gamma l))}|\theta_t - \theta_{t_0}| \\ &+ \frac{\gamma\|Vx_t\| + \mu\|FS\Delta_t^N Gx_t\|}{\bar{\gamma} - 1 + t_0(\tau - \gamma l)}|t - t_0| \\ &+ \frac{[1 - \theta_{t_0}(\bar{\gamma} - 1 + t_0\tau)]\widetilde{M}_0}{\theta_{t_0}(\bar{\gamma} - 1 + t_0(\tau - \gamma l))}\sum_{i=1}^N |r_{i,t} - r_{i,t_0}|.\end{aligned}$$

Since $\theta_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to (0, \|A\|^{-1}]$ is locally Lipschitzian, and $r_{i,t} : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to [a_i, b_i]$ is locally Lipschitzian for each $i = 1, \ldots, N$, we conclude that $x_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to C$ is locally Lipschitzian.

(iv) From the last inequality in (iii), the result follows immediately. □

We prove the following theorem for strong convergence of the net $\{x_t\}$ as $t \to 0$, which guarantees the existence of solutions of the variational inequality (3.2).

Theorem 3.1 *Let the net* $\{x_t\}$ *be defined via* (3.1). *If* $\lim_{t\to 0}\theta_t = 0$, *then* x_t *converges strongly to a point* $\tilde{x} \in \Omega$ *as* $t \to 0$, *which solves the VIP* (3.2). *Equivalently, we have* $P_\Omega(2I - A)\tilde{x} = \tilde{x}$.

Proof We first show the uniqueness of solutions of the VIP (3.2), which is indeed a consequence of the strong monotonicity of $A - I$. In fact, since A is a $\bar{\gamma}$-strongly positive bounded linear operator with $\bar{\gamma} \in (1, 2)$, we know that $A - I$ is $(\bar{\gamma} - 1)$-strongly monotone with constant $\bar{\gamma} - 1 \in (0, 1)$. Suppose that $\tilde{x} \in \Omega$ and $\hat{x} \in \Omega$ both are solutions to the VIP (3.2). Then we have

$$\langle (A - I)\tilde{x}, \tilde{x} - \hat{x}\rangle \le 0 \tag{3.21}$$

and

$$\langle (A - I)\hat{x}, \hat{x} - \tilde{x}\rangle \le 0. \tag{3.22}$$

Adding (3.21) and (3.22) yields

$$\langle (A - I)\tilde{x} - (A - I)\hat{x}, \tilde{x} - \hat{x}\rangle \le 0.$$

The strong monotonicity of $A - I$ implies that $\tilde{x} = \hat{x}$ and the uniqueness is proved.

Next, we prove that $x_t \to \tilde{x}$ as $t \to 0$. Observing $\mathrm{Fix}(T) = \mathrm{Fix}(S)$, from (3.1), we write, for given $p \in \Omega$,

$$\begin{aligned} &x_t - p \\ &\quad = x_t - w_t + w_t - p \end{aligned}$$

$$\begin{aligned}
&= x_t - w_t + (I - \theta_t A)S\Delta_t^N Gx_t + \theta_t\bigl(t\gamma Vx_t + (I - t\mu F)S\Delta_t^N Gx_t\bigr) - p \\
&= x_t - w_t + (I - \theta_t A)\bigl(S\Delta_t^N Gx_t - S\Delta_t^N Gp\bigr) \\
&\quad + \theta_t\bigl[t\gamma Vx_t + (I - t\mu F)S\Delta_t^N Gx_t - p\bigr] + \theta_t(I - A)p \\
&= x_t - w_t + (I - \theta_t A)\bigl(S\Delta_t^N Gx_t - S\Delta_t^N Gp\bigr) \\
&\quad + \theta_t\bigl[t(\gamma Vx_t - \mu Fp) + (I - t\mu F)S\Delta_t^N Gx_t - (I - t\mu F)p\bigr] + \theta_t(I - A)p,
\end{aligned}$$

where $w_t = (I - \theta_t A)S\Delta_t^N Gx_t + \theta_t(t\gamma Vx_t + (I - t\mu F)S\Delta_t^N Gx_t)$. Then, by Proposition 2.1(i), we have

$$\begin{aligned}
&\|x_t - p\|^2 \\
&\quad = \langle x_t - w_t, x_t - p\rangle + \bigl\langle (I - \theta_t A)(S\Delta_t^N Gx_t - S\Delta_t^N Gp, x_t - p\bigr\rangle \\
&\qquad + \theta_t\bigl[t\langle \gamma Vx_t - \mu Fp, x_t - p\rangle + \bigl\langle (I - t\mu F)S\Delta_t^N Gx_t - (I - t\mu F)p, x_t - p\bigr\rangle\bigr] \\
&\qquad + \theta_t\bigl\langle (I - A)p, x_t - p\bigr\rangle \\
&\quad \le (1 - \theta_t\bar{\gamma})\|x_t - p\|^2 + \theta_t\bigl[(1 - t\tau)\|x_t - p\|^2 + t\gamma l\|x_t - p\|^2 \\
&\qquad + t\bigl\langle (\gamma V - \mu F)p, x_t - p\bigr\rangle\bigr] + \theta_t\bigl\langle (I - A)p, x_t - p\bigr\rangle \\
&\quad = \bigl[1 - \theta_t\bigl(\bar{\gamma} - 1 + t(\tau - \gamma l)\bigr)\bigr]\|x_t - p\|^2 + \theta_t\bigl(t\bigl\langle (\gamma V - \mu F)p, x_t - p\bigr\rangle \\
&\qquad + \bigl\langle (I - A)p, x_t - p\bigr\rangle\bigr).
\end{aligned}$$

Therefore,

$$\|x_t - p\|^2 \le \frac{1}{\bar{\gamma} - 1 + t(\tau - \gamma l)}\bigl(t\bigl\langle (\gamma V - \mu F)p, x_t - p\bigr\rangle + \bigl\langle (I - A)p, x_t - p\bigr\rangle\bigr). \tag{3.23}$$

Since the net $\{x_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})}$ is bounded (due to Proposition 3.1(i)), we know that if $\{t_n\}$ is a subsequence in $(0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$ such that $t_n \to 0$ and $x_{t_n} \rightharpoonup x^*$, then from (3.23), we obtain $x_{t_n} \to x^*$. Let us show that $x^* \in \Omega$. Indeed, by Proposition 3.1(ii), we know that $\lim_{n\to\infty} \|x_{t_n} - Sx_{t_n}\| = 0$ and $\lim_{n\to\infty} \|x_{t_n} - Gx_{t_n}\| = 0$. Hence, according to Lemma 2.5, we get $x^* \in \operatorname{Fix}(S) \cap \Xi$. It is clear from the definition of S that $x^* \in \operatorname{Fix}(T) \cap \Xi$. Next we prove that $x^* \in \bigcap_{m=1}^N \operatorname{GMEP}(\Theta_m, \varphi_m, \mathcal{A}_m)$. As a matter of fact, utilizing (3.14) and (3.16), we obtain from $x_{t_n} \to x^*$, $y_{t_n} = Gx_{t_n} \to x^*$ and $\Delta_{t_n}^m y_{t_n} \to x^*$ for each $m = 1, \ldots, N$. Since $\Delta_{t_n}^m y_{t_n} = T_{r_{m,t_n}}^{(\Theta_m,\varphi_m)}(I - r_{m,t_n}\mathcal{A}_m)\Delta_{t_n}^{m-1} y_{t_n}$, $n \ge 0$, $m \in \{1, \ldots, N\}$, we have

$$\begin{aligned}
0 &\le \Theta_m\bigl(\Delta_{t_n}^m y_{t_n}, y\bigr) + \varphi_m(y) - \varphi_m\bigl(\Delta_{t_n}^m y_{t_n}\bigr) \\
&\quad + \bigl\langle \mathcal{A}_m\Delta_{t_n}^{m-1} y_{t_n}, y - \Delta_{t_n}^m y_{t_n}\bigr\rangle + \frac{1}{r_{m,t_n}}\bigl\langle y - \Delta_{t_n}^m y_{t_n}, \Delta_{t_n}^m y_{t_n} - \Delta_{t_n}^{m-1} y_{t_n}\bigr\rangle.
\end{aligned}$$

By (A2), we have

$$\begin{aligned}
\Theta_m\bigl(y, \Delta_{t_n}^m y_{t_n}\bigr) &\le \varphi_m(y) - \varphi_m\bigl(\Delta_{t_n}^m y_{t_n}\bigr) + \bigl\langle \mathcal{A}_m\Delta_{t_n}^{m-1} y_{t_n}, y - \Delta_{t_n}^m y_{t_n}\bigr\rangle \\
&\quad + \frac{1}{r_{m,t_n}}\bigl\langle y - \Delta_{t_n}^m y_{t_n}, \Delta_{t_n}^m y_{t_n} - \Delta_{t_n}^{m-1} y_{t_n}\bigr\rangle.
\end{aligned}$$

Let $z_t = ty + (1-t)x^*$ for all $t \in (0,1]$ and $y \in C$. This implies that $z_t \in C$. Then we have

$$\begin{aligned}
&\langle z_t - \Delta_{t_n}^m y_{t_n}, \mathcal{A}_m z_t \rangle \\
&\quad \geq \varphi_m(\Delta_{t_n}^m y_{t_n}) - \varphi_m(z_t) + \langle z_t - \Delta_{t_n}^m y_{t_n}, \mathcal{A}_m z_t \rangle - \langle z_t - \Delta_{t_n}^m y_{t_n}, \mathcal{A}_m \Delta_{t_n}^{m-1} y_{t_n} \rangle \\
&\qquad - \left\langle z_t - \Delta_{t_n}^m y_{t_n}, \frac{\Delta_{t_n}^m y_{t_n} - \Delta_{t_n}^{m-1} y_{t_n}}{r_{m,t_n}} \right\rangle + \Theta_m(z_t, \Delta_{t_n}^m y_{t_n}) \\
&\quad = \varphi_m(\Delta_{t_n}^m y_{t_n}) - \varphi_m(z_t) + \langle z_t - \Delta_{t_n}^m y_{t_n}, \mathcal{A}_m z_t - \mathcal{A}_m \Delta_{t_n}^m y_{t_n} \rangle \\
&\qquad + \langle z_t - \Delta_{t_n}^m y_{t_n}, \mathcal{A}_m \Delta_{t_n}^m y_{t_n} - \mathcal{A}_m \Delta_{t_n}^{m-1} y_{t_n} \rangle \\
&\qquad - \left\langle z_t - \Delta_{t_n}^m y_{t_n}, \frac{\Delta_{t_n}^m y_{t_n} - \Delta_{t_n}^{m-1} y_{t_n}}{r_{m,t_n}} \right\rangle + \Theta_m(z_t, \Delta_{t_n}^m y_{t_n}).
\end{aligned}$$

By (3.16), we have $\|\mathcal{A}_m \Delta_{t_n}^m y_{t_n} - \mathcal{A}_m \Delta_{t_n}^{m-1} y_{t_n}\| \to 0$ as $n \to \infty$. Furthermore, by the monotonicity of $\mathcal{A}_m$, we obtain $\langle z_t - \Delta_{t_n}^m y_{t_n}, \mathcal{A}_m z_t - \mathcal{A}_m \Delta_{t_n}^m y_{t_n} \rangle \geq 0$. Then, by (A4), we obtain

$$\langle z_t - x^*, \mathcal{A}_m z_t \rangle \geq \varphi_m(x^*) - \varphi_m(z_t) + \Theta_m(z_t, x^*).$$

Utilizing (A1), (A4), and the last inequality, we obtain

$$\begin{aligned}
0 &= \Theta_m(z_t, z_t) + \varphi_m(z_t) - \varphi_m(z_t) \\
&\leq t\Theta_m(z_t, y) + (1-t)\Theta_m(z_t, x^*) + t\varphi_m(y) + (1-t)\varphi_m(x^*) - \varphi_m(z_t) \\
&\leq t[\Theta_m(z_t, y) + \varphi_m(y) - \varphi_m(z_t)] + (1-t)\langle z_t - x^*, \mathcal{A}_m z_t \rangle \\
&= t[\Theta_m(z_t, y) + \varphi_m(y) - \varphi_m(z_t)] + (1-t)t\langle y - x^*, \mathcal{A}_m z_t \rangle,
\end{aligned}$$

and hence

$$0 \leq \Theta_m(z_t, y) + \varphi_m(y) - \varphi_m(z_t) + (1-t)\langle y - x^*, \mathcal{A}_m z_t \rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq \Theta_m(x^*, y) + \varphi_m(y) - \varphi_m(x^*) + \langle y - x^*, \mathcal{A}_m x^* \rangle.$$

This implies that $x^* \in \mathrm{GMEP}(\Theta_m, \varphi_m, \mathcal{A}_m)$ and hence $x^* \in \bigcap_{m=1}^N \mathrm{GMEP}(\Theta_m, \varphi_m, \mathcal{A}_m)$. Thus, $x^* \in \bigcap_{m=1}^N \mathrm{GMEP}(\Theta_m, \varphi_m, \mathcal{A}_m) \cap \mathrm{Fix}(T) \cap \Xi$.

