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# Periodic solutions for Liénard type equation with time-variable coefficient

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**Abstract**

By means of topological theory, a class of Liénard type equations with a deviating argument of the form  $(\varphi_p(x'(t)))' + f(x(t-\tau))x'(t-\tau) + \beta(t)g(x(t-\tau)) = e(t)$  is studied. It is notable that the coefficient  $\beta(t)$  in front of  $g(x(t-\tau))$  is allowed to change sign in this paper. Moreover, a numerical simulation is carried out to verify the validity of the obtained results. In addition, the generalized form of the above equation with time-varying delays is also discussed briefly.

**Keywords:** periodic solution;  $p$ -Laplacian; topological theory; time-varying coefficient

**1 Introduction**

Consider the Liénard type  $p$ -Laplacian differential equation

$$(\varphi_p(x'(t)))' + f(x(t-\tau))x'(t-\tau) + \beta(t)g(x(t-\tau)) = e(t), \quad (1.1)$$

where  $p > 1$  and  $\varphi_p : R \rightarrow R$  is given by  $\varphi_p(s) = |s|^{p-2}s$  for  $s \neq 0$  and  $\varphi_p(0) = 0$ ,  $\int_0^T \beta(s) ds \neq 0$ ,  $f, g, e \in C(R, R)$ , and  $e(t)$  is  $T$ -periodic,  $\int_0^T e(s) ds = 0$ ,  $T, \tau > 0$  are given constants. The equation can be applied to the dynamics of fluids. An example is Euler's equation, governing the flow of an ideal fluid in a conservative force field. While there are plenty of results on the existence of periodic solutions for the  $p$ -Laplacian equation (see [1–10] and references therein), studying delay Liénard equations with a variable coefficient in front of the nonlinear term is relatively uncommon. The main difficulty lies in finding *a priori bounds* for (1.1) requiring the coefficient of the nonlinear term to keep a fixed sign.

In the aforementioned literature, based on the Leray-Schauder degree theory, Amster *et al.* [1] considered the existence of at least one periodic solution for the  $p$ -Laplacian-like system with a fixed delay. Gao and Zhang [2] discussed an  $n$ -dimensional  $p$ -Laplacian-like neutral functional differential equation, and they established some criteria to guarantee the existence of periodic solutions for the equation by using Mawhin's continuation theorem. Manásevich and Sędziwy [3] were concerned with the existence and uniqueness of a limit circle for a generalized Liénard type equation, which involves the one-dimensional  $p$ -Laplacian operator and a positive small parameter. Manásevich and Mawhin [4] studied the existence of periodic solutions to some system cases involving the fairly general vector-valued  $p$ -Laplacian operator. Aizicovici *et al.* [5] investigated a nonlinear periodic

problem driven by the scalar  $p$ -Laplacian with a nonsmooth potential via the degree map; they proved the existence of at least three distinct nontrivial solutions, two of which have a constant sign. In Gaines and Mawhin’s monograph [6], coincidence degree theory was described and used to study alternative problems. Aizicovici *et al.* [7] conducted periodic problems driven by the scalar  $p$ -Laplacian with a multivalued right-hand side nonlinearity. Gao *et al.* [8, 9] considered the existence of periodic solutions for two kinds of Rayleigh type  $p$ -Laplacian equations by using the continuation theorem. Cheung [10] proposed the existence of periodic solutions of a  $p$ -Laplacian Rayleigh equation with two deviating arguments.

Based on topological theory and some analysis techniques, the existence of periodic solutions for (1.1) is investigated in the present paper. It is significant that the coefficient  $\beta(t)$  of the nonlinear term can change sign, which cannot be achieved in most of the previous papers. Furthermore, a numerical simulation is performed to validate the feasibility of the obtained results. In addition, the fixed delays in (1.1) are also extended to the time-varying delays and we briefly discuss them. Moreover, the approaches used to estimate *a priori bounds* of periodic solutions are different from the corresponding ones in the literature.

### 2 Preliminaries

Let us consider the problem: find  $u \in C_T^1$  such that

$$(\varphi_p(u'(t)))' = \tilde{f}(t, u(t - \tau), u'(t - \tau)). \tag{2.1}$$

**Lemma 2.1** (see Amster *et al.* [1]) *Let  $\Omega \in C_T^1$  be an open set. Assume that:*

(A<sub>1</sub>) *For  $\lambda \in (0, 1]$  the problem*

$$(\varphi_p(u'(t)))' = \lambda \tilde{f}(t, u(t - \tau), u'(t - \tau))$$

*has no solution on  $\partial\Omega$ .*

(A<sub>2</sub>) *The equation*

$$\tilde{F}(a) \triangleq \frac{1}{T} \int_0^T \tilde{f}(t, a, 0) dt = 0$$

*has no solution on  $\partial\Omega \cap R^n$ .*

(A<sub>3</sub>) *The Brouwer degree*

$$\deg(\tilde{F}, \Omega \cap R^n, 0) \neq 0.$$

*Then the problem (2.1) has at least one solution in  $C_T^1$ .*

### 3 Main results

For the sake of convenience, we only study the periodic solutions of (1.1) in the case  $\int_0^T \beta(t) dt > 0$  ( $\int_0^T \beta(t) dt < 0$  can be discussed in the same way).

**Theorem 3.1** *Assume that the following conditions hold:*

(H<sub>1</sub>) There are positive constants  $m_1, m_2$ , and  $d$  such that

$$m_1|u|^{p-1} \leq |g(u)| \leq m_2|u|^{p-1}, \quad \forall |u| > d.$$

(H<sub>2</sub>) There is a constant  $r \geq 0$  such that  $\lim_{|u| \rightarrow \infty} \frac{|F(u)|}{|u|^{p-1}} \leq r$ , where  $F(x) = \int_0^x f(s) ds$ .

(H<sub>3</sub>)

$$A := \begin{cases} \left[ \frac{(\beta_\infty + \varepsilon_1)m_2 T}{m_1 \int_0^T (\beta(t) + \beta_\infty + \varepsilon_1) dt} \right]^{\frac{1}{p-1}} 2^{\frac{2-p}{p-1}} < 1, & 1 < p \leq 2, \\ \left[ \frac{(\beta_\infty + \varepsilon_1)m_2 T}{m_1 \int_0^T (\beta(t) + \beta_\infty + \varepsilon_1) dt} \right]^{\frac{1}{p-1}} < 1, & p > 2, \end{cases}$$

where  $\varepsilon_1 > 0, \beta_\infty = \max_{t \in [0, T]} |\beta(t)|$ .

(H<sub>4</sub>)  $ug(u) > 0, \forall |u| > d$ .

Then (1.1) has at least one  $T$ -periodic solution, if  $C_p r \left[ \frac{T}{2(1-A)} \right]^{p-1} + \frac{\beta_\infty m_2 T}{2(1-A)^p} < 1$ , where

$$C_p = \begin{cases} 1, & 1 < p \leq 2, \\ 2^{p-2}, & p > 2. \end{cases}$$

*Proof* Consider the homotopic equation of (1.1) as follows:

$$(\varphi_p(x'(t)))' + \lambda f(x(t-\tau))x'(t-\tau) + \lambda \beta(t)g(x(t-\tau)) = \lambda e(t), \quad \lambda \in (0, 1]. \tag{3.1}$$

For  $\forall \varepsilon_1 > 0, \beta_\infty = \max_{t \in [0, T]} |\beta(t)|$ , (3.1) can be written in the following form:

$$\begin{aligned} & (\varphi_p(x'(t)))' + \lambda f(x(t-\tau))x'(t-\tau) + \lambda(\beta(t) + \beta_\infty + \varepsilon_1)g(x(t-\tau)) \\ & = \lambda(\beta_\infty + \varepsilon_1)g(x(t-\tau)) + \lambda e(t), \quad \lambda \in (0, 1]. \end{aligned} \tag{3.2}$$