Next, we prove that $x_t \to \tilde{x}$ as $t \to 0$. First, let us assert that x^* is a solution of the VIP (3.2). As a matter of fact, since

$$x_t = x_t - w_t + (I - \theta_t A)S\Delta_t^N Gx_t + \theta_t(t\gamma Vx_t + (I - t\mu F)S\Delta_t^N Gx_t),$$

we have

$$x_t - S\Delta_t^N Gx_t = x_t - w_t + \theta_t(I - A)S\Delta_t^N Gx_t + \theta_t t(\gamma Vx_t - \mu FS\Delta_t^N Gx_t).$$

Since Δ_t^N is nonexpansive (due to (2.1) and Proposition 2.2(ii)), G is nonexpansive (due to Proposition 2.3) and S is nonexpansive (due to Lemma 2.4), $I - S\Delta_t^N G$ is monotone. So, from the monotonicity of $I - S\Delta_t^N G$, it follows that, for $p \in \Omega$,

$$\begin{aligned} 0 &\le \langle (I - S\Delta_t^N G)x_t - (I - S\Delta_t^N G)p, x_t - p\rangle = \langle (I - S\Delta_t^N G)x_t, x_t - p\rangle \\ &= \langle x_t - w_t, x_t - p\rangle + \theta_t \langle (I - A)S\Delta_t^N Gx_t, x_t - p\rangle \\ &\quad + \theta_t t\langle \gamma Vx_t - \mu FS\Delta_t^N Gx_t, x_t - p\rangle \\ &\le \theta_t \langle (I - A)S\Delta_t^N Gx_t, x_t - p\rangle + \theta_t t\langle \gamma Vx_t - \mu FS\Delta_t^N Gx_t, x_t - p\rangle \\ &= \theta_t \langle (I - A)x_t, x_t - p\rangle + \theta_t \langle (I - A)(S\Delta_t^N G - I)x_t, x_t - p\rangle \\ &\quad + \theta_t t\langle \gamma Vx_t - \mu FS\Delta_t^N Gx_t, x_t - p\rangle. \end{aligned}$$

This implies that

$$\langle (A - I)x_t, x_t - p\rangle \le \langle (I - A)(S\Delta_t^N G - I)x_t, x_t - p\rangle + t\langle \gamma Vx_t - \mu FS\Delta_t^N Gx_t, x_t - p\rangle. \quad (3.24)$$

Now, replacing t in (3.24) with t_n and letting $n \to \infty$, noticing the boundedness of $\{\gamma Vx_{t_n} - \mu FS\Delta_{t_n}^N Gx_{t_n}\}$ and the fact that $(I - A)(S\Delta_{t_n}^N G - I)x_{t_n} \to 0$ as $n \to \infty$ (due to (3.4)), we obtain

$$\langle (A - I)x^*, x^* - p\rangle \le 0.$$

That is, $x^* \in \Omega$ is a solution of the VIP (3.2); hence $x^* = \tilde{x}$ by uniqueness. In summary, we have proven that each cluster point of $\{x_t\}$ (as $t \to 0$) equals $\tilde{x}$. Consequently, $x_t \to \tilde{x}$ as $t \to 0$.

The VIP (3.2) can be rewritten as

$$\langle (2I - A)\tilde{x} - \tilde{x}, \tilde{x} - p\rangle \ge 0, \quad \forall p \in \Omega.$$

Recalling Proposition 2.1(i), the last inequality is equivalent to the fixed point equation

$$P_\Omega (2I - A)\tilde{x} = \tilde{x}.$$

□

Taking $F = \frac{1}{2}I$, $\mu = 2$, and $\gamma = 1$ in Theorem 3.1, we get

Corollary 3.1 *Let* $\{x_t\}$ *be defined by*

$$x_t = P_C\big[(I - \theta_t A)S\Delta_t^N Gx_t + \theta_t\big(tVx_t + (1 - t)S\Delta_t^N Gx_t\big)\big].$$

If $\lim_{t\to 0} \theta_t = 0$, *then* $\{x_t\}$ *converges strongly as* $t \to 0$ *to a point* $\tilde{x} \in \Omega$, *which is the unique solution of the VIP* (3.2).

First, we prove the following result in order to establish the strong convergence of the sequence $\{x_n\}$ generated by the composite explicit relaxed extragradient-like scheme (3.3).

Theorem 3.2 *Let* $\{x_n\}$ *be the sequence generated by the explicit scheme* (3.3), *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following condition*:

(C1) $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset (0,1]$ *and* $\alpha_n \to 0$, $\beta_n \to 0$ *as* $n \to \infty$.

Let LIM *be a Banach limit. Then*

$$\mathrm{LIM}_n\langle (A-I)\tilde{x}, \tilde{x}-x_n\rangle \le 0,$$

where $\tilde{x} = \lim_{t\to 0^+} x_t$ *with* x_t *being defined by*

$$x_t = P_C\big[(I-\theta_t A)S\Delta^N Gx_t + \theta_t\big(t\gamma Vx_t + (I-t\mu F)S\Delta^N Gx_t\big)\big], \tag{3.25}$$

where $S, G, \Delta^N : C \to C$ *are defined by* $Sx = \lambda x + (1-\lambda)Tx$, $Gx = T_{\nu_1}^{\Phi_1}(I-\nu_1 F_1)T_{\nu_2}^{\Phi_2}(I-\nu_2 F_2)x$ *and* $\Delta^N x = T_{r_N}^{(\Theta_N,\varphi_N)}(I-r_N\mathcal{A}_N)\cdots T_{r_1}^{(\Theta_1,\varphi_1)}(I-r_1\mathcal{A}_1)x$ *with* $0\le k\le\lambda<1$ *and* $r_i\in[a_i,b_i]\subset(0,2\eta_i)$ *for each* $i=1,\dots,N$.

Proof First, note that from the condition (C1), without loss of generality, we may assume that $0<\beta_n\le\|A\|^{-1}$ for all $n\ge 0$.

Let $\{x_t\}$ be the net generated by (3.25). Since Δ^N is a nonexpansive self-mapping on C, by Theorem 3.1 with $\Delta_t^N = \Delta^N$, there exists $\lim_{t\to 0} x_t \in \Omega$. Denote it by $\tilde{x}$. Moreover, $\tilde{x}$ is the unique solution of the VIP (3.2). From Proposition 3.1(i) with $\Delta_t^N = \Delta^N$, we know that $\{x_t\}$ is bounded and so are the nets $\{Vx_t\}$, $\{\Delta^N Gx_t\}$, and $\{FS\Delta^N Gx_t\}$.

First of all, let us show that $\{x_n\}$ is bounded. To this end, take $p\in\Omega$. Then we get

$$\begin{aligned}
\|y_n - p\| &= \big\|\alpha_n\gamma Vx_n + (I-\alpha_n\mu F)S\Delta_n^N Gx_n - p\big\| \\
&= \big\|\alpha_n(\gamma Vx_n - \mu Fp) + (I-\alpha_n\mu F)S\Delta_n^N Gx_n - (I-\alpha_n\mu F)S\Delta_n^N Gp\big\| \\
&\le \alpha_n\gamma l\|x_n-p\| + \alpha_n\big\|(\gamma V-\mu F)p\big\| + (1-\alpha_n\tau)\|x_n-p\| \\
&= \big(1-\alpha_n(\tau-\gamma l)\big)\|x_n-p\| + \alpha_n\big\|(\gamma V-\mu F)p\big\|,
\end{aligned}$$

together with Lemma 2.9, implies that

$$\begin{aligned}
\|x_{n+1}-p\| &= \big\|P_C\big[(I-\beta_n A)S\Delta_n^N Gx_n + \beta_n y_n\big] - p\big\| \\
&\le \big\|(I-\beta_n A)S\Delta_n^N Gx_n + \beta_n y_n - p\big\| \\
&= \big\|(I-\beta_n A)S\Delta_n^N Gx_n - (I-\beta_n A)S\Delta_n^N Gp \\
&\quad + \beta_n(y_n-p) + \beta_n(I-A)p\big\| \\
&\le \big\|(I-\beta_n A)S\Delta_n^N Gx_n - (I-\beta_n A)S\Delta_n^N Gp\big\| \\
&\quad + \beta_n\|y_n-p\| + \beta_n\|I-A\|\|p\| \\
&\le (1-\beta_n\bar{\gamma})\|x_n-p\| + \beta_n\big[\big(1-\alpha_n(\tau-\gamma l)\big)\|x_n-p\| \\
&\quad + \alpha_n\big\|(\gamma V-\mu F)p\big\|\big] + \beta_n\|I-A\|\|p\| \\
&\le \big(1-\beta_n(\bar{\gamma}-1)\big)\|x_n-p\| + \beta_n\big(\big\|(\gamma V-\mu F)p\big\| + \|I-A\|\|p\|\big) \\
&= \big(1-\beta_n(\bar{\gamma}-1)\big)\|x_n-p\| + \beta_n(\bar{\gamma}-1)\frac{\|(\gamma V-\mu F)p\| + \|I-A\|\|p\|}{\bar{\gamma}-1} \\
&\le \max\left\{\|x_n-p\|, \frac{\|(\gamma V-\mu F)p\| + \|I-A\|\|p\|}{\bar{\gamma}-1}\right\}.
\end{aligned}$$

By induction

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|(\gamma V - \mu F)p\| + \|I - A\|\|p\|}{\bar{\gamma} - 1}\right\}, \quad \forall n \ge 0.$$

This implies that $\{x_n\}$ is bounded and so are $\{Vx_n\}$, $\{\Delta_n^N Gx_n\}$, $\{FS\Delta_n^N Gx_n\}$, and $\{y_n\}$. Thus, utilizing the control condition (C1), we get

$$\begin{aligned} \|x_{n+1} - S\Delta_n^N Gx_n\| &= \|P_C[(I - \beta_n A)S\Delta_n^N Gx_n + \beta_n y_n] - S\Delta_n^N Gx_n\| \\ &\le \|(I - \beta_n A)S\Delta_n^N Gx_n + \beta_n y_n - S\Delta_n^N Gx_n\| \\ &= \beta_n \|y_n - AS\Delta_n^N Gx_n\| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Utilizing similar arguments to those of (3.20), we have

$$\|\Delta_n^N Gx_n - \Delta^N Gx_n\| \le \widetilde{M}_1 \sum_{i=1}^N |r_{i,n} - r_i|,$$

where $\sup_{n\ge0}\{\sum_{i=1}^N [\|\mathcal{A}_i\Delta_n^{i-1}Gx_n\| + \frac{1}{r_{i,n}}\|T_{r_{i,n}}^{(\Theta_i,\varphi_i)}(I - r_{i,n}\mathcal{A}_i)\Delta_n^{i-1}Gx_n - (I - r_{i,n}\mathcal{A}_i)\Delta_n^{i-1}Gx_n\|]\} \le \widetilde{M}_1$ for some $\widetilde{M}_1 > 0$. Consequently, it is not hard to find that

$$\begin{aligned} &\|S\Delta^N Gx_t - x_{n+1}\| \\ &\quad \le \|S\Delta^N Gx_t - S\Delta^N Gx_n\| + \|S\Delta^N Gx_n - S\Delta_n^N Gx_n\| + \|S\Delta_n^N Gx_n - x_{n+1}\| \\ &\quad \le \|x_t - x_n\| + \|\Delta^N Gx_n - \Delta_n^N Gx_n\| + \|S\Delta_n^N Gx_n - x_{n+1}\| \\ &\quad \le \|x_t - x_n\| + \widetilde{M}_1 \sum_{i=1}^N |r_{i,n} - r_i| + \|S\Delta_n^N Gx_n - x_{n+1}\| \\ &\quad = \|x_t - x_n\| + \epsilon_n, \end{aligned} \tag{3.26}$$

where $\epsilon_n = \widetilde{M}_1 \sum_{i=1}^N |r_{i,n} - r_i| + \|x_{n+1} - S\Delta_n^N Gx_n\| \to 0$ as $n \to \infty$. Also, by observing that A is strongly positive, we have

$$\langle Ax_t - Ax_n, x_t - x_n\rangle = \langle A(x_t - x_n), x_t - x_n\rangle \ge \bar{\gamma}\|x_t - x_n\|^2. \tag{3.27}$$

For simplicity, we write $w_t = (I - \theta_t A)S\Delta^N Gx_t + \theta_t(t\gamma Vx_t + (I - t\mu F)S\Delta^N Gx_t)$. Then we obtain $x_t = P_C w_t$ and

$$\begin{aligned} x_t - x_{n+1} &= x_t - w_t + (I - \theta_t A)S\Delta^N Gx_t + \theta_t(t\gamma Vx_t + (I - t\mu F)S\Delta^N Gx_t) - x_{n+1} \\ &= (I - \theta_t A)S\Delta^N Gx_t - (I - \theta_t A)x_{n+1} + \theta_t(t\gamma Vx_t \\ &\quad + (I - t\mu F)S\Delta^N Gx_t - Ax_{n+1}) + x_t - w_t. \end{aligned}$$

Applying Lemma 2.1, we have

$$\begin{aligned} &\|x_t - x_{n+1}\|^2 \\ &\quad \le \|(I - \theta_t A)S\Delta^N Gx_t - (I - \theta_t A)x_{n+1}\|^2 \end{aligned}$$

$$
\begin{aligned}
&+ 2\theta_t\langle S\Delta^N Gx_t - t(\mu FS\Delta^N Gx_t - \gamma Vx_t) - Ax_{n+1}, x_t - x_{n+1}\rangle + 2\langle x_t - w_t, x_t - x_{n+1}\rangle \\
&\le \|(I-\theta_t A)S\Delta^N Gx_t - (I-\theta_t A)x_{n+1}\|^2 \\
&\quad + 2\theta_t\langle S\Delta^N Gx_t - t(\mu FS\Delta^N Gx_t - \gamma Vx_t) - Ax_{n+1}, x_t - x_{n+1}\rangle \\
&\le (1-\theta_t\bar{\gamma})^2\|S\Delta^N Gx_t - x_{n+1}\|^2 + 2\theta_t\langle S\Delta^N Gx_t - x_t, x_t - x_{n+1}\rangle \\
&\quad - 2\theta_t t\langle \mu FS\Delta^N Gx_t - \gamma Vx_t, x_t - x_{n+1}\rangle + 2\theta_t\langle x_t - Ax_{n+1}, x_t - x_{n+1}\rangle.
\end{aligned} \tag{3.28}
$$

Using (3.26) and (3.27) in (3.28), we obtain

$$
\begin{aligned}
&\|x_t - x_{n+1}\|^2 \\
&\le (1-\theta_t\bar{\gamma})^2\|S\Delta^N Gx_t - x_{n+1}\|^2 + 2\theta_t\langle S\Delta^N Gx_t - x_t, x_t - x_{n+1}\rangle \\
&\quad + 2\theta_t t\langle \gamma Vx_t - \mu FS\Delta^N Gx_t, x_t - x_{n+1}\rangle + 2\theta_t\langle x_t - Ax_{n+1}, x_t - x_{n+1}\rangle \\
&\le (1-\theta_t\bar{\gamma})^2(\|x_t - x_n\| + \epsilon_n)^2 + 2\theta_t\|S\Delta^N Gx_t - x_t\|\|x_t - x_{n+1}\| \\
&\quad + 2\theta_t t\|\gamma Vx_t - \mu FS\Delta^N Gx_t\|\|x_t - x_{n+1}\| + 2\theta_t\langle x_t - Ax_{n+1}, x_t - x_{n+1}\rangle \\
&= (\theta_t^2\bar{\gamma} - 2\theta_t)\bar{\gamma}\|x_t - x_n\|^2 + \|x_t - x_n\|^2 + (1-\theta_t\bar{\gamma})^2(2\|x_t - x_n\|\epsilon_n + \epsilon_n^2) \\
&\quad + 2\theta_t\|S\Delta^N Gx_t - x_t\|\|x_t - x_{n+1}\| + 2\theta_t t\|\gamma Vx_t - \mu FS\Delta^N Gx_t\|\|x_t - x_{n+1}\| \\
&\quad + 2\theta_t\langle x_t - Ax_{n+1}, x_t - x_{n+1}\rangle \\
&\le (\theta_t^2\bar{\gamma} - 2\theta_t)\langle Ax_t - Ax_n, x_t - x_n\rangle + \|x_t - x_n\|^2 + (1-\theta_t\bar{\gamma})^2(2\|x_t - x_n\|\epsilon_n + \epsilon_n^2) \\
&\quad + 2\theta_t\|S\Delta^N Gx_t - x_t\|\|x_t - x_{n+1}\| + 2\theta_t t\|\gamma Vx_t - \mu FS\Delta^N Gx_t\|\|x_t - x_{n+1}\| \\
&\quad + 2\theta_t\langle x_t - Ax_{n+1}, x_t - x_{n+1}\rangle \\
&= \theta_t^2\bar{\gamma}\langle Ax_t - Ax_n, x_t - x_n\rangle + \|x_t - x_n\|^2 + (1-\theta_t\bar{\gamma})^2(2\|x_t - x_n\|\epsilon_n + \epsilon_n^2) \\
&\quad + 2\theta_t\|S\Delta^N Gx_t - x_t\|\|x_t - x_{n+1}\| + 2\theta_t t\|\gamma Vx_t - \mu FS\Delta^N Gx_t\|\|x_t - x_{n+1}\| \\
&\quad + 2\theta_t[\langle x_t - Ax_{n+1}, x_t - x_{n+1}\rangle - \langle Ax_t - Ax_n, x_t - x_n\rangle] \\
&= \theta_t^2\bar{\gamma}\langle A(x_t - x_n), x_t - x_n\rangle + \|x_t - x_n\|^2 + (1-\theta_t\bar{\gamma})^2(2\|x_t - x_n\|\epsilon_n + \epsilon_n^2) \\
&\quad + 2\theta_t\|S\Delta^N Gx_t - x_t\|\|x_t - x_{n+1}\| + 2\theta_t t\|\gamma Vx_t - \mu FS\Delta^N Gx_t\|\|x_t - x_{n+1}\| \\
&\quad + 2\theta_t[\langle (I-A)x_t, x_t - x_{n+1}\rangle + \langle A(x_t - x_{n+1}), x_t - x_{n+1}\rangle \\
&\quad - \langle A(x_t - x_n), x_t - x_n\rangle].
\end{aligned} \tag{3.29}
$$

Applying the Banach limit LIM to (3.29), from $\epsilon_n \to 0$ we have

$$
\begin{aligned}
&\mathrm{LIM}_n\|x_t - x_{n+1}\|^2 \\
&\le \theta_t^2\bar{\gamma}\,\mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle + \mathrm{LIM}_n\|x_t - x_n\|^2 \\
&\quad + 2\theta_t\|S\Delta^N Gx_t - x_t\|\,\mathrm{LIM}_n\|x_t - x_{n+1}\| \\
&\quad + 2\theta_t t\|\gamma Vx_t - \mu FS\Delta^N Gx_t\|\,\mathrm{LIM}_n\|x_t - x_{n+1}\| \\
&\quad + 2\theta_t[\mathrm{LIM}_n\langle (I-A)x_t, x_t - x_{n+1}\rangle + \mathrm{LIM}_n\langle A(x_t - x_{n+1}), x_t - x_{n+1}\rangle \\
&\quad - \mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle].
\end{aligned} \tag{3.30}
$$

Utilizing the property $\mathrm{LIM}_n a_n = \mathrm{LIM}_n a_{n+1}$ of the Banach limit in (3.30), we obtain

$$\begin{aligned}
&\mathrm{LIM}_n\langle (A-I)x_t, x_t - x_n\rangle \\
&\quad = \mathrm{LIM}_n\langle (A-I)x_t, x_t - x_{n+1}\rangle \\
&\quad \le \frac{\theta_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle + \frac{1}{2\theta_t}\big[\mathrm{LIM}_n\|x_t - x_n\|^2 - \mathrm{LIM}_n\|x_t - x_{n+1}\|^2\big] \\
&\qquad + \|S\Delta^N Gx_t - x_t\|\,\mathrm{LIM}_n\|x_t - x_n\| + t\|\gamma Vx_t - \mu FS\Delta^N Gx_t\|\,\mathrm{LIM}_n\|x_t - x_n\| \\
&\qquad + \mathrm{LIM}_n\langle A(x_t - x_{n+1}), x_t - x_{n+1}\rangle - \mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle \\
&\quad \le \frac{\theta_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle + \|S\Delta^N Gx_t - x_t\|\,\mathrm{LIM}_n\|x_t - x_n\| \\
&\qquad + t\|\gamma Vx_t - \mu FS\Delta^N Gx_t\|\,\mathrm{LIM}_n\|x_t - x_n\|.
\end{aligned} \tag{3.31}$$

Since as $t \to 0$,

$$\theta_t\langle A(x_t - x_n), x_t - x_n\rangle \le \theta_t\|A\|\|x_t - x_n\|^2 \le \theta_t K \to 0, \tag{3.32}$$

where $\|A\|\|x_t - x_n\|^2 \le K$,

$$\begin{aligned}
&\|S\Delta^N Gx_t - x_t\| \to 0 \quad \big(\text{see (3.4)}\big) \quad \text{and} \\
&t\|\gamma Vx_t - \mu FS\Delta^N Gx_t\| \to 0 \quad \text{as } t \to 0,
\end{aligned} \tag{3.33}$$

we conclude from (3.31)-(3.33) that

$$\begin{aligned}
&\mathrm{LIM}_n\langle (A-I)\tilde{x}, \tilde{x} - x_n\rangle \\
&\quad \le \limsup_{t\to 0}\mathrm{LIM}_n\langle (A-I)x_t, x_t - x_n\rangle \\
&\quad \le \limsup_{t\to 0}\frac{\theta_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle + \limsup_{t\to 0}\|S\Delta^N Gx_t - x_t\|\,\mathrm{LIM}_n\|x_t - x_n\| \\
&\qquad + \limsup_{t\to 0} t\|\gamma Vx_t - \mu FS\Delta^N Gx_t\|\,\mathrm{LIM}_n\|x_t - x_n\| \\
&\quad = 0.
\end{aligned}$$

This completes the proof. □

Now, using Theorem 3.2, we establish the strong convergence of the sequence $\{x_n\}$ generated by the composite explicit relaxed extragradient-like scheme (3.3) to a point $\tilde{x} \in \Omega$, which is also the unique solution of the VIP (3.2).