Integrating both sides of (3.2) from 0 to  $T$  and using the integral mean value theorem, there exists a constant  $\xi \in (0, T)$  such that

$$g(x(\xi - \tau)) \int_0^T (\beta(t) + \beta_\infty + \varepsilon_1) dt = \int_0^T (\beta_\infty + \varepsilon_1)g(x(t-\tau)) dt. \tag{3.3}$$

Now, we claim that

$$|x(\xi - \tau)| \leq A|x|_\infty + B, \tag{3.4}$$

where

$$A = \begin{cases} \left[ \frac{(\beta_\infty + \varepsilon_1)m_2 T}{m_1 \int_0^T (\beta(t) + \beta_\infty + \varepsilon_1) dt} \right]^{\frac{1}{p-1}} 2^{\frac{2-p}{p-1}}, & 1 < p \leq 2, \\ \left[ \frac{(\beta_\infty + \varepsilon_1)m_2 T}{m_1 \int_0^T (\beta(t) + \beta_\infty + \varepsilon_1) dt} \right]^{\frac{1}{p-1}}, & p > 2, \end{cases}$$

$$B = \begin{cases} \left[ \frac{(\beta_\infty + \varepsilon_1)M_g T}{m_1 \int_0^T (\beta(t) + \beta_\infty + \varepsilon_1) dt} \right]^{\frac{1}{p-1}} 2^{\frac{2-p}{p-1}} + d, & 1 < p \leq 2, \\ \left[ \frac{(\beta_\infty + \varepsilon_1)M_g T}{m_1 \int_0^T (\beta(t) + \beta_\infty + \varepsilon_1) dt} \right]^{\frac{1}{p-1}} + d, & p > 2, \end{cases}$$

$$M_g = \max_{|u| \leq d} |g(u)|.$$

Case 1: If  $|x(\xi - \tau)| \leq d$ , then (3.4) holds clearly.

Case 2: If  $|x(\xi - \tau)| > d$ . Denote

$$E_1 = \{t : t \in [0, T], |x(t - \tau)| \leq d\}, \quad E_2 = \{t : t \in [0, T], |x(t - \tau)| > d\}.$$

From (3.3), we have

$$|g(x(\xi - \tau))| \int_0^T (\beta(t) + \beta_\infty + \varepsilon_1) dt \leq (\beta_\infty + \varepsilon_1) \left( \int_{E_1} + \int_{E_2} \right) |g(x(t - \tau))| dt,$$

which, together with assumption (H<sub>1</sub>), leads to

$$\begin{aligned} |x(\xi - \tau)|^{p-1} m_1 \int_0^T (\beta(t) + \beta_\infty + \varepsilon_1) dt &\leq (\beta_\infty + \varepsilon_1) \left( \int_{E_2} m_2 |x(t - \tau)|^{p-1} dt + M_g T \right) \\ &\leq (\beta_\infty + \varepsilon_1) m_2 T |x|_\infty^{p-1} + (\beta_\infty + \varepsilon_1) M_g T, \end{aligned}$$

i.e.,

$$\begin{aligned} |x(\xi - \tau)| &\leq \left[ \frac{(\beta_\infty + \varepsilon_1) m_2 T}{m_1 \int_0^T (\beta(t) + \beta_\infty + \varepsilon_1) dt} \right]^{\frac{1}{p-1}} 2^{\frac{2-p}{p-1}} |x|_\infty \\ &\quad + \left[ \frac{(\beta_\infty + \varepsilon_1) M_g T}{m_1 \int_0^T (\beta(t) + \beta_\infty + \varepsilon_1) dt} \right]^{\frac{1}{p-1}} 2^{\frac{2-p}{p-1}}, \quad 1 < p \leq 2, \\ |x(\xi - \tau)| &\leq \left[ \frac{(\beta_\infty + \varepsilon_1) m_2 T}{m_1 \int_0^T (\beta(t) + \beta_\infty + \varepsilon_1) dt} \right]^{\frac{1}{p-1}} |x|_\infty \\ &\quad + \left[ \frac{(\beta_\infty + \varepsilon_1) M_g T}{m_1 \int_0^T (\beta(t) + \beta_\infty + \varepsilon_1) dt} \right]^{\frac{1}{p-1}}, \quad p > 2. \end{aligned}$$

Thus, from Case 1 and Case 2, we see that (3.4) holds.