Theorem 3.3 *Let $\{x_n\}$ be the sequence generated by the explicit scheme* (3.3), *where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions*:

(C1) $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset (0,1]$, *and* $\alpha_n \to 0$, $\beta_n \to 0$ *as* $n \to \infty$;

(C2) $\sum_{n=0}^{\infty}\beta_n = \infty$.

If $\{x_n\}$ is weakly asymptotically regular (*i.e.*, $x_{n+1} - x_n \rightharpoonup 0$), *then x_n converges strongly to a point $\tilde{x} \in \Omega$, which is the unique solution of the VIP* (3.2).

Proof First, note that from the condition (C1), without loss of generality, we may assume that $\alpha_n\tau < 1$ and $\frac{2\beta_n(\bar{\gamma}-1)}{1-\beta_n} < 1$ for all $n \geq 0$.

Let x_t be defined by (3.25), that is,

$$x_t = P_C\bigl[(I-\theta_t A)S\Delta^N Gx_t + \theta_t(S\Delta^N Gx_t - t\bigl(\mu FS\Delta^N Gx_t - \gamma Vx_t\bigr)\bigr],$$

for $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$, where $Sx = \lambda x + (1-\lambda)Tx$ for $0 \leq k \leq \lambda < 1$, $Gx = T_{\nu_1}^{\Phi_1}(I - \nu_1 F_1)T_{\nu_2}^{\Phi_2}(I-\nu_2 F_2)x$ for $\nu_j \in (0, 2\zeta_j)$, $j = 1, 2$, $\Delta^N x = T_{r_N}^{(\Theta_N,\varphi_N)}(I - r_N\mathcal{A}_N)\cdots T_{r_1}^{(\Theta_1,\varphi_1)}(I - r_1\mathcal{A}_1)x$ for $r_i \in [a_i, b_i] \subset (0, 2\eta_i)$, $i = 1, \ldots, N$, and $\lim_{t\to 0} x_t := \tilde{x} \in \Omega$ (due to Theorem 3.1). Then $\tilde{x}$ is the unique solution of the VIP (3.2).

We divide the rest of the proof into several steps.

Step 1. We see that

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|(\gamma V - \mu F)p\| + \|I - A\|\|p\|}{\bar{\gamma} - 1}\right\}, \quad \forall n \geq 0,$$

for all $p \in \Omega$ as in the proof of Theorem 3.2. Hence $\{x_n\}$ is bounded and so are $\{Vx_n\}$, $\{\Delta^N Gx_n\}$, $\{\Delta_n^N Gx_n\}$, $\{FS\Delta_n^N Gx_n\}$, and $\{y_n\}$.

Step 2. We show that $\limsup_{n\to\infty}\langle(I-A)\tilde{x}, x_n - \tilde{x}\rangle \leq 0$. To this end, put

$$a_n := \bigl\langle (A - I)\tilde{x}, \tilde{x} - x_n\bigr\rangle, \quad \forall n \geq 0.$$

Then, by Theorem 3.2, we get $\mathrm{LIM}_n a_n \leq 0$ for any Banach limit LIM. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \limsup_{j\to\infty}(a_{n_j+1} - a_{n_j})$$

and $x_{n_j} \rightharpoonup v \in H$. This implies that $x_{n_j+1} \rightharpoonup v$ since $\{x_n\}$ is weakly asymptotically regular. Therefore, we have

$$w - \lim_{j\to\infty}(\tilde{x} - x_{n_j+1}) = w - \lim_{j\to\infty}(\tilde{x} - x_{n_j}) = (\tilde{x} - v),$$

and so

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \lim_{j\to\infty}\bigl\langle (A - I)\tilde{x}, (\tilde{x} - x_{n_j+1}) - (\tilde{x} - x_{n_j})\bigr\rangle = 0.$$

Then, by Lemma 2.10, we obtain $\limsup_{n\to\infty} a_n \leq 0$, that is,

$$\limsup_{n\to\infty}\bigl\langle (I - A)\tilde{x}, x_n - \tilde{x}\bigr\rangle = \limsup_{n\to\infty}\bigl\langle (A - I)\tilde{x}, \tilde{x} - x_n\bigr\rangle \leq 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$. Indeed, for simplicity, we write $w_n = (I - \beta_n A)S\Delta_n^N Gx_n + \beta_n y_n$ for all $n \geq 0$. Then $x_{n+1} = P_C w_n$. Utilizing (3.3) and $S\Delta_n^N G\tilde{x} = \tilde{x}$, we have

$$y_n - \tilde{x} = (I - \alpha_n\mu F)S\Delta_n^N Gx_n - (I - \alpha_n\mu F)S\Delta_n^N G\tilde{x} + \alpha_n(\gamma Vx_n - \mu F\tilde{x})$$

and

$$x_{n+1} - \tilde{x} = x_{n+1} - w_n + (I - \beta_n A)\big(S\Delta_n^N G x_n - S\Delta_n^N G\tilde{x}\big) + \beta_n(y_n - \tilde{x}) + \beta_n(I - A)\tilde{x}.$$

Applying Lemmas 2.1, 2.7, and 2.9, we obtain

$$\begin{aligned}
\|y_n - \tilde{x}\|^2 &= \big\|(I - \alpha_n\mu F)S\Delta_n^N G x_n - (I - \alpha_n\mu F)S\Delta_n^N G\tilde{x} + \alpha_n(\gamma V x_n - \mu F\tilde{x})\big\|^2 \\
&\le \big\|(I - \alpha_n\mu F)S\Delta_n^N G x_n - (I - \alpha_n\mu F)S\Delta_n^N G\tilde{x}\big\|^2 \\
&\quad + 2\alpha_n\langle\gamma V x_n - \mu F\tilde{x}, y_n - \tilde{x}\rangle \\
&\le (1 - \alpha_n\tau)^2\|x_n - \tilde{x}\|^2 + 2\alpha_n\|\gamma V x_n - \mu F\tilde{x}\|\|y_n - \tilde{x}\| \\
&\le \|x_n - \tilde{x}\|^2 + 2\alpha_n\|\gamma V x_n - \mu F\tilde{x}\|\|y_n - \tilde{x}\|,
\end{aligned}$$

and hence

$$\begin{aligned}
&\|x_{n+1} - \tilde{x}\|^2 \\
&\quad = \big\|(I - \beta_n A)\big(S\Delta_n^N G x_n - S\Delta_n^N G\tilde{x}\big) + \beta_n(y_n - \tilde{x}) + \beta_n(I - A)\tilde{x} + x_{n+1} - w_n\big\|^2 \\
&\quad \le \big\|(I - \beta_n A)\big(S\Delta_n^N G x_n - S\Delta_n^N G\tilde{x}\big)\big\|^2 + 2\beta_n\langle y_n - \tilde{x}, x_{n+1} - \tilde{x}\rangle \\
&\qquad + 2\beta_n\big\langle(I - A)\tilde{x}, x_{n+1} - \tilde{x}\big\rangle + 2\langle x_{n+1} - w_n, x_{n+1} - \tilde{x}\rangle \\
&\quad \le \big\|(I - \beta_n A)\big(S\Delta_n^N G x_n - S\Delta_n^N G\tilde{x}\big)\big\|^2 + 2\beta_n\langle y_n - \tilde{x}, x_{n+1} - \tilde{x}\rangle \\
&\qquad + 2\beta_n\big\langle(I - A)\tilde{x}, x_{n+1} - \tilde{x}\big\rangle \\
&\quad \le (1 - \beta_n\bar{\gamma})^2\|x_n - \tilde{x}\|^2 + 2\beta_n\|y_n - \tilde{x}\|\|x_{n+1} - \tilde{x}\| + 2\beta_n\big\langle(I - A)\tilde{x}, x_{n+1} - \tilde{x}\big\rangle \\
&\quad \le (1 - \beta_n\bar{\gamma})^2\|x_n - \tilde{x}\|^2 + \beta_n\big(\|y_n - \tilde{x}\|^2 + \|x_{n+1} - \tilde{x}\|^2\big) + 2\beta_n\big\langle(I - A)\tilde{x}, x_{n+1} - \tilde{x}\big\rangle \\
&\quad \le (1 - \beta_n\bar{\gamma})^2\|x_n - \tilde{x}\|^2 + \beta_n\big[\|x_n - \tilde{x}\|^2 + 2\alpha_n\|\gamma V x_n - \mu F\tilde{x}\|\|y_n - \tilde{x}\|\big] \\
&\qquad + \beta_n\|x_{n+1} - \tilde{x}\|^2 + 2\beta_n\big\langle(I - A)\tilde{x}, x_{n+1} - \tilde{x}\big\rangle \\
&\quad = \big[(1 - \beta_n\bar{\gamma})^2 + \beta_n\big]\|x_n - \tilde{x}\|^2 + 2\alpha_n\beta_n\|\gamma V x_n - \mu F\tilde{x}\|\|y_n - \tilde{x}\| \\
&\qquad + \beta_n\|x_{n+1} - \tilde{x}\|^2 + 2\beta_n\big\langle(I - A)\tilde{x}, x_{n+1} - \tilde{x}\big\rangle.
\end{aligned} \tag{3.34}$$

It then follows from (3.34) that

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\le \frac{(1 - \beta_n\bar{\gamma})^2 + \beta_n}{1 - \beta_n}\|x_n - \tilde{x}\|^2 + \frac{\beta_n}{1 - \beta_n}\big[2\alpha_n\|\gamma V x_n - \mu F\tilde{x}\|\|y_n - \tilde{x}\| \\
&\quad + 2\big\langle(I - A)\tilde{x}, x_{n+1} - \tilde{x}\big\rangle\big] \\
&= \left(1 - \frac{2\beta_n(\bar{\gamma} - 1)}{1 - \beta_n}\right)\|x_n - \tilde{x}\|^2 \\
&\quad + \frac{2\beta_n(\bar{\gamma} - 1)}{1 - \beta_n}\cdot\frac{1}{2(\bar{\gamma} - 1)}\big[2\alpha_n\|\gamma V x_n - \mu F\tilde{x}\|\|y_n - \tilde{x}\| \\
&\quad + \beta_n\bar{\gamma}^2\|x_n - \tilde{x}\|^2 + 2\big\langle(I - A)\tilde{x}, x_{n+1} - \tilde{x}\big\rangle\big] \\
&= (1 - \omega_n)\|x_n - \tilde{x}\|^2 + \omega_n\delta_n,
\end{aligned}$$

where $\omega_n = \frac{2\beta_n(\bar{\gamma}-1)}{1-\beta_n}$ and

$$\delta_n = \frac{1}{2(\bar{\gamma}-1)}\bigl[2\alpha_n\|\gamma Vx_n - \mu F\tilde{x}\|\|y_n - \tilde{x}\| + \beta_n\bar{\gamma}^2\|x_n - \tilde{x}\|^2 + 2\bigl\langle(I-A)\tilde{x}, x_{n+1} - \tilde{x}\bigr\rangle\bigr].$$

It can be readily seen from Step 2 and conditions (C1) and (C2) that $\omega_n \to 0$, $\sum_{n=0}^{\infty}\omega_n = \infty$ and $\limsup_{n\to\infty}\delta_n \leq 0$. By Lemma 2.8 with $r_n = 0$, we conclude that $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$. This completes the proof. □

Corollary 3.2 *Let* $\{x_n\}$ *be the sequence generated by the explicit scheme* (3.3). *Assume that the sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the conditions* (C1) *and* (C2) *in Theorem* 3.3. *If* $\{x_n\}$ *is asymptotically regular* (*i.e.*, $x_{n+1} - x_n \to 0$), *then* $\{x_n\}$ *converges strongly to a point* $\tilde{x} \in \Omega$, *which is the unique solution of the VIP* (3.2).

Putting $\mu = 2$, $F = \frac{1}{2}I$, and $\gamma = 1$ in Theorem 3.3, we obtain the following.

Corollary 3.3 *Let* $\{x_n\}$ *be generated by the following iterative scheme*:

$$\begin{cases} y_n = \alpha_n Vx_n + (1-\alpha_n)S\Delta_n^N Gx_n, \\ x_{n+1} = P_C[(I-\beta_n A)S\Delta_n^N Gx_n + \beta_n y_n], \quad \forall n \geq 0. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the conditions* (C1) *and* (C2) *in Theorem* 3.3. *If* $\{x_n\}$ *is weakly asymptotically regular* (*i.e.*, $x_{n+1} - x_n \rightharpoonup 0$), *then* $\{x_n\}$ *converges strongly to a point* $\tilde{x} \in \Omega$, *which is the unique solution of the VIP* (3.2).

Putting $\alpha_n = 0$, $\forall n \geq 0$ in Corollary 3.3, we get the following.

Corollary 3.4 *Let* $\{x_n\}$ *be generated by the following iterative scheme*:

$$x_{n+1} = P_C\bigl[\bigl(I - \beta_n(A-I)\bigr)S\Delta_n^N Gx_n\bigr], \quad \forall n \geq 0.$$

Assume that the sequence $\{\beta_n\}$ *satisfies the conditions* (C1) *and* (C2) *in Theorem* 3.3 *with* $\alpha_n = 0$, $\forall n \geq 0$. *If* $\{x_n\}$ *is weakly asymptotically regular* (*i.e.*, $x_{n+1} - x_n \rightharpoonup 0$), *then* $\{x_n\}$ *converges strongly to a point* $\tilde{x} \in \Omega$, *which is the unique solution of the VIP* (3.2).

Remark 3.1 If $\{\alpha_n\}$, $\{\beta_n\}$ in Corollary 3.2 and $\{r_{i,n}\}_{i=1}^N$ in Δ_n^N satisfy conditions (C2) and

- (C3) $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty}|\beta_{n+1} - \beta_n| < \infty$; or
- (C4) $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ and $\lim_{n\to\infty}\frac{\beta_n}{\beta_{n+1}} = 1$ or, equivalently, $\lim_{n\to\infty}\frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}} = 0$ and $\lim_{n\to\infty}\frac{\beta_n - \beta_{n+1}}{\beta_{n+1}} = 0$; or,
- (C5) $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ and $|\beta_{n+1} - \beta_n| \leq o(\beta_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty}\sigma_n < \infty$ (the perturbed control condition);
- (C6) $\sum_{n=0}^{\infty}|r_{i,n+1} - r_{i,n}| < \infty$ for each $i = 1, \ldots, N$,

then the sequence $\{x_n\}$ generated by (3.3) is asymptotically regular. Now we give only the proof in the case when $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_{i,n}\}_{i=1}^N$ satisfy the conditions (C2), (C5), and (C6). By Step 1 in the proof of Theorem 3.3, there exists a constant $\widetilde{M}_2 > 0$ such that for all $n \geq 0$, $\mu\|FS\Delta_n^N Gx_n\| + \gamma\|Vx_n\| \leq \widetilde{M}_2$, $\|A\|\|S\Delta_n^N Gx_n\| + \|y_n\| \leq \widetilde{M}_2$, and

$$\sum_{i=1}^{N}\biggl[\bigl\|\mathcal{A}_i\Delta_n^{i-1}Gx_n\bigr\| + \frac{1}{r_{i,n}}\bigl\|T_{r_{i,n}}^{(\Theta_i,\varphi_i)}(I - r_{i,n}\mathcal{A}_i)\Delta_n^{i-1}Gx_n - (I - r_{i,n}\mathcal{A}_i)\Delta_n^{i-1}Gx_n\bigr\|\biggr] \leq \widetilde{M}_2.$$

Utilizing similar arguments to those of (3.20), we obtain

$$\big\| \Delta_n^N Gx_n - \Delta_{n-1}^N Gx_{n-1} \big\| \le \widetilde{M}_2 \sum_{i=1}^N |r_{i,n} - r_{i,n-1}| + \|x_n - x_{n-1}\|.$$

So, we obtain, for all $n \ge 0$,

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad = \big\| \alpha_n \gamma (Vx_n - Vx_{n-1}) + \gamma(\alpha_n - \alpha_{n-1}) Vx_{n-1} \\
&\qquad + (I - \alpha_n \mu F) S\Delta_n^N Gx_n - (I - \alpha_n \mu F) S\Delta_{n-1}^N Gx_{n-1} \\
&\qquad + \mu(\alpha_n - \alpha_{n-1}) FS\Delta_{n-1}^N Gx_{n-1} \big\| \\
&\quad \le \alpha_n \gamma l \|x_n - x_{n-1}\| + (1 - \alpha_n \tau) \big\| \Delta_n^N Gx_n - \Delta_{n-1}^N Gx_{n-1} \big\| \\
&\qquad + |\alpha_n - \alpha_{n-1}| \big(\gamma \|Vx_{n-1}\| + \mu \big\| FS\Delta_{n-1}^N Gx_{n-1} \big\| \big) \\
&\quad \le \alpha_n \gamma l \|x_n - x_{n-1}\| + (1 - \alpha_n \tau) \Bigg[\|x_n - x_{n-1}\| + \widetilde{M}_2 \sum_{i=1}^N |r_{i,n} - r_{i,n-1}| \Bigg] \\
&\qquad + |\alpha_n - \alpha_{n-1}| \big(\gamma \|Vx_{n-1}\| + \mu \big\| FS\Delta_{n-1}^N Gx_{n-1} \big\| \big) \\
&\quad \le \big(1 - \alpha_n(\tau - \gamma l)\big) \|x_n - x_{n-1}\| + \Bigg(\sum_{i=1}^N |r_{i,n} - r_{i,n-1}| + |\alpha_n - \alpha_{n-1}| \Bigg) \widetilde{M}_2,
\end{aligned}$$

and hence

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le \big\| (I - \beta_n A) S\Delta_n^N Gx_n + \beta_n y_n - (I - \beta_{n-1} A) S\Delta_{n-1}^N Gx_{n-1} - \beta_{n-1} y_{n-1} \big\| \\
&\quad \le \big\| (I - \beta_n A) \big(S\Delta_n^N Gx_n - S\Delta_{n-1}^N Gx_{n-1} \big) \big\| \\
&\qquad + |\beta_n - \beta_{n-1}| \|A\| \big\| S\Delta_{n-1}^N Gx_{n-1} \big\| + \beta_n \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}| \|y_{n-1}\| \\
&\quad \le (1 - \beta_n \bar{\gamma}) \big\| S\Delta_n^N Gx_n - S\Delta_{n-1}^N Gx_{n-1} \big\| \\
&\qquad + \beta_n \Bigg[\big(1 - \alpha_n(\tau - \gamma l)\big) \|x_n - x_{n-1}\| + \Bigg(\sum_{i=1}^N |r_{i,n} - r_{i,n-1}| + |\alpha_n - \alpha_{n-1}| \Bigg) \widetilde{M}_2 \Bigg] \\
&\qquad + |\beta_n - \beta_{n-1}| \big(\|A\| \big\| S\Delta_{n-1}^N Gx_{n-1} \big\| + \|y_{n-1}\| \big) \\
&\quad \le (1 - \beta_n \bar{\gamma}) \Bigg[\|x_n - x_{n-1}\| + \widetilde{M}_2 \sum_{i=1}^N |r_{i,n} - r_{i,n-1}| \Bigg] \\
&\qquad + \beta_n \Bigg[\big(1 - \alpha_n(\tau - \gamma l)\big) \|x_n - x_{n-1}\| \\
&\qquad + \Bigg(\sum_{i=1}^N |r_{i,n} - r_{i,n-1}| + |\alpha_n - \alpha_{n-1}| \Bigg) \widetilde{M}_2 \Bigg] + |\beta_n - \beta_{n-1}| \widetilde{M}_2 \\
&\quad \le (1 - \beta_n \bar{\gamma}) \Bigg[\|x_n - x_{n-1}\| + \Bigg(\sum_{i=1}^N |r_{i,n} - r_{i,n-1}| + |\alpha_n - \alpha_{n-1}| \Bigg) \widetilde{M}_2 \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&\quad + \beta_n\Bigg[\|x_n - x_{n-1}\| + \Bigg(\sum_{i=1}^{N}|r_{i,n} - r_{i,n-1}| + |\alpha_n - \alpha_{n-1}|\Bigg)\widetilde{M}_2\Bigg] + |\beta_n - \beta_{n-1}|\widetilde{M}_2 \\
&= \big(1 - \beta_n(\bar{\gamma} - 1)\big)\Bigg[\|x_n - x_{n-1}\| + \Bigg(\sum_{i=1}^{N}|r_{i,n} - r_{i,n-1}| + |\alpha_n - \alpha_{n-1}|\Bigg)\widetilde{M}_2\Bigg] \\
&\quad + |\beta_n - \beta_{n-1}|\widetilde{M}_2 \\
&\le \big(1 - \beta_n(\bar{\gamma} - 1)\big)\|x_n - x_{n-1}\| + \Bigg(\sum_{i=1}^{N}|r_{i,n} - r_{i,n-1}| + |\alpha_n - \alpha_{n-1}|\Bigg)\widetilde{M}_2 \\
&\quad + \big(o(\beta_n) + \sigma_{n-1}\big)\widetilde{M}_2. \qquad (3.35)
\end{aligned}
$$

By taking $a_{n+1} = \|x_{n+1} - x_n\|$, $\omega_n = \beta_n(\bar{\gamma} - 1)$, $\omega_n\delta_n = \widetilde{M}_2 o(\beta_n)$ and $r_n = (\sum_{i=1}^{N}|r_{i,n} - r_{i,n-1}| + |\alpha_n - \alpha_{n-1}| + \sigma_{n-1})\widetilde{M}_2$, from (3.35) we have

$$a_{n+1} \le (1 - \omega_n)a_n + \omega_n\delta_n + r_n.$$

Consequently, utilizing the conditions (C2), (C5), (C6), and Lemma 2.8, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

In view of this observation, we have the following.

Corollary 3.5 *Let* $\{x_n\}$ *be the sequence generated by the explicit scheme* (3.3), *where the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{r_{i,n}\}_{i=1}^{N}$ *satisfy the conditions* (C1), (C2), (C5), *and* (C6) (*or the conditions* (C1), (C2), (C3), *and* (C6), *or the conditions* (C1), (C2), (C4), *and* (C6)). *Then* $\{x_n\}$ *converges strongly to a point* $\tilde{x} \in \Omega$, *which is the unique solution of the VIP* (3.2).