Let  $\xi - \tau = kT + \bar{\xi}$ , where  $k$  is an integer and  $\bar{\xi} \in [0, T]$ ; noticing (3.4), we get

$$|x(t)| \leq A|x|_\infty + B + \int_{\bar{\xi}}^t |x'(s)| ds, \quad t \in [\bar{\xi}, \bar{\xi} + T],$$

and

$$|x(t - T)| \leq A|x|_\infty + B + \int_{t-T}^{\bar{\xi}} |x'(s)| ds, \quad t \in [\bar{\xi}, \bar{\xi} + T].$$

Combining the above two inequalities, we obtain

$$\begin{aligned} |x|_\infty &= \max_{t \in [\bar{\xi}, \bar{\xi} + T]} |x(t)| \\ &\leq \max_{t \in [\bar{\xi}, \bar{\xi} + T]} \left\{ A|x|_\infty + B + \frac{1}{2} \left( \int_{\bar{\xi}}^t |x'(s)| ds + \int_{t-T}^{\bar{\xi}} |x'(s)| ds \right) \right\} \\ &\leq A|x|_\infty + B + \frac{1}{2} \int_0^T |x'(s)| ds. \end{aligned}$$

In view of (H<sub>3</sub>), we have

$$|x|_\infty \leq \frac{B}{1-A} + \frac{\int_0^T |x'(s)| ds}{2(1-A)}. \tag{3.5}$$

On the other hand, noticing  $C_p r [\frac{T}{2(1-A)}]^{p-1} + \frac{\beta_\infty m_2 T}{2(1-A)^p} < 1$ , we easily see that there is a sufficiently small constant  $\varepsilon_2 > 0$  such that

$$C_p (r + \varepsilon_2) \left[ \frac{T}{2(1-A)} \right]^{p-1} + \frac{\beta_\infty m_2 T}{2(1-A)^p} < 1. \tag{3.6}$$

By assumption (H<sub>2</sub>) and for such  $\varepsilon_2$ , we know there exists a constant  $\rho > d$  (independent of  $\lambda$ ) such that

$$|F(u)| \leq (r + \varepsilon_2) |u|^{p-1}, \quad |u| > \rho. \tag{3.7}$$

Let

$$\Delta_1 = \{t : t \in [0, T], |x(t - \tau)| \leq \rho\}, \quad \Delta_2 = \{t : t \in [0, T], |x(t - \tau)| > \rho\}.$$

Multiplying both sides of (3.1) by  $x(t)$  and integrating them with  $[0, T]$ , noticing (3.7), we get

$$\begin{aligned} \int_0^T |x'(t)|^p dt &= - \int_0^T (\varphi_p(x'(t)))' x(t) dt \\ &= \lambda \int_0^T f(x(t - \tau)) x'(t - \tau) x(t) dt \\ &\quad + \lambda \int_0^T \beta(t) g(t, x(t - \tau)) x(t) dt - \lambda \int_0^T e(t) x(t) dt \\ &\leq \left( \int_{\Delta_1} + \int_{\Delta_2} \right) |F(x(t - \tau)) x'(t)| dt \\ &\quad + \beta_\infty |x|_\infty \left( \int_{E_1} + \int_{E_2} \right) |g(x(t - \tau))| dt + e_\infty |x|_\infty T \\ &\leq [M_\rho + (r + \varepsilon_2) |x|_\infty^{p-1}] \int_0^T |x'(t)| dt + \beta_\infty m_2 T |x|_\infty^p \\ &\quad + \beta_\infty M_g T |x|_\infty + e_\infty T |x|_\infty, \end{aligned} \tag{3.8}$$

where  $M_\rho = \max_{|u| \leq \rho} |F(u)|$ ,  $e_\infty = \max_{t \in [0, T]} |e(t)|$ .