4 Applications

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. For a given nonlinear mapping $\mathcal{A} : C \to H$, we consider the variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle \mathcal{A}x^*, y - x^*\rangle \ge 0, \quad \forall y \in C. \qquad (4.1)$$

We will denote by $\mathrm{VI}(C, \mathcal{A})$ the set of solutions of the VIP (4.1).

Recall that if u is a point in C, then the following relation holds:

$$u \in \mathrm{VI}(C, \mathcal{A}) \quad \Leftrightarrow \quad u = P_C(I - \lambda\mathcal{A})u, \quad \lambda > 0.$$

In the meantime, it is easy to see that the following relation holds:

$$\text{SVI (1.13) with } F_2 = 0 \quad \Leftrightarrow \quad \text{VIP (4.1) with } \mathcal{A} = F_1. \qquad (4.2)$$

An operator $\mathcal{A} : C \to H$ is said to be an α-inverse strongly monotone operator if there exists a constant $\alpha > 0$ such that

$$\langle \mathcal{A}x - \mathcal{A}y, x - y\rangle \ge \alpha\|\mathcal{A}x - \mathcal{A}y\|^2, \quad \forall x, y \in C.$$

As an example, we recall that the α-inverse strongly monotone operators are firmly nonexpansive mappings if $\alpha \geq 1$ and that every α-inverse strongly monotone operator is also $\frac{1}{\alpha}$-Lipschitz-continuous (see [17]).

Let us observe also that, if $\mathcal{A}$ is α-inverse strongly monotone, the mappings $P_C(I - \lambda\mathcal{A})$ are nonexpansive for all $\lambda \in (0, 2\alpha]$ since they are compositions of nonexpansive mappings.

Throughout the rest of this paper, we always assume the following:

$F : C \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$;
$F_j : C \to H$ is ζ_j-inverse strongly monotone for $j = 1, 2$ and $T_i : C \to C$ is a k_i-strictly pseudocontractive mapping for each $i = 1, \dots, N$;
A is a $\bar{\gamma}$-strongly positive bounded linear operator on H with $\bar{\gamma} \in (1, 2)$ and
$V : C \to H$ is an l-Lipschitzian mapping with $l \geq 0$;
$0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \gamma l < \tau$ with $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;
$G : C \to C$ is a mapping defined by $Gx = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x$ with $0 < \nu_j < 2\zeta_j$ for $j = 1, 2$, and the fixed point set of G is denoted by Ξ;
$\Delta_t^N : C \to C$ is a mapping defined by $\Delta_t^N x = (I - r_{N,t}\mathcal{A}_N) \cdots (I - r_{1,t}\mathcal{A}_1)x$, $t \in (0, 1)$ with $\mathcal{A}_i = I - T_i$ and $\{r_{i,t}\} \subset [a_i, b_i] \subset (0, 1 - k_i)$ for each $i = 1, \dots, N$;
$\Delta_n^N : C \to C$ is a mapping defined by $\Delta_n^N x = (I - r_{N,n}\mathcal{A}_N) \cdots (I - r_{1,n}\mathcal{A}_1)x$ with $\{r_{i,n}\} \subset [a_i, b_i] \subset (0, 1 - k_i)$ and $\lim_{n\to\infty} r_{i,n} = r_i$, for each $i = 1, \dots, N$;
$\Omega = \bigcap_{i=1}^N \operatorname{Fix}(T_i) \cap \Xi \neq \emptyset$ and P_Ω is the metric projection of H onto Ω;
$\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset (0, 1]$ and $\{\theta_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})} \subset (0, 1)$.

We now introduce the following composite implicit relaxed extragradient-like scheme that generates a net $\{x_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})}$ in one implicit manner:

$$x_t = P_C\bigl[(I - \theta_t A)\Delta_t^N G x_t + \theta_t\bigl(t\gamma V x_t + (I - t\mu F)\Delta_t^N G x_t\bigr)\bigr]. \tag{4.3}$$

Moreover, we also propose the following composite explicit relaxed extragradient-like scheme, which generates a sequence in another explicit way:

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu F)\Delta_n^N G x_n, \\ x_{n+1} = P_C[(I - \beta_n A)\Delta_n^N G x_n + \beta_n y_n], \quad \forall n \geq 0, \end{cases} \tag{4.4}$$

where $x_0 \in C$ is an arbitrary initial guess.

Theorem 4.1 *Let the net* $\{x_t\}$ *be defined via* (4.3). *If* $\lim_{t\to 0}\theta_t = 0$, *then* x_t *converges strongly to a point* $\tilde{x} \in \Omega$ *as* $t \to 0$, *which solves the VIP*

$$\bigl\langle (A - I)\tilde{x}, \tilde{x} - p\bigr\rangle \leq 0, \quad \forall p \in \Omega. \tag{4.5}$$

Equivalently, we have $P_\Omega(2I - A)\tilde{x} = \tilde{x}$.

Proof First of all, it is easy to see that $T_i : C \to C$ is k_i-strictly pseudocontractive if and only if

$$\langle T_i x - T_i y, x - y\rangle \leq \|x - y\|^2 - \frac{1 - k_i}{2}\bigl\|(I - T_i)x - (I - T_i)y\bigr\|^2, \quad \forall x, y \in C.$$

It is clear that in this case the mapping $I - T_i$ is $\frac{1-k_i}{2}$-inverse strongly monotone. Moreover, by Lemma 2.3(i) we know that if $T_i : C \to C$ is k_i-strictly pseudocontractive, then T_i is Lipschitz-continuous with constant $\frac{1+k_i}{1-k_i}$, *i.e.*, $\|T_ix - T_iy\| \le \frac{1+k_i}{1-k_i}\|x - y\|$ for all $x, y \in C$.

In Theorem 3.1, we put $\Theta_i = 0$, $\varphi_i = 0$, $\Phi_1 = \Phi_2 = 0$, $T = I$, $\lambda = 0$, $\mathcal{A}_i = I - T_i$ and $\eta_i = \frac{1-k_i}{2}$ for each $i = 1, \dots, N$. Then $T : C \to C$ is a k-strictly pseudocontractive mapping with $k = 0$ and $\mathcal{A}_i$ is η_i-inverse strongly monotone with $\eta_i = \frac{1-k_i}{2}$ for each $i = 1, \dots, N$. In this case, $S = I$ and $\mathrm{GMEP}(\Theta_i, \varphi_i, \mathcal{A}_i) = \mathrm{VI}(C, \mathcal{A}_i)$ for each $i = 1, \dots, N$. Next let us show $\mathrm{VI}(C, \mathcal{A}_i) = \mathrm{Fix}(T_i)$ for each $i = 1, \dots, N$. Indeed, we have, for $\nu > 0$,

$$\begin{aligned}
u \in \mathrm{VI}(C, \mathcal{A}_i) \quad &\Leftrightarrow \quad \langle \mathcal{A}_iu, y - u\rangle \ge 0, \quad \forall y \in C \\
&\Leftrightarrow \quad \langle u - \nu\mathcal{A}_iu - u, u - y\rangle \ge 0, \quad \forall y \in C \\
&\Leftrightarrow \quad u = P_C(u - \nu\mathcal{A}_iu) \\
&\Leftrightarrow \quad u = P_C(u - \nu u + \nu T_iu) \\
&\Leftrightarrow \quad \langle u - \nu u + \nu T_iu - u, u - y\rangle \ge 0, \quad \forall y \in C \\
&\Leftrightarrow \quad \langle u - T_iu, u - y\rangle \le 0, \quad \forall y \in C \\
&\Leftrightarrow \quad u = T_iu \\
&\Leftrightarrow \quad u \in \mathrm{Fix}(T_i).
\end{aligned}$$

Hence, we conclude that

$$\Omega = \bigcap_{i=1}^{N} \mathrm{GMEP}(\Theta_i, \varphi_i, \mathcal{A}_i) \cap \mathrm{Fix}(T) \cap \Xi = \bigcap_{i=1}^{N} \mathrm{VI}(C, \mathcal{A}_i) \cap \mathrm{Fix}(I) \cap \Xi = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \Xi.$$

Also, observe that

$$\begin{aligned}
S\Delta_t^N Gx_t &= ST_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}\mathcal{A}_N)\cdots T_{r_{1,t}}^{(\Theta_1,\varphi_1)}(I - r_{1,t}\mathcal{A}_1)Gx_t \\
&= T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}\mathcal{A}_N)\cdots T_{r_{1,t}}^{(\Theta_1,\varphi_1)}(I - r_{1,t}\mathcal{A}_1)Gx_t \\
&= P_C(I - r_{N,t}\mathcal{A}_N)\cdots P_C(I - r_{2,t}\mathcal{A}_2)P_C\big[(1 - r_{1,t})Gx_t + r_{1,t}T_1Gx_t\big] \\
&= P_C(I - r_{N,t}\mathcal{A}_N)\cdots P_C(I - r_{2,t}\mathcal{A}_2)(I - r_{1,t}\mathcal{A}_1)Gx_t \\
&= \cdots \\
&= (I - r_{N,t}\mathcal{A}_N)\cdots(I - r_{2,t}\mathcal{A}_2)(I - r_{1,t}\mathcal{A}_1)Gx_t.
\end{aligned}$$

In this case, the implicit scheme (3.1) reduces to (4.3). Consequently, utilizing Theorem 3.1 we obtain the desired result. □

Remark 4.1 Theorem 4.1 extends and improves Ceng *et al.*'s hierarchical fixed point problem (1.7) for a nonexpansive mapping (see [32], Theorem 3.1) and Jung's hierarchical fixed point problem (1.10) for a strict pseudocontraction (see [1], Theorem 3.1) to the hierarchical fixed point problem (4.5) with the constraint of SVI (1.13).

Taking $F = \frac{1}{2}I$, $\mu = 2$, and $\gamma = 1$ in Theorem 4.1, we get

Corollary 4.1 *Let* $\{x_t\}$ *be defined by*

$$x_t = P_C\big[(I - \theta_t A)\Delta_t^N Gx_t + \theta_t\big(tVx_t + (1-t)\Delta_t^N Gx_t\big)\big].$$

If $\lim_{t\to 0}\theta_t = 0$, *then* $\{x_t\}$ *converges strongly as* $t \to 0$ *to a point* $\tilde{x} \in \Omega$, *which is the unique solution of the VIP* (4.5).

Next, by utilizing Theorem 3.2, we prove the following result in order to establish the strong convergence of the sequence $\{x_n\}$ generated by the composite explicit relaxed extragradient-like scheme (4.4).

Theorem 4.2 *Let* $\{x_n\}$ *be the sequence generated by the explicit scheme* (4.4), *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following condition*:

(C1) $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset (0,1]$, *and* $\alpha_n \to 0$, $\beta_n \to 0$ *as* $n \to \infty$.

Let LIM *be a Banach limit. Then*

$$\mathrm{LIM}_n\big\langle (A-I)\tilde{x}, \tilde{x} - x_n\big\rangle \le 0,$$

where $\tilde{x} = \lim_{t\to 0^+} x_t$ *with* x_t *being defined by*

$$x_t = P_C\big[(I - \theta_t A)\Delta^N Gx_t + \theta_t\big(t\gamma Vx_t + (I - t\mu F)\Delta^N Gx_t\big)\big], \tag{4.6}$$

where $G, \Delta^N : C \to C$ *are defined by* $Gx = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x$ *and* $\Delta^N x = (I - r_N\mathcal{A}_N)\cdots(I - r_1\mathcal{A}_1)x$ *with* $\nu_j \in (0, 2\zeta_j)$, $j = 1,2$ *and* $r_i \in [a_i, b_i] \subset (0, 1-k_i)$ *for each* $i = 1,\dots,N$.

Proof In Theorem 3.2, we put $\Theta_i = 0$, $\varphi_i = 0$, $\Phi_1 = \Phi_2 = 0$, $T = I$, $\lambda = 0$, $\mathcal{A}_i = I - T_i$ and $\eta_i = \frac{1-k_i}{2}$ for each $i = 1,\dots,N$. Then $\mathcal{A}_i$ is η_i-inverse strongly monotone with $\eta_i = \frac{1-k_i}{2}$ for each $i = 1,\dots,N$. Utilizing similar arguments to those in the proof of Theorem 4.1, we get $\mathrm{GMEP}(\Theta_i, \varphi_i, \mathcal{A}_i) = \mathrm{VI}(C, \mathcal{A}_i) = \mathrm{Fix}(T_i)$, $\Delta_n^N Gx_n = (I - r_{N,n}\mathcal{A}_N)\cdots(I - r_{1,n}\mathcal{A}_1)Gx_n$ and $\Delta^N Gx_t = (I - r_N\mathcal{A}_N)\cdots(I - r_1\mathcal{A}_1)Gx_t$. Thus, it is easy to see that the schemes (3.3) and (3.25) reduce to the ones (4.4) and (4.6), respectively. Consequently, by utilizing Theorem 3.2, we derive the desired result. □

Remark 4.2 Theorem 4.2 extends and improves Jung's hierarchical fixed point problem (1.10) for a strict pseudocontraction (see [1], Theorem 3.2) to the hierarchical fixed point problem (4.5) with the constraint of SVI (1.13).

Now, using Theorem 4.2, we establish the strong convergence of the sequence $\{x_n\}$ generated by the composite explicit relaxed extragradient-like scheme (4.4) to a point $\tilde{x} \in \Omega$, which is also the unique solution of the VIP (4.5).

Theorem 4.3 *Let* $\{x_n\}$ *be the sequence generated by the explicit scheme* (4.4), *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following conditions*:

(C1) $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset (0,1]$, *and* $\alpha_n \to 0$, $\beta_n \to 0$ *as* $n \to \infty$;

(C2) $\sum_{n=0}^{\infty}\beta_n = \infty$.

If $\{x_n\}$ *is weakly asymptotically regular* (*i.e.*, $x_{n+1} - x_n \rightharpoonup 0$), *then* x_n *converges strongly to a point* $\tilde{x} \in \Omega$, *which is the unique solution of the VIP* (4.5).

Proof In Theorem 3.3, we put $\Theta_i = 0$, $\varphi_i = 0$, $\Phi_1 = \Phi_2 = 0$, $T = I$, $\lambda = 0$, $\mathcal{A}_i = I - T_i$, and $\eta_i = \frac{1-k_i}{2}$ for each $i = 1, \dots, N$. Then $\mathcal{A}_i$ is η_i-inverse strongly monotone with $\eta_i = \frac{1-k_i}{2}$ for each $i = 1, \dots, N$. Utilizing similar arguments to those in the proof of Theorem 4.1, we get $\Omega = \bigcap_{i=1}^N \operatorname{Fix}(T_i) \cap \Xi$ and $\Delta_n^N G x_n = (I - r_{N,n}\mathcal{A}_N) \cdots (I - r_{1,n}\mathcal{A}_1) G x_n$. So, it is easy to see that the scheme (3.3) reduces to the ones (4.4). Thus, in terms of Theorem 3.3 we derive the desired result. □

Remark 4.3 Theorem 4.3 extends and improves Ceng *et al.*'s hierarchical fixed point problem (1.7) for a nonexpansive mapping (see [32], Theorem 3.2) and Jung's hierarchical fixed point problem (1.10) for a strict pseudocontraction (see [1], Theorem 3.3) to the hierarchical fixed point problem (4.5) with the constraint of SVI (1.13).

Corollary 4.2 *Let* $\{x_n\}$ *be the sequence generated by the explicit scheme* (4.4). *Assume that the sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the conditions* (C1) *and* (C2) *in Theorem* 4.3. *If* $\{x_n\}$ *is asymptotically regular* (*i.e.*, $x_{n+1} - x_n \to 0$), *then* $\{x_n\}$ *converges strongly to a point* $\tilde{x} \in \Omega$, *which is the unique solution of the VIP* (4.5).

Putting $\mu = 2$, $F = \frac{1}{2}I$, and $\gamma = 1$ in Theorem 4.3, we obtain the following.

Corollary 4.3 *Let* $\{x_n\}$ *be generated by the following iterative scheme*:

$$\begin{cases} y_n = \alpha_n V x_n + (1 - \alpha_n)\Delta_n^N G x_n, \\ x_{n+1} = P_C[(I - \beta_n A)\Delta_n^N G x_n + \beta_n y_n], \quad \forall n \geq 0. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the conditions* (C1) *and* (C2) *in Theorem* 4.3. *If* $\{x_n\}$ *is weakly asymptotically regular* (*i.e.*, $x_{n+1} - x_n \rightharpoonup 0$), *then* $\{x_n\}$ *converges strongly to a point* $\tilde{x} \in \Omega$, *which is the unique solution of the VIP* (4.5).

Putting $\alpha_n = 0$, $\forall n \geq 0$ in Corollary 4.3, we get the following.

Corollary 4.4 *Let* $\{x_n\}$ *be generated by the following iterative scheme*:

$$x_{n+1} = P_C\big[\big(I - \beta_n(A - I)\big)\Delta_n^N G x_n\big], \quad \forall n \geq 0.$$

Assume that the sequence $\{\beta_n\}$ *satisfies the conditions* (C1) *and* (C2) *in Theorem* 4.3 *with* $\alpha_n = 0$, $\forall n \geq 0$. *If* $\{x_n\}$ *is weakly asymptotically regular* (*i.e.*, $x_{n+1} - x_n \rightharpoonup 0$), *then* $\{x_n\}$ *converges strongly to a point* $\tilde{x} \in \Omega$, *which is the unique solution of the VIP* (4.5).

Remark 4.4 If $\{\alpha_n\}$, $\{\beta_n\}$ in Corollary 4.2 and $\{r_{i,n}\}_{i=1}^N$ in Δ_n^N satisfy conditions (C2) and

(C3) $\sum_{n=0}^\infty |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^\infty |\beta_{n+1} - \beta_n| < \infty$; or

(C4) $\sum_{n=0}^\infty |\alpha_{n+1} - \alpha_n| < \infty$ and $\lim_{n\to\infty} \frac{\beta_n}{\beta_{n+1}} = 1$; or, equivalently, $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}} = 0$ and $\lim_{n\to\infty} \frac{\beta_n - \beta_{n+1}}{\beta_{n+1}} = 0$; or,

(C5) $\sum_{n=0}^\infty |\alpha_{n+1} - \alpha_n| < \infty$ and $|\beta_{n+1} - \beta_n| \leq o(\beta_{n+1}) + \sigma_n$, $\sum_{n=0}^\infty \sigma_n < \infty$ (the perturbed control condition);

(C6) $\sum_{n=0}^{\infty} |r_{i,n+1} - r_{i,n}| < \infty$ for each $i = 1, \dots, N$,
then the sequence $\{x_n\}$ generated by (4.4) is asymptotically regular. Now we give only the proof in the case when $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_{i,n}\}_{i=1}^N$ satisfy the conditions (C2), (C5), and (C6). Indeed, in Remark 3.1, we put $\Theta_i = 0$, $\varphi_i = 0$, $\Phi_1 = \Phi_2 = 0$, $T = I$, $\lambda = 0$, $\mathcal{A}_i = I - T_i$, and $\eta_i = \frac{1-k_i}{2}$ for each $i = 1, \dots, N$. Utilizing Remark 3.1, we derive the claim.

In view of Remark 4.4, we have the following.

Corollary 4.5 *Let* $\{x_n\}$ *be the sequence generated by the explicit scheme* (4.4), *where the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{r_{i,n}\}_{i=1}^N$ *satisfy Remark* 4.4(C1), (C2), (C5), *and* (C6) (*or Remark* 4.4(C1), (C2), (C3), *and* (C6), *or Remark* 4.4(C1), (C2), (C4), *and* (C6)). *Then* $\{x_n\}$ *converges strongly to a point* $\tilde{x} \in \Omega$, *which is the unique solution of the VIP* (4.5).

Remark 4.5 Corollary 4.5 extends and improves Jung's hierarchical fixed point problem (1.10) for a strict pseudocontraction (see [1], Corollary 4.5) to the hierarchical fixed point problem (4.5) with the constraint of SVI (1.13).

5 Concluding remarks

We introduced and analyzed one composite implicit relaxed extragradient-like scheme and another composite explicit relaxed extragradient-like scheme for finding a common solution of a finite family of generalized mixed equilibrium problems (GMEPs) with the constraints of a system of generalized equilibrium problems (SGEP) and the hierarchical fixed point problem (HFPP) for a strictly pseudocontractive mapping by virtue of the general composite implicit and explicit schemes for a nonexpansive mapping $T : H \to H$ (see [32]) and the general composite implicit and explicit ones for a strict pseudocontraction $T : H \to H$ (see [1]). Our Theorems 3.1-3.3 and Corollary 3.5 improve, extend, supplement, and develop Theorems 3.1 and 3.2 of [32], Theorems 3.1-3.3 and Corollary 3.5 of [1] and Theorem 3.1 of [23] in the following aspects.

(i) Ceng *et al.*'s general composite implicit scheme for a nonexpansive mapping $T : H \to H$ (see (3.1) in [32]) and Jung's general composite implicit one for a strict pseudocontraction $T : H \to H$ (see (3.1) in [1]) extends to developing the composite implicit relaxed extragradient-like scheme (3.1) for a finite family of GMEPs with constraints of SGEP (1.12) and the HFPP for a strict pseudocontraction. Moreover, Ceng *et al.*'s general composite explicit scheme for a nonexpansive mapping $T : H \to H$ (see (3.5) in [32]) and Jung's general composite explicit one for a strict pseudocontraction (see (3.3) in [1]) extends to developing the composite explicit relaxed extragradient-like one (3.3) for a finite family of GMEPs with constraints of SGEP (1.12) and the HFPP for a strict pseudocontraction.

(ii) The argument techniques in our Theorems 3.1-3.3 and Corollary 3.5 are very different from those techniques in [32] Theorems 3.1-3.2 and [1] Theorems 3.1-3.3 and Corollary 3.5 because we make use of the properties of the resolvent $T_r^{(\Theta,\varphi)}$ (see, *e.g.*, Proposition 2.2 and the argument of (3.9), (3.14), (3.16), and (3.20)), the ones of the strong positive bounded linear operators (see Lemma 2.9), the ones of the Banach limit LIM (see Lemma 2.10), the equivalence of the fixed point equation $x^* = T_{\nu_1}^{\Phi_1}(I - \nu_1 F_1) T_{\nu_2}^{\Phi_2}(I - \nu_2 F_2) x^*$ to the SGEP (1.12) for ζ_j-inverse strongly monotone mappings $F_j : C \to H$, $j = 1, 2$ (see Proposition 2.3) and the contractive coefficient estimates for the contractions T^λ associating with nonexpansive mappings (see Lemma 2.7).