Substituting (3.5) into (3.8) yields

$$\begin{aligned} \int_0^T |x'(t)|^p dt &\leq (r + \varepsilon_2) \left[ \frac{B}{1-A} + \frac{\int_0^T |x'(t)| dt}{2(1-A)} \right]^{p-1} \int_0^T |x'(t)| dt \\ &\quad + \beta_\infty m_2 T \left[ \frac{B}{1-A} + \frac{\int_0^T |x'(t)| dt}{2(1-A)} \right]^p \\ &\quad + \left[ M_\rho + \frac{\beta_\infty M_g T}{2(1-A)} + \frac{e_\infty T}{2(1-A)} \right] \int_0^T |x'(t)|^p dt \end{aligned}$$

$$\begin{aligned}
 & + (\beta_\infty M_g + e_\infty) \frac{BT}{1-A} \\
 \leq & C_p(r + \varepsilon_2) \left[ \left( \frac{B}{1-A} \right)^{p-1} + \left( \frac{\int_0^T |x'(t)| dt}{2(1-A)} \right)^{p-1} \right] \int_0^T |x'(t)| dt \\
 & + \frac{\beta_\infty m_2 T}{2(1-A)^p} \left( \int_0^T |x'(t)| dt \right)^p \\
 & + \left[ M_\rho + \frac{\beta_\infty M_g T}{2(1-A)} + \frac{e_\infty T}{2(1-A)} \right] T^{\frac{1}{q}} \left( \int_0^T |x'(t)|^p dt \right)^{\frac{1}{p}} \\
 & + (\beta_\infty M_g + e_\infty) \frac{BT}{1-A} + \frac{2^{p-1} \beta_\infty m_2 B^p T}{(1-A)^p} \\
 \leq & \left[ C_p(r + \varepsilon_2) \left[ \frac{T}{2(1-A)} \right]^{p-1} + \frac{\beta_\infty m_2 T}{2(1-A)^p} \right] \int_0^T |x'(t)|^p dt \\
 & + \theta_1 \left( \int_0^T |x'(t)|^p dt \right)^{\frac{1}{p}} + \theta_2, \tag{3.9}
 \end{aligned}$$

where

$$\begin{aligned}
 C_p &= \begin{cases} 1, & 1 < p \leq 2, \\ 2^{p-2}, & p > 2, \end{cases} \\
 \theta_1 &= C_p(r + \varepsilon_2) \left( \frac{B}{1-A} \right)^{p-1} T^{\frac{1}{q}} + \left[ M_\rho + \frac{\beta_\infty M_g T}{2(1-A)} + \frac{e_\infty T}{2(1-A)} \right] T^{\frac{1}{q}}, \\
 \theta_2 &= (\beta_\infty M_g + e_\infty) \frac{BT}{1-A} + \frac{2^{p-1} \beta_\infty m_2 B^p T}{(1-A)^p}, \\
 e_\infty &= \max_{t \in [0, T]} |e(t)|.
 \end{aligned}$$

In view of (3.6) and  $\frac{1}{p} < 1$ , it follows from (3.9) that there is a constant  $M > 0$  such that

$$\int_0^T |x'(t)|^p dt \leq M, \tag{3.10}$$

which implies that there exists a constant  $M_1 > 0$  such that

$$|x'|_\infty \leq M_1. \tag{3.11}$$

From (3.5) and (3.10), we can see that there exists a constant  $M_2$  such that

$$|x|_\infty \leq \frac{B}{1-A} + \frac{T^{\frac{1}{q}} M^{\frac{1}{p}}}{2(1-A)} \triangleq M_2. \tag{3.12}$$

According to (3.11) and (3.12), set  $\Omega = \{x : |x'|_\infty < M_1 + 1, |x|_\infty < M_2 + 1\}$ , then we see that (1.1) has no solution on  $\partial\Omega$  for  $\lambda \in (0, 1]$ , and when  $x(t) = M_2 + 1$  or  $-M_2 - 1$ , from (H<sub>4</sub>), we can get

$$-\frac{1}{T} \int_0^T \beta(t)g(M_2 + 1) dt < 0,$$

$$-\frac{1}{T} \int_0^T \beta(t)g(-M_2 - 1) dt < 0.$$

So, condition (A<sub>2</sub>) of Lemma 2.1 is also satisfied.

Denote

$$H(x, \mu) = \mu x + \frac{1 - \mu}{T} \int_0^T \beta(t)g(x) dt,$$

and when  $x \in \partial\Omega \cap R$ ,  $\mu \in [0, 1]$ , in view of  $\int_0^T \beta(s) ds > 0$ , we have

$$xH(x, \mu) = \mu x^2 + \frac{(1 - \mu)x}{T} \int_0^T \beta(t)g(x) dt > 0.$$

Thus,  $H(x, \mu)$  is a homotopic transformation and

$$\begin{aligned} \deg \{ \tilde{F}, \Omega \cap R, 0 \} &= \deg \left\{ -\frac{1}{T} \int_0^T \beta(t)g(x) dt, \Omega \cap R, 0 \right\} \\ &= \deg \{ x, \Omega \cap R, 0 \} \neq 0. \end{aligned}$$

Therefore, condition (A<sub>3</sub>) of Lemma 2.1 is also satisfied. By using Lemma 2.1, we conclude that (1.1) has at least one  $T$ -periodic solution  $x(t)$  on  $\bar{\Omega}$  with  $|x|_\infty \leq M_2$ . This completes the proof of Theorem 3.1. □

#### 4 Generalization

As a matter of fact, (1.1) can also be extended to the time-varying case, which admits the following form:

$$(\varphi_p(x'(t)))' + f(x(t - \tau))x'(t - \tau) + \beta(t)g(x(t - \delta(t))) = e(t), \tag{4.1}$$

where  $\delta(t + T) \equiv \delta(t)$ ,  $\delta'(t) < 1$ , and  $\int_0^T \beta(s) ds \neq 0$  (the case of  $\int_0^T \beta(s) ds > 0$  will be considered in this section, and the other case,  $\int_0^T \beta(s) ds < 0$ , can be studied in the same way). Then, according to Mawhin’s continuation theorem, the following result is obtained with a similar process to Theorem 3.1.

**Theorem 4.1** *Assume that the following conditions hold:*

(C<sub>1</sub>) *There exist constants  $r_1 > 0$ ,  $r_2 > 0$ , and  $d_1 \geq 0$  such that*

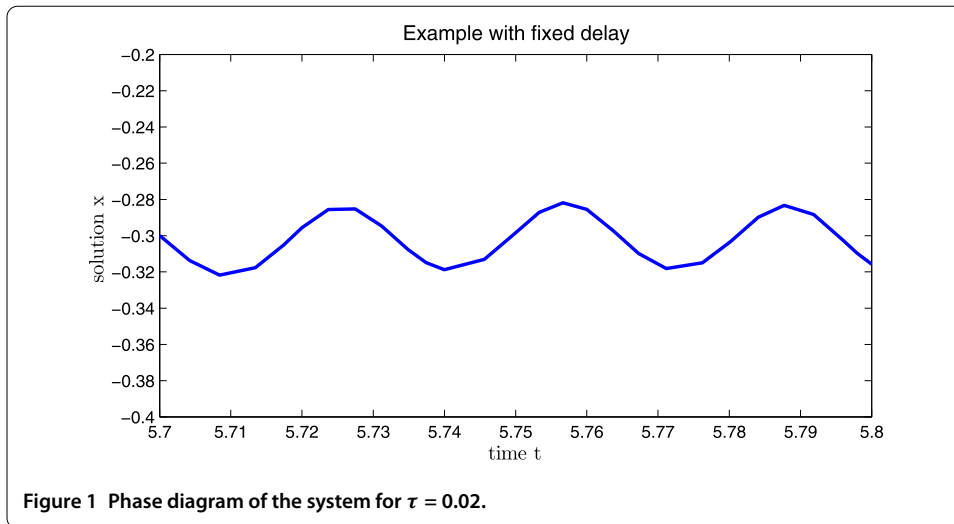
$$r_1|u|^{p-1} \leq |g(u)| \leq r_2|u|^{p-1} \quad \text{and} \quad ug(u) > 0, \quad \forall |u| > d_1.$$