(iii) The problem of finding a common solution $\tilde{x} \in \bigcap_{i=1}^{N} \mathrm{GMEP}(\Theta_i, \varphi_i, \mathcal{A}_i) \cap \mathrm{Fix}(T) \cap \Xi$ of SGEP (1.12), the fixed point problem of a k-strict pseudocontraction T and a finite family of GMEPs in our Theorems 3.1-3.3 and Corollary 3.5 is more general and more flexible than the one of finding a fixed point of a nonexpansive mapping $T : H \to H$ in [32] Theorems 3.1 and 3.2, the one of finding a fixed point of a strictly pseudocontractive mapping $T : H \to H$ in [1] Theorems 3.1-3.3 and Corollary 3.5, and the one of finding a common solution of GMEP (1.11), SGEP (1.12), and the fixed point problem of a k-strict pseudocontraction T in [17] Theorem 3.1. It is worth pointing out that the problem of finding $\tilde{x} \in (\bigcap_{i=1}^{N} \mathrm{GMEP}(\Theta_i, \varphi_i, \mathcal{A}_i)) \cap \mathrm{Fix}(T) \cap \Xi$ extends the fixed point problems in [1, 32] from the domain H of the mapping T to the domain C for the one of finding $\tilde{x} \in (\bigcap_{i=1}^{N} \mathrm{GMEP}(\Theta_i, \varphi_i, \mathcal{A}_i)) \cap \mathrm{Fix}(T) \cap \Xi$ and generalizes the fixed point problems in [1, 32] to the setting of SGEP (1.12) and a finite family of GMEPs. In the meantime, the problem of finding $\tilde{x} \in (\bigcap_{i=1}^{N} \mathrm{GMEP}(\Theta_i, \varphi_i, \mathcal{A}_i)) \cap \mathrm{Fix}(T) \cap \Xi$ extends the problem of finding $\tilde{x} \in \mathrm{GMEP}(\Theta, \varphi, \mathcal{A}) \cap \mathrm{Fix}(T) \cap \Xi$ in [23] from one GMEP to a finite family of GMEPs.

(iv) Our Theorems 3.1-3.3 and Corollary 3.5 generalize [32] Theorems 3.1 and 3.2 from a nonexpansive mapping $T : H \to H$ to a k-strict pseudocontraction $T : C \to C$ and extend [32] Theorems 3.1 and 3.2 to the setting of SGEP (1.12) and a finite family of GMEPs. Moreover, Theorems 3.1-3.3 and Corollary 3.5 generalize Theorems 3.1-3.3 and Corollary 3.5 of [1] from a strict pseudocontraction $T : H \to H$ to the setting of SGEP (1.12) and a finite family of GMEPs. In the meantime, the operators T_t in the implicit scheme (3.1) of Jung [1] are replaced by the composite ones $S\Delta_t^N G$ in our implicit scheme (3.1) and the operators T_n in the explicit scheme (3.3) of Jung [1] are replaced by the composite ones $S\Delta_n^N G$ in our explicit scheme (3.3).

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details

[1]Department of Mathematics, Shanghai Normal University, and Scientific Computing Key Laboratory of Shanghai Universities, Shanghai, 200234, China. [2]Department of Information Management, and Innovation Center for Big Data and Digital Convergence, Yuan Ze University, Chung-Li, 32003, Taiwan. [3]Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung, 80708, Taiwan.

Acknowledgements

In this research, the first author was partially supported by the Innovation Program of Shanghai Municipal Education Commission (15ZZ068), the Ph.D. Program Foundation of Ministry of Education of China (20123127110002), and the Program for Outstanding Academic Leaders in Shanghai City (15XD1503100). The second author and third author were supported partly by the National Science Council of the Republic of China.

Received: 8 May 2015 Accepted: 28 August 2015 Published online: 17 September 2015

References

1. Jung, JS: A general composite iterative method for strictly pseudocontractive mappings in Hilbert spaces. Fixed Point Theory Appl. **2014**, Article ID 173 (2014)
2. Ceng, LC, Petrusel, A, Wong, MM, Yu, SJ: Strong convergence of implicit viscosity approximation methods for pseudocontractive mappings in Banach spaces. Optimization **60**(6), 659-670 (2011)
3. Ceng, LC, Petrusel, A, Yao, JC: Strong convergence of modified implicit iterative algorithms with perturbed mappings for continuous pseudocontractive mappings. Appl. Math. Comput. **209**(2), 162-176 (2009)
4. Jung, JS: Strong convergence of iterative methods for *k*-strictly pseudo-contractive mappings in Hilbert spaces. Appl. Math. Comput. **215**, 3746-3753 (2010)
5. Ceng, LC, Petrusel, A, Lee, CS, Wong, MM: Two extragradient approximation methods for variational inequalities and fixed point problems of strict pseudo-contractions. Taiwan. J. Math. **13**(2A), 607-632 (2009)
6. Ceng, LC, Shyu, DS, Yao, JC: Relaxed composite implicit iteration process for common fixed points of a finite family of strictly pseudocontractive mappings. Fixed Point Theory Appl. **2009**, Article ID 402602 (2009)

7. Morales, CH, Jung, JS: Convergence of paths for pseudo-contractive mappings in Banach spaces. Proc. Am. Math. Soc. **128**, 4311-4319 (2000)
8. Ceng, LC, Petrusel, A, Szentesi, S, Yao, JC: Approximation of common fixed points and variational solutions for one-parameter family of Lipschitz pseudocontractions. Fixed Point Theory **11**(2), 203-224 (2010)
9. Ceng, LC, Wen, CF: Hybrid method with perturbation for Lipschitzian pseudocontractions. J. Appl. Math. **2012**, Article ID 250538 (2012)
10. Lions, JL: Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires. Dunod, Paris (1969)
11. Ceng, LC, Petrusel, A, Yao, JC: Relaxed extragradient methods with regularization for general system of variational inequalities with constraints of split feasibility and fixed point problems. Abstr. Appl. Anal. **2013**, Article ID 891232 (2013)
12. Korpelevich, GM: The extragradient method for finding saddle points and other problems. Matecon **12**, 747-756 (1976)
13. Peng, JW, Yao, JC: A new hybrid-extragradient method for generalized mixed equilibrium problems. Taiwan. J. Math. **12**, 1401-1432 (2008)
14. Yao, YH, Liou, YC, Kang, SM: Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method. Comput. Math. Appl. **59**, 3472-3480 (2010)
15. Ceng, LC, Wang, CY, Yao, JC: Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities. Math. Methods Oper. Res. **67**(3), 375-390 (2008)
16. Ceng, LC, Latif, A, Ansari, QH, Yao, JC: Hybrid extragradient method for hierarchical variational inequalities. Fixed Point Theory Appl. **2014**, Article ID 222 (2014)
17. Ceng, LC, Ansari, QH, Yao, JC: Relaxed extragradient iterative methods for variational inequalities. Appl. Math. Comput. **218**, 1112-1123 (2011)
18. Ceng, LC, Guu, SM, Yao, JC: Finding common solutions of a variational inequality, a general system of variational inequalities, and a fixed-point problem via a hybrid extragradient method. Fixed Point Theory Appl. **2011**, Article ID 626159 (2011)
19. Ceng, LC, Wong, MM: Relaxed extragradient method for finding a common element of systems of variational inequalities and fixed point problems. Taiwan. J. Math. **17**(2), 701-724 (2013)
20. Ceng, LC, Wong, MM, Latif, A: Generalized extragradient iterative method for systems of variational inequalities. J. Inequal. Appl. **2012**, Article ID 88 (2012)
21. Ceng, LC, Ansari, QH, Wong, MM, Yao, JC: Mann type hybrid extragradient method for variational inequalities, variational inclusions and fixed point problems. Fixed Point Theory **13**(2), 403-422 (2012)
22. Ceng, LC, Latif, A, Wen, CF, Al-Mazrooei, AE: Hybrid steepest-descent methods for triple hierarchical variational inequalities. J. Funct. Spaces **2015**, Article ID 980352 (2015)
23. Ceng, LC, Yao, JC: A relaxed extragradient-like method for a generalized mixed equilibrium problem, a general system of generalized equilibria and a fixed point problem. Nonlinear Anal. **72**, 1922-1937 (2010)
24. Ceng, LC, Ansari, QH, Schaible, S, Yao, JC: Iterative methods for generalized equilibrium problems, systems of general generalized equilibrium problems and fixed point problems for nonexpansive mappings in Hilbert spaces. Fixed Point Theory **12**(2), 293-308 (2011)
25. Ceng, LC, Al-Homidan, S: Algorithms of common solutions for generalized mixed equilibria, variational inclusions, and constrained convex minimization. Abstr. Appl. Anal. **2014**, Article ID 132053 (2014)
26. Ceng, LC, Ansari, QH, Schaible, S: Hybrid extragradient-like methods for generalized mixed equilibrium problems, system of generalized equilibrium problems and optimization problems. J. Glob. Optim. **53**, 69-96 (2012)
27. Ceng, LC, Ansari, QH, Schaible, S: Relaxed extragradient-like method for general system of generalized mixed equilibria and fixed point problem. Taiwan. J. Math. **16**(2), 445-478 (2012)
28. Cai, G, Bu, SQ: Strong and weak convergence theorems for general mixed equilibrium problems and variational inequality problems and fixed point problems in Hilbert spaces. J. Comput. Appl. Math. **247**, 34-52 (2013)
29. Ceng, LC, Wen, CF, Liou, YC: Multi-step iterative algorithms with regularization for triple hierarchical variational inequalities with constraints of mixed equilibria, variational inclusions, and convex minimization. J. Inequal. Appl. **2014**, Article ID 414 (2014)
30. Ceng, LC, Ansari, QH, Yao, JC: Some iterative methods for finding fixed points and for solving constrained convex minimization problems. Nonlinear Anal. **74**, 5286-5302 (2011)
31. Tian, M: A general iterative algorithm for nonexpansive mappings in Hilbert spaces. Nonlinear Anal. **73**, 689-694 (2010)
32. Ceng, LC, Guu, SM, Yao, JC: A general composite iterative algorithm for nonexpansive mappings in Hilbert spaces. Comput. Math. Appl. **61**, 2447-2455 (2011)
33. Marino, G, Xu, HK: A general iterative method for nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **318**, 43-56 (2006)
34. Yamada, I: The hybrid steepest-descent method for variational inequality problems over the intersection of the fixed-point sets of nonexpansive mappings. In: Butnariu, D, Censor, Y, Reich, S (eds.) Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, pp. 473-504. North-Holland, Amsterdam (2001)
35. Ceng, LC, Yao, JC: A hybrid iterative scheme for mixed equilibrium problems and fixed point problems. J. Comput. Appl. Math. **214**(1), 186-201 (2008)
36. Latif, A, Ceng, LC, Ansari, QH: Multi-step hybrid viscosity method for systems of variational inequalities defined over sets of solutions of equilibrium problem and fixed point problems. Fixed Point Theory Appl. **2012**, Article ID 186 (2012)
37. Xu, HK, Kim, TH: Convergence of hybrid steepest-descent methods for variational inequalities. J. Optim. Theory Appl. **119**(1), 185-201 (2003)
38. Ceng, LC, Latif, A, Al-Mazrooei, AE: Composite viscosity methods for common solutions of general mixed equilibrium problem, variational inequalities and common fixed points. J. Inequal. Appl. **2015**, Article ID 217 (2015)
39. Ceng, LC, Latif, A, Yao, JC: On solutions of system of variational inequalities and fixed point problems in Banach spaces. Fixed Point Theory Appl. **2013**, Article ID 176 (2013)
40. Ceng, LC, Yao, JC: Relaxed and hybrid viscosity methods for general system of variational inequalities with split feasibility problem constraint. Fixed Point Theory Appl. **2013**, Article ID 43 (2013)

41. Ceng, LC, Kong, ZR, Wen, CF: On general systems of variational inequalities. Comput. Math. Appl. **66**, 1514-1532 (2013)
42. Marino, G, Xu, HK: Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. J. Math. Anal. Appl. **329**, 336-346 (2007)
43. Goebel, K, Kirk, WA: Topics on Metric Fixed-Point Theory. Cambridge University Press, Cambridge (1990)
44. Xu, HK: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**(2), 240-256 (2002)