(C<sub>2</sub>) *There are constant  $r_3 > 0$  and  $d_2 \geq 0$  such that  $|f(u)| \leq r_3|u|^{p-1}$ ,  $\forall |u| > d_2$ .*

(C<sub>3</sub>) *There is a constant  $\varepsilon_2 > 0$  such that*

$$M := \begin{cases} \left[ \frac{(\beta^-|_{\infty+\varepsilon_2})r_2T}{r_1 \int_0^T (\beta^+ + \varepsilon_2)dt} \right]^{\frac{1}{p-1}} 2^{\frac{2-p}{p-1}} < 1, & 1 < p \leq 2, \\ \left[ \frac{(\beta^-|_{\infty+\varepsilon_2})r_2T}{r_1 \int_0^T (\beta^+ + \varepsilon_2)dt} \right]^{\frac{1}{p-1}} < 1, & p > 2, \end{cases}$$

where  $\beta^+ = \max_{t \in [0, T]} \{\beta(t), 0\}$ ,  $\beta^- = \max_{t \in [0, T]} \{-\beta(t), 0\}$ .



Then (4.1) has at least one  $T$ -periodic solution if

$$C_p r_3 T^{p-1} + 2^{p-1} r_2 \theta |\beta|_\infty T^p < (1 - M)^{p-1},$$

where  $\theta = \max_{t \in [0, T]} \frac{1}{1 - \delta'(\gamma(t))}$  and  $\gamma(t)$  denotes the inverse of function  $t - \delta(t)$ .

### 5 Example

For convenience, as an application of Theorem 3.1, we consider the following example:

$$(\varphi_4(x'(t)))' + f(x(t - \tau))x'(t - \tau) + \left(\sin t + \frac{1}{2}\right)g(x(t - \tau)) = \cos 200t, \tag{5.1}$$

where  $p = 4$ ,  $f(t) = \frac{t^2 \sin t}{500,000}$ ,  $\beta(t) = \sin t + \frac{1}{2}$ ,  $g(u) = \frac{u^3}{10,000} + 1$ ,  $e(t) = \cos 200t$ , then  $F(u) = \int_0^u f(s) ds = \frac{1}{500,000}(-u^2 \cos u + 2u \sin u + 2 \cos u - 2)$ , and  $\beta_\infty = \frac{3}{2}$ . So we can choose  $m_1 = \frac{1}{10,000}$ ,  $m_2 = \frac{1}{5,000}$ ,  $d = 10\sqrt[3]{10}$ ,  $r = \frac{1}{500,000}$ ,  $\varepsilon_1 = \frac{1}{2}$  so that the conditions of Theorem 3.1 hold. Therefore, by Theorem 3.1 we find that (5.1) has at least one  $\frac{\pi}{100}$ -periodic solution, which can also be illustrated by numerical simulation.

By using MATLAB<sup>®</sup> 7.12.0 (R2011a) toolkit: *dde23*, which can be used to solve delay differential equations with constant delays, (5.1) is simulated on  $tspan = [5.7, 5.8]$  with history =  $[0.5, -0.3]$  for  $t \leq 0$  (see Figure 1). It can be found from Figure 1 that the equation admits one periodic solution with periodicity 0.0314, which is around  $\frac{\pi}{100}$ . Therefore, the results achieved in this paper are significant.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to this work. All authors read and approved the final version of the manuscript.

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